Artificial Intelligence Review
https://doi.org/10.1007/s10462-023-10474-8

A survey of fractional calculus applications in artificial neural networks

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Abstract

Artificial neural network (ANN) is the backbone of machine learning, specifically deep learning. The interpolating and learning ability of an ANN makes it an ideal tool for modelling, control and various other complex tasks. Fractional calculus (FC) involving derivatives and integrals of arbitrary non-integer order has recently been popular for its capability to model memory-type systems. There have been many attempts to explore the possibilities of combining these two fields, the most popular combination being the use of fractional derivative in the learning algorithm. This paper reviews the use of fractional calculus in various artificial neural network architectures, such as radial basis functions, recurrent neural networks, backpropagation NNs, and convolutional neural networks. These ANNs are popularly known as fractional-order artificial neural networks (FANNs). A detailed review of the various concepts related to FANNs, including activation functions, training algorithms based on fractional derivative, stability, synchronization, hardware implementations of FANNs, and real-world applications of FANNs, is presented. The study also highlights the advantage of combining fractional derivatives with ANN, the impact of fractional derivative order on performance indices like mean square error, the time required for training and testing FANN, stability, and synchronization in FANN. The survey reports interesting observations: combining FC to an ANN endows it with the memory feature; Caputo definition of fractional derivative is the most commonly used in FANNs; fractional derivative-based activation functions in ANN provide additional adjustable hyperparameters to the networks; the FANN has more degree of freedom for adjusting parameters compared to an ordinary ANN; use of multiple types of activation functions can be employed in FANN, and many more.

Keywords Fractional calculus · Fractional-order neural networks · Gradient descent · Least mean square · Architectures · Stability · Synchronization

Abbreviations

ANN	Artificial neural network
FC	Fractional calculus

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Published online: 25 April 2023

FANN	Fractional-order artificial neural network
NN	Feural network
CNN	Convolutional neural network
RNN	Recurrent neural network
DRNN	Deep recurrent neural network
R-L	Riemann-Liouville (derivative)
G-L	Grunwald-Letnikov (derivative)
M-L	Mittag-Leffer function
RBF	Radial basis function
FGD	Fractional gradient descent-based
BP	Backpropagation
LMS	Least mean square
NLMS	Normalized least mean square
Fx-NLMS	Filtered-x normalized least mean squares
MSE	Mean squared error
FNCC	Fractional normalized convex combination
LSTM	Long-short term memory
ARFIMA	Auto regressive fractional integrated moving average
GRNN	Generalized regression radial basis neural network
MAE	Mean absolute error
MAPE	Mean absolute percentage error
FNPK	Fractional neutron point kinetics
F-ROM	Fractional reduced-order model
LMI	Linear matrix nnequality
FHNN	Fractional Hopfield NN
FODHNN	FO discrete Hopfield NN
MWCFONN	Multiple weighted coupled FANN
FOQVNN	FO quaternion-valued NN
fPIRNN	FO physics-informed RNN
FQVBAMNN	Fractional-order quaternion-valued bidirectional associative memory neural network
BAM	Bidirectional associative memory
FMNN	Fractional-order memristor neural network
FMBAMNN	Fractional-order memristive BAM neural network
MWCFCNN	Multi-weighted complex structure on fractional-order coupled neural network
FPGA	Field programmable gate array
FODHNN	Fractional-order discrete hopfield neural network
WT	Wavelet transform
MLP	Multilayer perceptron
AFDA	Adaptive fractional differential algorithm
FODPSO	Fractional order darwinian particle swarm optimization
DCNN	Deep convolutional neural network
FOC	Fractional optimal control
PSNR	Peak signal to noise ratio
CSFCO	Cat swarm fractional calculus optimization
CSO	Cat swarm optimization
MAPSO-EFFO	Aadaptive particle swarm optimization and the enhanced fruit fly optimization

VdPDO	Van der pol-duffing oscillator
FLANN	Functional link ANN
MOESP	Multivariable output error state space
FCVNN	Fractional complex-valued neural Network
FBAMNN	Fractional-order BAM neural network
MDFNN	Memristor-based delay fractional-order neural network
FMBFII	Free-matrix-based fractional-order integral inequality
FDTNN	Fractional-order discrete-time neural network
FDNNDA	Fractional-order delayed neural network with discontinuous activations
FOMCVNN	Fractional-order memristive complex-valued neural network
PCA	Principal component analysis
FOS	Fractional order system
SiLU	Sigmoid weighted linear units
dSiLU	Derivative of SiLU
RreLU	Randomized leaky ReLU
SreLU	S-shaped ReLU
PreLU	Parametric rectified linear units
LreLU	Leaky ReLU
ELU	Exponential linear units
PELU	Parametric exponential linear unit
SELU	Scaled exponential linear units
EliSH	Exponential linear squashing
FRBF	Fractional radial basis function
FORNN	Fractional recurrent NN
FBNN	Fractional deep backpropagation NN
FCNN	Fractional convolutional NN
GELU	Gaussian error linear unit
NLP	Natural language processing
MNIST	Modified national institute of standards and technology database

1 Introduction

An artificial neural network (ANN) is endowed with features like parallel computing, handling large datasets, and performing nonlinear operations Sivanandam and Deepa (2007). Due to these features, ANN processing exploits various areas like medical analysis, education, agriculture, industry, weather forecasting, tourism, textiles, manufacturing industry, defense, governance, marketing, and many more fields for various applications involving recognition, prediction, decision-making, classification, region-based traffic flow prediction, image processing, an image measuring system, etc (Sivanandam and Deepa 2007; Cheng et al. 2023; Zhang et al. 2022; Liu et al. 2022a, b). It can also be used for the analysis of nonlinear systems in presence of noise and provide accurate results.

In general, any ANN consists of three layers, an input layer, a hidden layer, and a output layer. Each layer can have only a dozen units or nodes or millions of units depending upon the complexity of the system. When the complexity of function and data increases, the number of hidden layers is increased, NNs thus formed by the addition of more layers are called deep neural networks. This deep NN is used for the analysis and processing of big and complex data. ANNs are trained using a training set. For example, to train an ANN to recognize the picture of a lion, thousands of images of lions are shown, these thousand images are called training sets. Once the ANN is trained, to check whether ANN can recognize a Lion or not, some unknown images are given and checked whether ANN can identify or not. If ANN is identified incorrectly, then backpropagation algorithm is used to adjust whatever it has learned during training. Backpropagation is employed by updating the weights of the connections in ANN nodes based on the error obtained. This process continues until the ANN can correctly recognize a lion in an image with minimal possible error rates.

The branch of mathematics dealing with integrodifferential operators with arbitrary non-integer order (real or complex) is popularly known as fractional calculus (FC). The history of FC dates back to seventeenth century. There is now abundant literature available clearly pointing towards the superiority of fractional calculus as a mathematical tool in physics, chemistry, biology, engineering, finance, and many more (Hilfer 2000; Shen 2018; Ionescu et al. 2017; Debnath 2003; Bukhari et al. 2020; Li et al. 2023; Zhou and Zhang 2022). The fractional derivative operators, by definition, involve an integration, making them nonlocal operators as opposed to classical integer-order derivative operators. This non-locality property of fractional derivatives is exploited in modelling systems with memory and spatially distributed dynamics (Loverro 2004; Valério et al. 2013; De Oliveira and Tenreiro Machado 2014; Gutierrez et al. 2010).

When the ANNs are modeled by employing fractional differential equations (FDE), they are named fractional artificial neural networks (FANNs). The FDE describes the dynamic behavior of the ANNs neurons accurately. Hence, engineering systems can be modeled more efficiently and accurately by FANN. Recently in ANN, authors have employed FC either in activation functions (Zamora Esquivel et al. 2019; Ivanov 2018) or in the backpropagation algorithm for updating weights (Khan et al. 2018a, b; Bao et al. 2018; Sheng et al. 2020). Application of FANN is observed in various areas like image processing, biomedical, finance, control systems, energy, system identification, digital signal processing, 5G wireless technology, etc. FANN has been successfully used to operate several underwater unmanned vehicles (Liu et al. 2022). The main advantages of using FC in these applications are efficient information processing and accurate results.

Experiments have shown the improved performance of FANN in terms of various parameters like better accuracy, stability, and synchronization. In Viera-Martin et al. (2022), Maiti et al. (2022), Raubitzek et al. (2022), a review related to the ANN involved with FC is carried out. Specifically, in Viera-Martin et al. (2022), the analytical and numerical methods employed to solve the differential equations and fractional differential equations are presented. It also overviews the optimization algorithms employed for the training of ANN involved with FC and the control strategies employed to synchronize and stabilize ANN involved with FC. In Maiti et al. (2022), the critical review of various advances in the fractional-order form of hopfield, cellular, memristive, complex-valued, and other neural networks, and their applications has been presented. An review of how fractional derivatives could improve machine learning techniques has been presented in Raubitzek et al. (2022).

In several science and engineering fields, a neural network is used as a mathematical modeling tool for complex dynamic systems. An ANN 'remembers' the input-output applied to it by virtue of its learning capability. On the other hand, the nonlocal fractional derivative operators are well known for faithfully modelling systems with memory. These two facts consequently lead to a promising possibility that the use of fractional derivatives in an ANN might very well enhance its learning capabilities and accuracy (Zhang et al. 2017a, b). The survey confirms this fact. ANNs are used extensively in machine learning and deep learning. So any improvement in the performance of an ANN will contribute to better AI and ML algorithms. Due to memory effect and heredity characteristics fractional calculus is mainly employed in neural networks to model the reality of the system. This provides system stability, synchronization, and parameters training by applying optimum algorithms. Nowadays memristive networks have gained much attention (Huang et al. 2012; Kaslik and Sivasundaram 2012; Chen et al. 2013, 2014; Stamova 2014; Zhang et al. 2015; Wu and Zeng 2017; Bao et al. 2018; Li et al. 2019).

In Rakkiyappan et al. (2015), Rakkiyappan et al. (2016) authors have proposed fractional-order-based neural networks. In these papers, it is demonstrated that fractional-order feedforward neural networks are easier and faster to convergent compared to integer-order feedforward neural networks. Due to faster convergence, memory, and heredity properties, authors have proposed fractional-order neural networks to model complex dynamic systems to improve stability and synchronization (Chen 2013).

In addition, this survey also reports the use of fractional activation functions, the effect of fractional derivative order on the stability and synchronization of the ANNs, deep FANN and its applications, and hardware implementation of FANNs. In this paper, we present a state of the art review of fractional-order artificial neural networks, use of fractional activation functions, the impact of fractional order on the stability, and synchronization of FANN. This work, probably for the first time, presents a detailed review on the following topics related to FANN:

(1) Training algorithms in FANN.
(2) Stability and synchronization in FANN.
(3) Sensitivity analysis of FANN.
(4) Hardware implementation of FANN.
(5) Applications of FANN.

The paper is organized as follows. Section 2 presents the fundamentals of fractional calculus. Introduction to fractional artificial neural networks is given in section 3. Section 4 details the training algorithms used for FANN. Section 5 presents the review of stability and synchronization analysis for FANN. Review on the topic of sensitivity analysis of FANNs is given in section 6. Section 7 discusses the hardware implementation of FANN. Section 8 lists major applications of ANNs involved with fractional derivatives. The overview of the survey including the challenges and probable future directions in the area of combining fractional calculus and artificial neural networks is discussed in section 9. Conclusion is given in section 10.

2 Preliminaries

In this section, a brief about fractional calculus and fractional artificial neural networks is presented. We review the theory of arbitrary-order integrals and derivatives, which generalize and integrate the concepts of integer-order differentiation and n-fold integration.

2.1 Fractional calculus (FC)

Fractional calculus (FC) is a powerful mathematical tool for modeling a wide range of complex real-world and engineering systems (Carpinteri and Mainardi 1997; Machado et al. 2011). The three popular fundamental definitions of fractional derivatives (FDs) are:

(1) Grunwald-Letnikov (GL) derivative of fractional order $\alpha \in \mathbb{R}^+$ Das (2011): It is defined as

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \, {}^\alpha C_j f(t-jh), \tag{1} $$

where, $[x]$ is the integer part of x and ${}^\alpha C_j$ is the binomial coefficient.

(2) Riemann–Liouville (RL) derivative of fractional order $\alpha \in \mathbb{R}^+$ Das (2011): It is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2} $$

for $n-1 < \alpha < n, n \in \mathbb{Z}^+$ and $\Gamma(\cdot)$ is the Gamma function.

(3) Caputo derivative of fractional order $\alpha \in \mathbb{R}^+$ Das (2011): It is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{3} $$

for $n-1 < \alpha < n, n \in \mathbb{Z}^+$, where $f^n(\tau)$ is the *nth*-order derivative of the function *f*(*t*).

It should be noted that for a causal (initially relaxed) function, all the three definitions coincide. The Caputo definition, though more restrictive, is widely employed as it allows the use of physical initial conditions.

Apart from the above definitions, Weyl, Marchaud, Hadamard, Chen, Davidson-Essex, Coimbra, Canavati, Riesz, and Osler derivative formulae have been defined. Fractional integrals' definitions like R-L (left-sided), R-L (right-sided), Hadamard, Weyl, Chen (left-sided), Chen (right-sided), Erd-elyi (left-sided), Erd-elyi (right-sided) integral, Kober (left-sided), and Kober (right-sided) have been defined De Oliveira and Tenreiro Machado (2014). Different tools for the usage of FC in different areas have been described in Li et al. (2017). A new fractional green function related to the fractional telegraph equation has been described Figueiredo Camargo et al. (2008). The fundamental theorem of FC in various versions like R-L, Liouville, Caputo, Weyl, and Riesz has been explained Grigoletto and de Oliveira (2013). A new conformable fractional derivative has been derived in Abdeljawad (2015).

Simplification of discrete FC for difference equations has been described in Atici and Eloe (2009). A newly discovered simple definition of the fractional derivative and fractional integral has been described Khalil et al. (2014). From 1974 to 2010, including documents and events, FC's past has been described (Machado et al. 2011). The work based on fractional differential equations and fractional calculus in different areas by various authors till 2013 has been reviewed in Karniadakis et al. (2015). Two new criteria for fractional operators have been derived, which can be accessed by the G-L, R-L, and Caputo derivatives, and the Riesz are proposed in Ortigueira and Tenreiro Machado (2015). Fractional derivatives of basic functions like exponential, sine, cosine, etc. are computed and analyzed in Kleinz and Osler (2000). Geometric and physical interpretations for various fractional integrations and differentiations have been proposed in Podlubny (2002). In Borredon et al. (1999), basic definitions of fractional derivatives and integrals have been defined with illustrative examples. The evolution of FC, its history, and contribution of different authors in FC has been reported in Machado et al. (2014). The basics, theory, and applications of FC have been represented in Ortigueira (2011), Samko et al. (1993), Diethelm (2010), Davies (2002), Herrmann (2018), Liu et al. (2022). The definition of R-L fractional derivative has been derived in Samko et al. (1993). Riesz potential operators' new definitions are discussed in Ortigueira (2006a), Ortigueira (2006b) and the designing of problems based on digital Riesz FC has been illustrated in Tseng and Lee (2014).

In Song et al. (2022), the new discrete Hadamard FC is presented using time-scale theory. The boundedness theorem of general FC has been presented in Fan et al. (2022). It is widely trendy in continuous time random walks. Fractional difference equations have recently gained popularity in FC. By utilizing the Laplace transform, solutions of semi-linear equations are proposed in Baleanu and Wu (2019). Here, new fractional sum equations are developed using Picard's approach.

2.2 Mittag-Leffler function

Mittag-Leffler function is a generalization of the exponential function, which has a major role in FC Haubold et al. (2011). This function can be represented in generalized form as follows,

$$E_{\alpha,\beta}(Z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + \beta)}, \tag{4}$$

where $\alpha, \beta \in \mathbb{R}^+, \mathbb{R}^+ \in \mathbb{C}$, $\Gamma(.)$ represents the Euler's gamma function, and $\mathbb{C}$ represents the set of complex numbers. Substitution β=1, gives 1-parameter MLF:

$$E_{\alpha}(Z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(1 + \alpha k)}. \tag{5}$$

The Mittag-Leffler function is widely used in various applications (Haubold et al. 2011). It plays a crucial role in stability studies of FANN.

3 ANN based on fractional derivatives (FANN)

Due to long-term memory feature of fractional derivatives, ANNs are modeled by employing fractional differential equations (FDE). Such networks are named fractional artificial neural networks (FANNs). In an ANN, the activation functions play an important role to provide accurate output. However, if we employ fractional derivative-based activation functions in ANN, it provides additional adjustable hyperparameters to the networks. In this section, we present a brief review of activation functions. For example, the Step function is not compatible with some NN architectures, specifically, the backpropagation architecture Werbos (1974). ReLU can destroy neurons during training so that they do not activate for any data set Karpathy and Fei-Fei (2015). To solve this problem, ELUs in Clevert et al. (2015), Leaky ReLUs in Maas et al. (2013), the PReLU in He et al. (2015), and the Swish activation functions have been proposed. It is also observed that by combining different types of activation functions using fractional derivatives and modifying the order of the fractions, multiple types of activation functions can be employed in ANN Zamora Esquivel et al. (2019).

Table 1 describes the summary of the activation functions and their fractional derivative. The functions with exponential terms (trigonometric, hyperbolic) can be represented in terms of Mittag-Leffler (M-L). In Ivanov (2018), feedforward neural networks have been employed with M-L functions, thereby increasing the degree of freedom for adjusting parameters Ivanov (2018).

4 Training algorithms in FANN

FANN is trained by using algorithms that employ FO derivatives in weight updation. In this section, we present a detailed commonly used algorithms for FANN structure.

4.1 Gradient descent method

The gradient descent method is the most preferred method for ANN as it is an optimization algorithm to search local minima for differentiable functions. The meaning of gradient is the rate of change of output with respect to input. In other words, it is a differentiable function. It has been used to calculate the difference between the weights with respect to the change in error. The algorithm aims to find the coefficients in such a way that the value of the loss function or cost function is minimum. The weight is updated as follows,

$$w^{\lambda+1} = w^{\lambda} - \eta \Delta E_n, \tag{6}$$

where τ represents the iteration number, η denotes the learning rate parameter and ΔE_n represents the difference between the error function Bishop (2006).

In Wei et al. (2020), authors have proposed a method in which lower integral term is iterated and higher order terms are divided, and designed in a variable fractional order. RBFs, RNNs, backpropagation NNs and CNNs employ gradient descent algorithms for updating weights in the training process. Since the gradient descent weight update process has a derivative in the weight updating formula and if fractional order derivative is employed then the gradient descent process is termed as the fractional gradient descent process. And all the neural networks employing the fractional weight update process are termed as Fractional order ANN architecture. Following are the general steps to employ the fractional gradient descent algorithm in FANN:

Table 1 Summary of fractional derivative of activation functions Zamora Esquivel et al. (2019)

Sr. No.	Activation Function	Mathematical Representation	Fractional derivative of activation function
1	ReLU	$[f(x) = x]$ for (x>0), otherwise $[f(x) = 0]$	$g(x) = D^a f(x) = \frac{\Gamma(2)}{\Gamma(2-a)} x^{(1-a)}$
2	Sigmoid	$f(x) = log(1 + e^x)$	$g(x) = D^a ln(1 + e^x)$ $G(x) = \lim_{h\to 0} \frac{1}{h^a} \sum_{n=0}^{\infty} (-1^n) \frac{\Gamma(a+1) ln(1+e^{x-nh})}{\Gamma(n+1)\Gamma(1-n+a)}$
3	Hyperbolic Tangent	f(x) = tanh(x)	$g(x) = D^a tanh(x)$ $G(x) = \lim_{h\to 0} \frac{1}{h^a} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(a+1) tanh(x-nh)}{\Gamma(n+1)\Gamma(1-n+a)}$

(1) Obtain output y(n) of FANN.
(2) Let d(n) is desired response of FANN.
(3) Obtain estimation error ΔE_n of FANN.

$$\Delta E_n = d(n) - y(n). \tag{7}$$

(4) The weight vector is updated as follows by following Eq. 6

$$w^{\lambda+1} = w^{\lambda} - \eta \Delta E_n.$$

(5) By using the fractional gradient descent algorithm, weights are updated as follows,

$$w^{\lambda+1} = w^{\lambda} - \eta_v D^v \Delta E_n, \tag{8}$$

where η_v and D^v are step sizes used for the fractional gradient descent algorithm and fractional derivative of order v respectively.

(6) Weights are updated till minimum error is obtained. FANN can be simulated by varying orders of v. For various values of v, the performance of FANN can be computed in terms of MSE, number of epochs, training time required to train the networks, etc.

Fractional gradient method is employed in various types of neural networks like radial basis functions, recurrent neural networks, convolution neural networks, backpropagation networks, complex-valued networks. These methods are reviewed in brief in the following subsections.

4.1.1 FC in radial basis functions (RBF)

An implementation of RBF-NN using fractional gradient descent-based learning algorithm (FGD) has been presented in Khan et al. (2018a). This algorithm is a combination of conventional and modified R-L derivative-based fractional gradient descent methods. Fig. 1 represents the architecture of RBF.

Φi is the i^{th} hidden neuron's basis function. It uses the Gaussian kernel function. *Wi* represents the synaptic weights between the hidden layer and the output neuron, and *Xn* represents the input to RBFNN.

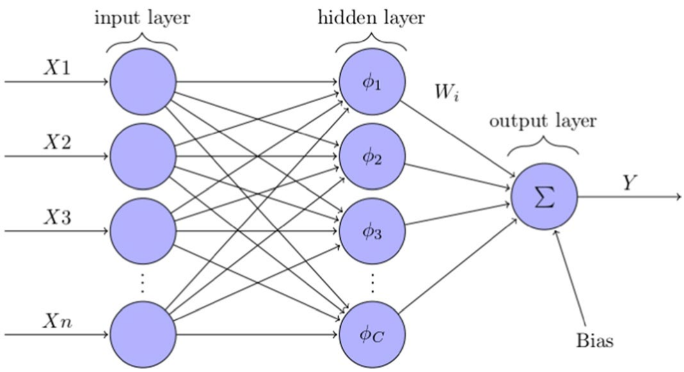

Fig. 1 RBF neural network architecture Khan et al. (2018a)

Reference Khan et al. (2018a) has applied this weight updation expression to the following applications:

- System identification: FRBF has required fewer epochs than RBF during the testing phase for a given SNR.
- Pattern classification: FRBF has achieved slightly better accuracy compared to RBF and required a lower number of epochs and the convergence rate has been improved.
- Time series Prediction: While testing, FRBF MSE is lower compared to RBF.
- Function Approximation: Improved accuracy during training and testing and better convergence.

Gradient descent algorithms based on the Caputo fractional derivative for the training of RBF have been implemented in Xue (2021) . The algorithm considers momentum while training NN. Results illustrate that the proposed design based on Caputo fractional derivative has better performance.

4.1.2 FC in Recurrent NN

In Khan et al. (2018b), Pu et al. (2017), recurrent NN has been implemented using a gradient descent algorithm, proposing a new weight-updating formula. Fig. 2 illustrates the architecture of recurrent NN. Weights have been updated by fractional gradients and have been obtained by using the chain rule. The proposed algorithm implemented for three different applications illustrate improved results in terms of better accuracy and less MSE compared to the same algorithm without fractional calculus. In Yao and Wang (2020), a new algorithm has been proposed for echo state networks that have a fractional order. The Echo State Network belongs to the class of RNN, where the RNN's hidden layer is replaced with a dynamical reservoir. In this network, only output weights have been trained. Other weights are considered to be random values (Jaeger and Haas 2004; Jaeger 2001; Zhang et al. 2014).

Reference Khan et al. (2018b) has applied this weight-updated formula to the following applications:

- Non-linear system identification: Slightly improved accuracy (less MSE) has been obtained for fractional order RNN (FORNN) compared to RNN.
- Pattern classification: During testing of the network, better accuracy of FORNN has been obtained as compared to RNN.

It has required slightly more time for training for almost all types of FRNN compared to integer-order RNN.

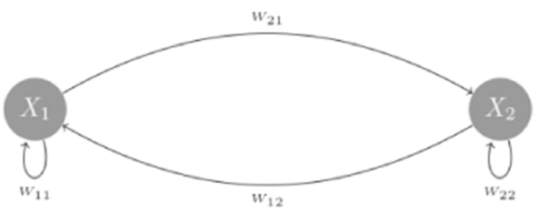

Fig. 2 A simple two-neuron recurrent neural network Khan et al. (2018b)

4.1.3 FC in backpropagation (BP) neural networks

A deep backpropagation (BP) neural network model using Caputo derivative fractional-order and L_2 regularization is described in Bao et al. (2018). L_1 (Lasso Regression) and L_2 (Ridge Regression) are two types of regularization described in the literature. In L_2 regularization, the squared magnitude of the coefficient is multiplied by λ and it is constant. The value of λ should be properly selected. It will result in under-fitting if its value is high. This regularization has been used to minimize over-fitting problems. Different conditions have been derived to meet the convergence of the algorithm. Finally, the impact of fractional order and regularization effect on convergence has been discussed.

The Caputo derivative-based training process in the same network but without using L_2 regularization has been proposed in Wang et al. (2017), Chen and Zhao (2019). An architecture of multi-layers has been implemented to recognize handwritten digits from the Modified national institute of standards and technology database (MNIST) dataset. In Chen et al. (2020), an algorithm for backpropagation networks has been proposed that uses a combination of extremal optimization of initial weight parameters and fractional order gradient descent weight update process. The given network has been used to recognize MNIST handwritten digits. Results show that the proposed combined algorithm has more accuracy (in training as well as testing) compared to the integer order network.

4.1.4 FC in convolutional neural networks (CNN)

CNN's architecture consists of several layers, including convolution and pooling layers. The layer is serially connected to the other layer as shown in Fig. 3 and loss functions are added.

In Sheng et al. (2020), CNN has been implemented using Caputo derivative-based fractional gradient method for backpropagation. Between two different layers, integer order gradients have been used, but within layers, fractional gradients have been used, which will update parameters, keeping the chain rule. Caputo FO gradient method has been used by authors in Chen et al. (2022) to enhance the dynamic updating efficiency of CNN's biases and weights.

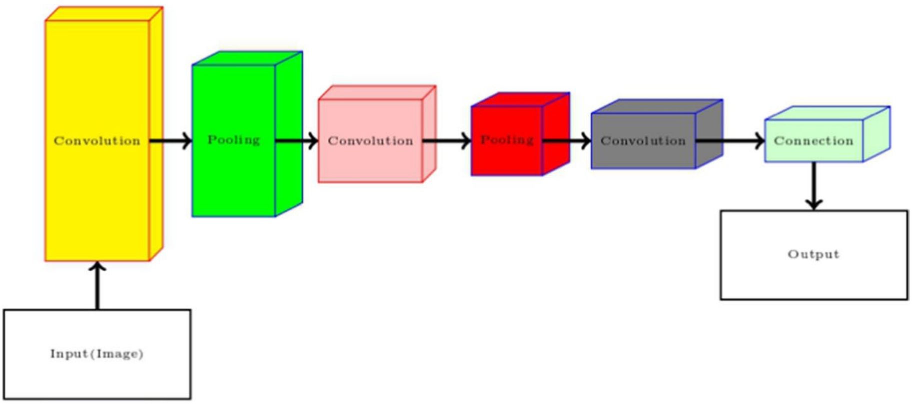

Fig. 3 An architecture of CNN Sheng et al. (2020)

4.1.5 Complex valued FANN

Fractional gradient descent can be used for complex-valued neural networks, as in Wang et al. (2017), where authors have trained complex neural networks using the Caputo derivative-based gradient descent algorithm. In Khan et al. (2019), an adaptive stochastic gradient descent has been proposed, which is fractional in nature, and has more flexibility for designing and has more control parameters. The proposed method has been compared with the stochastic gradient descent method and found to be fast convergent and accurate.

Table 2 summarizes the results of various FANNs that use the fractional order gradient descent method. For each FANN, the type of FC using the weight update formula and applications are depicted. Also, the performance of FANN in terms of MSE and accuracy compared to integer order NN is reported.

The used notations used in Table 2 are: e_k: Estimation error; α: Mixing parameter of gradient FO; W_{k_j}: Weight at k^{th} neuron; δ_k: Local gradient for k^{th} neuron; Y_j: Output of j^{th} neuron; E: Total error of NN; E_{L_2}: Error after introducing L_2 Regularization; η: Learning rate; L: Loss function; μ: Iterative step size; W^l_{ji}: Weight matrix between l^{th} layer and $(l+1)^{th}$ layer; t: Integer representing t^{th} iteration; W^l_k: Weight matrix between fully connected layers

In Khan et al. (2018a), Khan et al. (2018b), it is shown with extensive simulation studies that FANN requires less time for training compared to integer-order NN, albeit at the cost of increased time for testing.

Table 2 Summary of weight updating formula of various FANN architectures

Sr. No.	Type of ANN	Type of FC used	Final weight updation formula by using FC and gradient descent	Performance of FANN
1	FRBF Khan et al. (2018a)	Modified R-L derivative	$W_i(n+1) = W_i(n)$ $+e_k[\alpha n + (1-\alpha)n_v W_i^{(1-V)}(t)]\Phi_i(x, x_i)$.	Improved parameters (accuracy, MSE) of FRBF are obtained compared to RBF.
2	FORNN Khan et al. (2018b)	R-L derivative	$Wk_j(n+1) = Wk_j(n) + \sum_{t=n-h-1}^{n} \delta_k(t)$ $*\ [\alpha n + (1-\alpha)\eta_f W_{K_j}^{(1-f)}(t)]Y_j(t-1)$.	Slightly improved accuracy (less MSE) is obtained for FORNN compared to RNN.
3	FBNN Bao et al. (2018)	Caputo derivative	$(W^l_{ji})^{(t+1)} = (W^l_{ji})^{(t)} - \eta D^v_{(w^l_{ji})^t} E$, without L_2 Regularization $(W^l_{ji})^{(t+1)} = (W^l_{ji})^{(t)} - \eta D^v_{(w^l_{ji})^t} E_{L2}$, with L_2 Regularization	Improved accuracy during training and testing of FBNN including L_2 regularization.
4	FCNN Sheng et al. (2020)	Caputo derivative	$W^{(l)}_{[k+1]} = W^{(l)}_{[k]} - \mu \frac{\delta^\alpha L}{\delta W^{[l]\alpha}}$.	Slightly improved accuracy during training and testing of FBNN is obtained for few cases. As FO increases, better accuracy is obtained.

4.2 Particle swarm optimization (PSO)

Swarm-based algorithms, a potent family of optimization approaches, have been developed as a result of studying the group behavior of social animals. In PSO, swarms of particles move around the parameter space, and their movements are guided by their own best-known position in the parameter space as well as the entire swarm's best-known position. When improved positions are obtained, it helps in guiding the movements of the swarm or trajectory (Marini and Walczak 2015).

The PSO algorithm for optimization of the training process in FANN has been implemented in Tlelo-Cuautle et al. (2022). In Zhang and Yang (2020), FO-based CNN has been modeled using the G-L derivative and the PSO algorithm has been applied for the computation of smaller control energy. Zhang and Yang have investigated the ideal quasi-synchronization issue for delayed FMNN. A stochastic inertia weight-PSO has been implemented to synchronize Caputo fractional-order based FMNN in Chang et al. (2020). In Waseem et al. (2020), a feed-forward ANN using Darwinian-PSO (D-PSO) optimized algorithm has been used to solve non-linear differential equations of second order. The use of algorithms has aided in determining the optimal weight values in ANN.

4.3 Levenberg–Marquardt algorithm (LM)

To solve the convergence issues while working with the gradient descent algorithm in ANN, Gauss–Newton iteration can be employed. The LM algorithm is a newly developed, improved technique that combines Gauss–Newton iteration and gradient descent algorithm. The following update rule defines it Ranganathan (2004).

$$x_{i+1} = x_i - (H + \lambda I)^{-1} \nabla f(x_i), \tag{9}$$

where $x_i = (x_1, x_2, \dots, x_n)$ is a vector, H is the Hessian matrix calculated at x_i, and I is an identity matrix. It uses the following update rule.

- If the value of error after updation is less, the value of λ is reduced.
- If the value of error after updation is more, the value of λ is increased.

In Hadian-Rasanan et al. (2020), different kinds of Lane–Emden equations are approximated using FANN. The FO of the Legendre function has been utilized as a hidden layer activation function and the LM optimization approach has been applied in ANN training for weight optimization. FO differential equations have been implemented using the FANN in Hadian Rasanan et al. (2020). In this, FO Jacobi functions have been used by FANN as the activation functions for the hidden layer and LM optimization technique for training the network.

4.4 Interior point algorithm (IPA)

Constrained optimization issues can be effectively solved using IPA. It is based on Karmarkar's algorithm Karmarkar (1984). In Lodhi et al. (2019), FO differential equations have been implemented using the FANN, and IPA technique is used for training the network in the weight update process. Modeling has been performed in an unsupervised fashion. To verify the accuracy of the approach, various Bagley–Torvik equation options have been taken into account for simulation.

Nonlinear systems that are created using arbitrary-order Riccati equations have been implemented using FANN in Lodhi et al. (2019). With the use of the error function, FANN has been used to construct the system's energy function. IPA has been used as an optimized algorithm for the energy function's design parameter.

5 Stability and synchronization in FANN

Stability and synchronization are the two important features of any system involving ANN. In this section, we present a review of the work carried out in the FANN system and the important results.

5.1 Stability analysis in FANN

In neural networks, problems like oscillation, divergence, or instability arise due to time delay. Moreover, dynamic systems face a problem of stability. The existence, uniqueness, and global asymptotic stability (GAS) of the equilibrium point are some of the most studied issues in nonlinear circuit theory. A GAS neural network is actually guaranteed to calculate the global optimal solution regardless of the beginning situation, which further suggests that the network is free of erroneous suboptimal responses (Forti and Tesi 1995). Due to the availability of numerous stable equilibrium points, one real drawback is that erroneous suboptimal answers are likely to emerge. There should be a single globally asymptotically stable equilibrium point. To prevent the occurrence of false answers and to ensure convergence toward the global ideal solution, GAS is an essential characteristic (Forti et al. 1994).

In literature, the stability analysis for neural networks has been performed including time delays, specifically the stability of fractional-order based neural networks. Here, time delays in the time derivatives of states have been analyzed. In this section, we present research carried out in the area of stability in FANN. Various stability methods are available in the literature such as fractional Lyapunov method, Razumikhin-type theorem, Barbalat lemma, Matrix measure approach, extended impulsive differential inequality, Holder, Gronwall Bellman inequality, inequality scaling skills, Cauchy–Schiwartz inequality, and others have been used to achieve stability in FANN.

These stability methods are employed in different types of neural networks based on different fractional order derivatives as mentioned in Table 3. Most of the analysis is carried out using Caputo, R-L, and G-L derivatives based on fractional order (Defined in section 2). In all these different types of FANN, stability obtained is either robust, global robust, asymptotic, uniform, exponential, finite-time, or Mittag-Leffler. The methods to obtain stability of FANN are described as follows:

(1) Mittag–Leffler stability:

A review of the literature reveals that a combination of Mittag–Leffler functions (one or two-sided) and some inequality theorems or properties, such as the fractional Lyapunov method, can be used to achieve stability. As in Ali et al. (2019), the Mittag-

Table 3 Summary of various stability methods of FANN

Reference Number	Type of NN used	Type of FC used	Type of stability	Methods to obtain stability
Guo and Xin (2018)	BAM NN	Differential equations	Global robust	Lyapunov–Krasovskii functionals, linear matrix inequalities
Gai et al. (2016)	Multi-variable FANN	Caputo	Robust	Linear matrix inequality, Lyapunov direct method
Zhang et al. (2017)	FO hopfield FANN	Caputo	Robust	Lyapunov functionals
Wang et al. (2017)	BAM NN	R-L	Global asymptotic	Fractional Barbalat's lemma, Razumikhin-type
Velmurugan et al. (2017)	FO complex valued NN	Caputo	Global asymptotic	Lyapunov functions and fractional Halanay inequality
Wang et al. (2017)	Impulsive FO complex-valued NN	Caputo	Global asymptotic	Lyapunov(fractional) method and Mittag–Leffler functions
Chen et al. (2017)	FMNN	Caputo	Global asymptotic	Spectral radii of matrices, maximum modulus principle
Liang et al. (2015)	FO cellular NN	Caputo	Asymptotic	Lyapunov function
Popa (2023)	FANN	Caputo	Asymptotic	Linear matrix inequalities, algebraic inequalities
Wu et al. (2021)	FANN	R-L	Asymptotic	Lyapunov direct method, Jensen's integral inequality, linear matrix inequality
Wang et al. (2014)	Fo Hopfield NN	Caputo	Asymptotic	Lyapunov (fractional) method
Pratap et al. (2019)	FO CNN	Caputo	Asymptotic	Cauchy-Schwartz and Gronwall inequality
Zhang et al. (2018)	Fractional Neutral-type Delayed NN	R-L	Asymptotic	Matrix or algebraic inequalities
Kaslik and Rădulescu (2017)	Complex valued FANN	Caputo	Asymptotic	Linear matrix inequality

Table 3 (continued)

Reference Number	Type of NN used	Type of FC used	Type of stability	Methods to obtain stability
Li et al. (2019)	FO Neutral-type Delayed Projective NN	R-L	Asymptotic	Linear matrix inequality
Kaslik and Sivasundaram (2012)	FANN (Hopfield type)	Caputo	Asymptotic	Matrix inequality
Wu et al. (2021)	FANN	R-L	Asymptotic	Lyapunov direct method, LMI, Jensen's integral inequality
Wei et al. (2017)	FOMCVNN	R-L	Asymptotic	Lyapunov functions, nonlinear measure method, contraction mapping theory
Rakkiyappan et al. (2017)	FCVNN	Caputo	Global asymptotic	Lyapunov (fractional) method
Wang et al. (2019)	FCVNN	Caputo -Podlubny	Asymptotic	Linearization method and Laplace transform
Ali et al. (2019)	BAM NN	Caputo	Global Mittag-Leffler or global asymptotic	Lyapunov functional having complex valued
Hymavathi et al. (2022)	FCVNN	Caputo	Global exponential	Linear matrix inequalities
Kumar and Das (2019)	BAM NN	Differential equations	Alpha exponential	Matrix measure approach, extended impulsive differential inequality
Chen et al. (2013)	FO delayed NN	Caputo	Uniform (alpha exponential)	Sufficient criterion (new differential inequality of FO)
Wu et al. (2021)	FO complex valued NN	Caputo	Uniform	Banach fixed point theorem
Wu et al. (2017)	2-state FO mixed delay NN	Caputo	Quasi-uniform	Holder, Cauchy-Schwartz, Gronwall inequality

Table 3 (continued)

Reference Number	Type of NN used	Type of FC used	Type of stability	Methods to obtain stability
Gu et al. (2019)	FO time-delayed inertial NN	R-L	Global uniform	Lyapunov (fractional) direct method
Chen et al. (2016)	Delayed Hopfield NN	Caputo	Finite-time	Inequality technique
Rakkiyappan et al. (2014)	FOMCVNN	Caputo	Finite-time	Holder, Gronwall–Bellman inequality, inequality scaling skills
Wu et al. (2015)	Fo Cellular NN	Caputo	Finite-time	Inequalities like Holder, Gronwall
Yang et al. (2021)	FANN	Caputo	Finite-time	Gronwall integral inequality
You et al. (2020)	Discrete FO complex valued NN	Caputo	Finite-time	Arzela–Ascoli's and fixed-point theorem, inequality scaling skills
Hu et al. (2020)	FCVNN	Caputo	Finite-time	Gronwall, Cauchy- Schiwartz inequality, inequality scaling skills
Du and Lu (2021)	FMNN	Caputo	Finite-time	Gronwall, Cauchy- inclusion, FO Gronwall inequality
Zhang et al. (2013)	FMNN	Differential equations	Global exponential	Inequality techniques, Lyapunov functional
Yu et al. (2012)	FANN	Caputo	Alpha exponential	Bellman–Gronwall Inequality
Zhang et al. (2016)	Fo Hopfield NN	Caputo	Mittag–Leffler	Lyapunov functionals
Yang et al. (2018)	FANN	Caputo	Global Mittag-Leffler	Lyapunov direct method (without delay) and fractional Razumikhin-type theorem (with delay)
Zhang et al. (2017)	FANN	Caputo	Global	LMI
You et al. (2020)	Discrete FCVNN	Caputo	Global Mittag-Leffler	Lyapunov's (fractional) direct method
Tyagi et al. (2016)	FCVNN (Hopfield)	Caputo	Global Mittag-Leffler	Lyapunov direct method

Table 3 (continued)

Reference Number	Type of NN used	Type of FC used	Type of stability	Methods to obtain stability
Yang et al. (2016)	Impulsive FANN	Caputo	Mittag–Leffler	Inequality techniques (Bernoulli, Dresden, Holder, Cauchy–Schwartz)
Zhang et al. (2015)	FANN (Hopfield)	Caputo	Mittag-Leffler	Lipschitz conditions, Lyapunov's (fractional) direct method
Popa (2023)	FANN	Caputo	Mittag–Leffler	Halanay-type lemma, linear matrix inequalities
Liu et al. (2017)	FORNN	Caputo	Mittag–Leffler	Activation functions' geometrical properties, nonsingular M-matrix's properties
Liu et al. (2017)	FO competitive NN	Caputo	Mittag-Leffler	Inequality techniques
Liu et al. (2018)	FMNN	Caputo	Global Mittag-Leffler	Lyapunov method
Zhang et al. (2017b)	FO impulsive NN	Caputo	Global Mittag-Leffler stability	Lyapunov direct method, contraction mapping principle
Wu et al. (2016)	FANN (Hopfield)	Caputo	Global Mittag-Leffler	Fractional Lyapunov method
Wu et al. (2016)	FANN	Caputo	Mittag-Leffler	Fractional Lyapunov method
Wu and Zeng (2017)	FMNN	Caputo	Global Mittag-Leffler	Fractional Lyapunov method
Wu et al. (2016)	BAM NN	Caputo	Global Mittag-Leffler	Gronwall-like inequality of Caputo, Lyapunov approach
Yang et al. (2018)	FQVNN	Caputo	Global Mittag-Leffler	Inequality techniques, fractional Lyapunov method
Wang et al. (2018)	FANN	Caputo	Global Mittag-Leffler	Lyapunov functional technique (Lur'e Postnikov)
Wu et al. (2016)	FANN	Caputo	Mittag-Leffler	Fractional Lyapunov approach
Hioual et al. (2022)	FDTNN	Caputo	Mittag–Leffler	Discrete Laplace transform method, Banach contraction mapping

Table 3 (continued)

Reference Number	Type of NN used	Type of FC used	Type of stability	Methods to obtain stability
Jia et al. (2020)	FMNN	Caputo	Global Mittag-Leffler	Lyapunov function, Gronwall's integral inequality
Stamova and Simeonov (2017)	FO cellular NN	Caputo	Mittag-Leffler	Fractional Lyapunov approach
Li et al. (2021)	Impulsive FCNN	Caputo	Mittag-Leffler	Lyapunov method matrix in-equalities
Li et al. (2021)	FCNN	Caputo	β Mittag-Leffler	Lyapunov method matrix in-equalities

Leffler stability has a faster convergence speed than the exponential stability near the origin.

(2) Uniform stability:

The uniform stability of the FO complex valued NN has been proposed in Li et al. (2021). Two different forms of activation functions have been considered. The Banach fixed point theorem has been used to obtain stability for NN.

(3) Asymptotic stability:

Asymptotic stability implies that solutions that begin close enough to the equilibrium not only stay close enough but also eventually converge to the equilibrium (Liang et al. 2015).

(4) Finite-time stability:

In Rakkiyappan et al. (2014), a finite-time stability, which has faster convergence, improved robustness, and disturbance rejection properties has been obtained.

(5) Global Robust stability:

The stability of a class of multiple variable FANN with uncertainties is proposed in Gai et al. (2016). Several issues relating to the robust stability of such networks are discussed. In such networks, inequality theorems and properties such as the linear matrix and the fractional Lyapunov direct method are used to achieve robust stability.

(6) Exponential stability:

A new FO differential inequality has been introduced. Using this new inequality, α -exponential stability is obtained. The network's convergence rate is determined by the order of the differential equation (Yu et al. 2012).

Figure 4 presents the number of papers available in the literature for stability analysis in various FANN. From Table 3 and Fig. 4, the literature survey reveals that a lot of research work is carried out on Mittag–Leffler stability using the fractional Lyapunov method as compared to other methods of stability, it is so because the fractional Lyapunov Method is used to analyze nonlinear systems without the need for differential equation simplification. If the Lyapunov function is chosen as the system's candidate, the

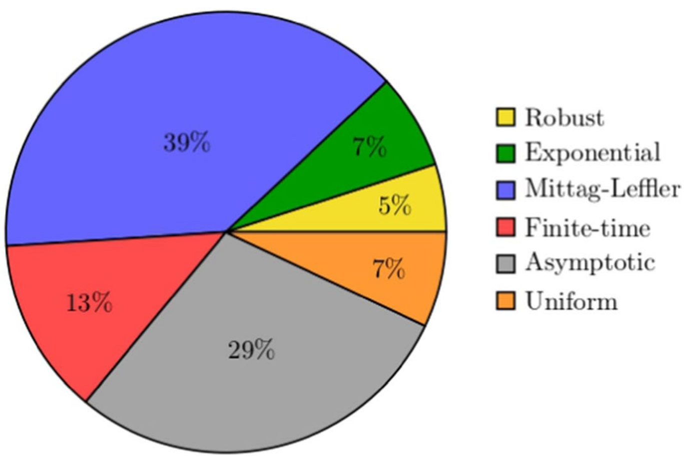

Fig. 4 Research contributions towards stability analysis in FANN

system is found to be stable. This stability serves as a tool for analyzing the stability of nonlinear systems and addresses a variety of stability issues (Sabatier et al. 2010).

The general procedure to achieve stability using Mittag–Leffler functions is as follows:

1. Decide or select a type of FANN.
2. Choose fractional calculus definitions like Caputo, R-L, G-L, etc.
3. Define Mittag–Leffler functions.
4. Apply Laplace transform to Mittag–Leffler functions.
5. Describe FANN with differential equations (basic definitions like Caputo).
6. Apply the Laplace transform to step 5.
7. To obtain FANN stability, use inequality theorems such as the fractional Lyapunov method.

5.2 Lyapunov method

Following steps are used to implement Lyapunov stability in both direct and indirect methods (The direct approach is applicable for the simplification of large-dimensional problems while the indirect approach is applicable for the simplification of more complex problems) Trigeassou et al. (2011),

1. Define linear FO differential equation.
$$\begin{aligned} &D^{m_N}(y(t)) + a_{N-1}D^{m_{N-1}}(y(t)) + \cdots + a_1 D^{m_1}(y(t)) + a_0 y(t) \\ &\quad = b_M D^{m_M}(u(t)) + \cdots + b_1 D^{m_1}(u(t)) + b_0 u(t), \end{aligned} \tag{10}$$
where $m_1 < m_2 < \cdots < m_N$ are fractional derivation orders, $M \le N$, $y(t)$ is output and $u(t)$ is input.
2. Apply Laplace transform to the equations of the system as defined in Eq. 10.
3. Determine a frequency state model (matrix form) of discrete nature.
4. Describe direct and indirect Lyapunov approaches.

5.3 Summary of stability

To obtain stability, the following steps are used.

1. Any fractional definition can be used to define the system equation. Initial conditions are also defined. The given function should be locally Lipschitz.
2. Various definitions are used to define the equilibrium point of a system equation for cases when this point is at the origin and when it is not at the origin.
3. To prove the existence and uniqueness of the above system, the following theorem is used.
$$|\, f(t,x_1) - f(t,x_2) \,| \le l \,|\, (x_1 - x_2) \,|, \tag{11}$$
where $f(t, x)$ is a continuous real-valued function. It is defined in the G domain. In this domain, the above Lipschitz condition with respect to x is defined. l is constant and positive.
4. A relation is obtained between FO systems and the Lipschitz condition. Theorems like triangle inequality are used.

5. This method is extended for FO systems, which generate stability like Mittag-Leffler stability. To achieve this stability, a method like the fractional Lyapunov method can be used. For the analysis of systems, the Laplace and inverse Laplace transforms can be used.

From this research work, some important conclusions can be drawn as follows,

- In Zhang et al. (2015), Zhang et al. (2017), Wang et al. (2014), Pratap et al. (2019), it is illustrated that the convergence rate is dependent upon fractional order. It increases as the fractional order is varied from $0-1$.
- From Wei et al. (2017), it is observed that fractional order has an impact on finite-time stability, it is proved that for higher values of fractional order finite-time stability is achieved at a faster rate.
- In Yu et al. (2012), fractional order exponential stability is defined as follows,

$$\| x(t) - y(t) \| \leq M \| x_0 - y_0 \| \, exp(-\lambda t\alpha), \tag{12}$$

  where, t is the positive value, $\| \ \|$ represents the Euclidean norm. Where M and λ are two positive constants, x_0 and y_0 are two initial values.
- Linear Matrix Inequality (LMI) based condition of stability has been defined in Yang et al. (2018). It is observed that by using these conditions, more calculations are required as this stability is dependent on the whole matrix of the system.
- A simple LMI-based stability condition for delayed FANN has been investigated in Jia et al. (2020). A linear state feedback controller has been designed. But controller design is not feasible for high-dimensional FANNs.
- A finite-time stability of FO complex-valued neural networks (CVNN) has been studied in Hu et al. (2020). The results are a little bit conservative.
- In Wang et al. (2019), stability analysis of fractional CVNN (FCVNN) demonstrates that the computational complexity increases as the system are divided into real and imaginary parts since it is a complex-valued FANN.
- In Liang et al. (2015), the asymptotic stability of FO cellular NN has been investigated and results obtained are a little bit conservative.
- Research on the stability of FO neutral-type delayed NN is in the development and exploitation stages in comparison to integer order NN Chen et al. (2013).
- According to Pratap et al. (2019), the majority of the results on stability and synchronisation has dealt with LMI. But as the number of neurons increases, computational complexity also increases. By using matrix elements, the number of matrices required is less.

5.4 Synchronization analysis in FANN

A synchronization is a tool that can be used for controlling the chaos of practical applications Ma et al. (2015). Pecora and Carroll introduced chaos synchronization. There has been a lot of research on the synchronization of NN after this introduction. In this section, we present a review of synchronization analysis in FANN.

Each neuron has its own state which changes With respect to time. The state is represented by $x_1, x_2, \cdots, x_n$. The update of this state value depends upon interconnections and the initial value of the state Yueh and Cheng (2006). If neuron i has state $x_i(t)$ and neuron j has state $x_j(t)$, NN is said to be synchronized if and only if

$$\lim_{t\to+\infty} \| x_i(t) - x_j(t) \| = 0. \tag{13}$$

A NN's parameters cannot be calculated properly if there are external distractions and variations in the model. There are unpredictable changes in networks' responses, including stability and synchronization. Real-time applications require fast synchronization.

In any system, synchronization error can be computed as follows Bao et al. (2015),

1. Obtain $x(t)$ of the drive system of NN.
2. Obtain $y(t)$ of the response system of NN.
3. Obtain $e(t)$ which is the synchronization error of the system. It is the difference between $y(t)$ and $x(t)$.
4. Controllers are built to achieve the lowest possible error. Here $e(t) \to 0$ as $t \to \infty$ $(i = 1, 2, \cdots, n)$, indicates that the drive and response systems are synchronized.
5. If $\lim_{t\to+\infty} \| y(t) - x(t) \| = 0$, systems $x(t)$ and $y(t)$ are global asymptotic synchronization.

Various methods are available in the literature to achieve synchronization such as SMC, linear feedback, state feedback control, output feedback, data feedback, adaptive controls, and period intermittent control methods. Some inequality theories, as well as stability conditions, are also employed to achieve system synchronization.

These synchronization methods have been employed in different types of FANN as illustrated in Table 4. It is observed that the maximum research work is carried out in the Caputo and R-L-based fractional derivative neural networks. In all these FANN, synchronization achieved is either Quasi or asymptotic or Mittag-Leffler or finite-time or exponential or projective synchronization. Numerous methods to obtain synchronization of FANN are described as follows:

1. Projective synchronization:
   Projective synchronization allows for faster communication and it has a proportional feature. Because of these characteristics, this method is preferred over others Juan et al. (2014).
2. Global Mittag Leffler synchronization:
   The global Mittag-Leffler function is used to achieve FANN synchronization (Li et al. 2020; Kao et al. 2021; Xiao et al. 2020).
3. Finite-time synchronization:
   In this type of synchronization, the upper bound of the synchronization's setting time is estimated (Li et al. 2020; Kao et al. 2021).
4. Adaptive synchronization:
   Adaptive synchronization of FANN is obtained by using adaptive and feedback control (Bao et al. 2015; Wu and Huang 2019; Xiao et al. 2016).
5. Quasi-Uniform synchronization:
   In practice, there are fewer cases for synchronization implementations in which the synchronization error approaches zero with respect to time. These numbers change. This is referred to as Quasi synchronization (Kaslik and Rădulescu 2017).

Table 4 Summary of various synchronization methods of FANN

Reference number	Type of NN used	Type of FC used	Type of synchronization	Methods to obtain synchronization
Juan et al. (2014)	FANN	Caputo	Global projective	Adaptive control and open-loop control
Zhang et al. (2017)	FMNN	R-L	Projective	Stability theorems like Barbalat's lemma and Razumikhin-type
Bao and Cao (2015)	FMNN	Caputo	Projective	New hybrid controller
Zhang et al. (2018)	FANN	Caputo	Projective	Combination of control like open loop and adaptive
Gu et al. (2019)	FMNN	Caputo	Projective	Combination of control like open loop and time-delayed feedback
Zhang et al. (2018)	FO delayed NN	Caputo	Projective	Construction of Lyapunov function, implementation of fractional inequality theorems
Wang (2017)	RBFNN	R-L	Projective	Adaptive SMC
Ding and Shen (2016)	Nonidentical FANN	Caputo	Projective	SMC
He et al. (2021)	FANN	Caputo	Projective	SMC
Ma et al. (2015)	2 coupled FCVNN	Caputo	Projective	Adaptive control technique
He et al. (2020)	Delayed FO competitive NN	Caputo	Projective	Adaptive control method
Zhang et al. (2021)	FO delayed NN	Caputo	Projective	Lyapunov direct method
Zhang and Deng (2019)	FOMCVNN	Caputo	Finite-time projective	Set-valued map, the differential inclusion theory, integral inequalities like Gronwall–Bellman
Hui et al. (2020)	FMNN	Caputo	Finite-Time Projective	Set-valued map, FO differential inclusion
Zheng et al. (2017)	MDFNN	Caputo	Finite-time projective	Set-valued map, the differential inclusion theory, integral inequalities like Gronwall–Bellman, linear feedback controller
Velmurugan and Rakkiyappan (2016)	FMNN (Hub structure)	Caputo	Hybrid projective	Design scaling matrix

Table 4 (continued)

Reference number	Type of NN used	Type of FC used	Type of synchronization	Methods to obtain synchronization
Song et al. (2018)	FMNN	Caputo	Combination of passive projective and H∞	Adaptive SMC
He et al. (2019)	Delayed and Disturbed	Caputo	Quasi-matrix and quasi-inverse -matrix projective	Various projective matrix and controllers
Li et al. (2019)	FCVNN	Caputo	Quasi projective and complete	Controllers like linear feedback and adaptive
Yang et al. (2018)	FCVRNN	Caputo	Quasi-projective	Linear controller
Liu and Yu (2021)	FANN	Caputo	Quasi-projective	Discrete fractional Halanay inequality, Lyapunov functional method
Chen et al. (2018)	FANN	Caputo	Global Mittag-Leffler projective	Combination of control like open loop and adaptive
Wu et al. (2017)	Nonidentical FANN	Caputo	Projective	SMC
Li et al. (2020)	Nonidentical FANN	Caputo	Global Mittag-Leffler projective	SMC
Kao et al. (2021)	Delayed FMNN	Caputo	Mittag–Leffler	Hybrid adaptive controller
Xiao et al. (2020)	FQVBAMNN	Caputo	Global Mittag-Leffler	Linear controllers
Pratap et al. (2018)	FANN	Caputo	Mittag-Leffler	M-Matrix theory and controller like delayed feedback
Jia and Zeng (2020)	FMNN	Caputo	Global Mittag-Leffler lag quasi	Control like linear feedback pinning

Table 4 (continued)

Reference number	Type of NN used	Type of FC used	Type of synchronization	Methods to obtain synchronization
Li et al. (2018)	FMRNN	Caputo	Finite-time	2 novel state feed-back controllers
Zhang and Yang (2020)	MBAMNN	Caputo	Impulsive finite-time	2 impulsive controllers
Zheng et al. (2020)	FCVNN	Caputo	Finite-time	Novel, effective control techniques
Shao et al. (2021)	FO hyper-chaotic systems via adaptive NN	Caputo	Robust finite-time	SMC
Ren and Wu (2018)	Variable-order FANN	Caputo	Finite-time	Fractional Lyapunov method, LMI, Mittag–Leffler function
Meng and Wang (2018)	FMBAMNN	Caputo	Finite-time Mittag-Leffler	Feedback controller, Lyapunov theory differential inequalities
Hui et al. (2023)	FMNN	Caputo	Finite-time	Gronwall–Bellman inequality
Shang et al. (2023)	FOQVNN	finite-time	Finite-time	Lyapunov function
Zhang et al. (2019)	FCVNN	R-L	Finite-time	Adaptive error feedback control technique
Bao et al. (2015)	FMNN	Caputo	Adaptive	Control like adaptive, linear delay feedback
Wu and Huang (2019)	FCVNN	Caputo	Pinning adaptive and Exponential	Adaptive pinning controller
Xiao et al. (2016)	FMNN	Caputo	Adaptive	Control like adaptive, linear delay feedback inequality
Yang et al. (2017)	FMNN	Caputo	Quasi-uniform	Inequalities like Holder, Cp inequality and Gronwall-Bellman
Zhang et al. (2018)	FMNN	Caputo	Lag quasi	Laplace transform, linear feedback control
Liu et al. (2019)	FMNN (Hopfield)	Caputo	Robust	Lyapunov function, Mittag-Leffler stability
Chen et al. (2015)	FMNN	Caputo	Global asymptotic	Linear error feedback control
Hu et al. (2018)	Nonidentical FANN	R-L	Global asymptotic	New feedback controller

Table 4 (continued)

Reference number	Type of NN used	Type of FC used	Type of synchronization	Methods to obtain synchronization
Chang et al. (2020)	FMNN	Caputo	Asymptotic	LMI, algorithm like SIWPSO used for controller
Pratap et al. (2020)	MWCFCNN	R-L, Caputo	Asymptotical	State feedback controller
Zhang et al. (2018)	FMNN	Caputo	Global asymptotic	Lyapunov direct method, comparison principle
Gu et al. (2019)	FO time-delayed inertial NN	R-L	Global asymptotic	Feedback controllers
Wu and Huang (2019)	FMNN	Caputo	Global asymptotic	One-norm Lyapunov function, fractional Halanay inequality
Gu et al. (2020)	FDTNN	R-L	Asymptotical	Feedback controllers
Zhang and Bao (2018)	FBAMNN (3 neuron complex value)	Caputo	Asymptotical	Theory like Lyapunov, comparison, linear feedback control
Li et al. (2021)	FCNN	Caputo	Asymptotical	Kronecker product, method, FO Lyapunov, direct method
Zhang and Yang (2018)	FMBAMNN	Caputo	Exponential	Impulsive controller
Yin et al. (2017)	FANN	R-L, Caputo	Global O($t^{-\alpha}$)	Output feedback controller
Chowdhury et al. (2020)	FO nonautonomous NN	Caputo	Global O($t^{-\alpha}$)	Data-sampling control
Wang et al. (2019)	Variable order FANN	Caputo	Global Mittag-Leffler	State-feedback controller
Wang et al. (2020)	FO nonautonomous NN	Caputo	Global Mittag-Leffler	Linear feedback control
Wang et al. (2021)	FO coupled NN	Caputo	Mittag-Leffler	Convex Lyapunov function, general convex quadratic function, FO Lyapunov direct method
Hu et al. (2019)	FANN	Caputo	Global	FMBFII, Lyapunov–Krasovskii functions, FO integral Jensen's inequality

Table 4 (continued)

Reference number	Type of NN used	Type of FC used	Type of synchronization	Methods to obtain synchronization
Zhang et al. (2018)	FDNNDA	Caputo	Global	Controllers like adaptive and state feedback
Zhu et al. (2008)	Time-Delayed FANN	G-L	Chaos	Nonlinear control theory
Cheng et al. (2019)	FANN	Caputo	Outer	Data-sampling methods (centralized and decentralized)
Yu et al. (2012)	2-layer network	R-L	Intralayer and interlayer	Lyapunov stability theory
Lin et al. (2023)	MWCFONN	Caputo	Output	Lyapunov functional method, inequality techniques

6. Robust synchronization:
   A synchronization of a fractional-order memristor neural network (FMNN) through the development of a Lyapunov function and efficient conditions has been proposed in Liu et al. (2019). The system is synchronized robustly using discontinuous R.H.S. based on FO and Mittag-Leffler stability.
7. Asymptotical synchronization:
   LMI and algebraic methods are established by the state feedback controller. These are robust asymptotical synchronization analysis methods (Pratap et al. 2020).
8. Exponential synchronization:
   In Zhang and Yang (2018), exponential synchronization of fractional-order memristive bidirectional associative memory neural networks (FMBAMNN) including time delay has been obtained. An exponential function is used to study the FO differential system. Various cases after the implementation of impulsive effects have been described by design. A wide range of these effects has been affected by α.
9. Global synchronization:
   In Yin et al. (2017), two types of output feedback controllers have been implemented to obtain Global O $(t^{-\alpha})$ synchronization of FANN. FANN is said to be Global O $(t^{-\alpha})$ synchronized if FANN is Globally O $(t^{-\alpha})$ stable.
10. Chaos synchronization:
    If a chaotic system generates a signal, the signal cannot be synchronized with another system Pecora and Carroll (2015). The chaos synchronization of FANN with time delay is determined Zhu et al. (2008).
11. Outer synchronization:
    In Cheng et al. (2019), a data sampling method and fractional calculus have been employed to achieve FANN outer-synchronization with a deviating argument.

Figure 5 presents the no. of papers available in the literature for synchronization analysis in various FANN. From Table 4 and Fig. 5, it is observed that a lot of research work is carried out on projective synchronization as compared to other methods of synchronization.

From these papers following facts have been observed,

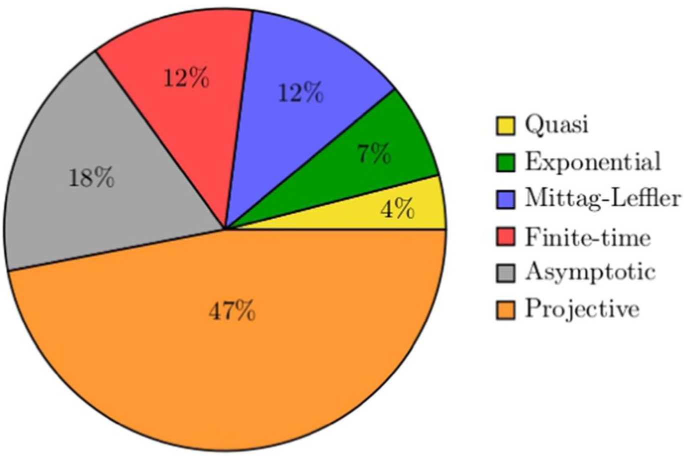

Fig. 5 Research contributions towards synchronization analysis in FANN

- The synchronization of fractional-order memristor-based neural networks is impacted by the fractional order of the derivative (Bao et al. 2015).
- If $0 < \mu < \alpha$, the impulsive effect of μ affects the synchronization of FMBAMNN. Here, $\mu \in (0, 1)$ is a positive constant If $\mu > \alpha$, in fact, causes instability of the system. To control the unfavorable impulse, a feedback controller has been used (Zhang and Yang 2018).
- When $\alpha = 1$, reference (Zheng et al. 2017) has demonstrated that projective synchronization can be converted to complete synchronization and anti-synchronization of driver response systems when $\alpha = -1$.
- In Bao and Cao (2015), Zhang et al. (2018), Ding and Shen (2016), Zheng et al. (2017), it is demonstrated that for the value of $\alpha = 1$ and $\alpha = -1$, the projective synchronization can be converted to complete synchronization and anti-synchronization of driver response systems respectively.
- Synchronization of FANN with time delay has been affected by fractional order (Gu et al. 2019).
- Settling time is dependent on α. The greater the value of α, the greater the value of settling time (Li et al. 2018; Zhang and Yang 2020).
- The convergence speed of matrix-projective synchronization erroris faster for larger values of α (He et al. 2020).
- Adaptive control is dependent on α. Fractional order has affected synchronized adaptive feedback gains. For one reason, FMNN can provide more secure communication compared to integer order NN (Bao and Cao 2015; Ding and Shen 2016).
- For the following condition expression, the master and slave systems are α-exponentially synchronized (Yu et al. 2012):

$$\| x(t) - y(t) \| \leq M \| x_0 - y_0 \| exp(-\lambda t \alpha), \tag{14}$$

  where M and λ are two positive constants, x_0 and y_0 are two initial values.

6 Sensitivity analysis of FANN

Sensitivity analysis in ANN is crucial as it helps in identifying the variables and parameters that influence the output model and its performance. It also allows us to assess the impact of the selected input variables on the output variables and insignificant variables. Sensitivity analysis is performed by observing the impact of perturbed input within the linear region on the output of the network (Novak et al. 2018).

The measure of the network sensitivity is termed as "quotient of error increase." It indicates how many times the network error will increase after the removal of a given variable in relation to the network error with all the analyzed variables. The network error is determined as follows,

$$Error = \frac{1}{n} \sum_{i=1}^{n} (y_i - y_p), \tag{15}$$

where y_i is the observed value and y_p is the theoretical value of the output variable determined based on the model. By removing the input variables from the network input, running the training procedure again, and calculating a new network error $Error_i$, the relevance of the input variables may be evaluated. The ratio of the error obtained at the network startup for a data set without one variable $Error_i$ and the $Error$ obtained for a dataset with all the variables is defined as the quotient of error as follows:

$$W = \frac{Error_i}{Error}. \tag{16}$$

It signifies that with the greater quotient error after the rejection of the variable, the network is more sensitive to the lack of this variable. Deleting the variable has no impact on the network quality and may even make it better if the error quotient is 1 or less (Mrzygłód et al. 2020).

Sensitivity is one of the important characteristics of FANN. Fractional Hopfield NN (FHNN) can be employed as a defense against chip cloning threats for anticounterfeiting because of features like FO-stability and FO-sensitivity (Pu et al. 2017). Initial value sensitivity is a property of both chaotic and traditional nonlinear dynamic systems. The FO discrete Hopfield NN (FODHNN), which is employed as a pseudorandom number sequence generator in the encryption process, has a better encryption effect due to the sensitivity of initial values and complicated chaotic behaviors of FO discrete systems (Chen et al. 2020). The most common characteristic of nonlinear systems is chaos. It is distinguished by a high sensitivity to the dynamics of the system's initial conditions. High sensitivity to external stimuli is demonstrated by the NN model (Allehiany et al. 2021).

Sensitivity analysis is performed for various FANN structures. Sensitivity analysis has been performed and the system performance of photovoltaic, Fuel Cells, H2 storage tanks, and Electrolyzers based on the cost variation have been computed in Guo et al. (2021). Reference (Wang et al. 2022) focuses on the sensitivity of the FO parameters to the performance of the proposed FORNN with physics-informed battery knowledge. Experiments under dynamic operation conditions are conducted to analyze various sensitivities such as FO gradient sensitivity, impedance sensitivity, loss weight sensitivity, etc. Parameter sensitivity was first analyzed and then FO physics-informed RNN (fPIRNN) was trained with the sampled federal urban driving schedule; partial differential equation; data in accordance. Using sampled groups of the various parameters, Pearson correlation coefficients are produced to quantify sensitivity (Wang et al. 2022). The most sensitive parameter of the delayed FANNs is found via a sensitivity analysis of the crucial bifurcation values. The method proposed here can be used to solve various FO differential equation bifurcations (Li et al. 2022).

Recently, there have been some studies reporting the key sensitivity analysis for FANNs. The results will alter significantly if the encryption and decryption keys are even slightly different. Even though the key only deviates slightly, the cipher picture cannot be accurately deciphered. A lot of security analyses including key space analysis, key sensitivity analysis, histogram analysis, information entropy analysis, and correlation analysis are examined to show the applicability of the proposed cryptosystem in Roohi et al. (2020). This sensitivity analysis has been addressed in Wang et al. (2020). This analysis is also performed in a three-neuron FODHNN. It serves as a generator of pseudo-random chaotic sequences. With the help of the secret key created by a five-parameter external key and the plain image's hash code, its initial value may be determined. The FODHNN discrete step size and order are both included in the external key. This guarantees a substantial secret key space, enhances algorithm sensitivity to the plain image, and results in good key sensitivity for the encryption and decryption procedures (Chen et al. 2020). Reference Li et al. (2022) considers two FO chaotic

NNs to explore the application of this FANN in image encryption. The key sensitivity of the encryption algorithm is performed to evaluate its security.

7 Hardware implementation of FANN

A lot of research work has been carried out in the design of FANN for various applications, analysis of stability and synchronization, but the challenge is in the implementation of FANN. Very few works describe the hardware implementation of FANN as in Pu (2016), Pu et al. (2017). Here authors have implemented a Hopfield neural network circuit with fractional order. Specifically, the analog circuit of FANN has been implemented as a reference with the help of a fractor. Its model has been depicted, in which each neuron has one op amp in addition to its other components. Every FO neuron has the same circuit configuration. It is demonstrated mathematically that the FANN's operation rule is the FO partial differential equation. The performance of the circuit has been tested by simulating it. A multilayer FANN circuit for any fractional order has been implemented in Yifei (2005). Its electrical circuit has been depicted in which every neuron has been formed by three operational amplifiers and associative components (resistor and capacitor) according to the circuit. FC has been defined using the Caputo definition.

In Tolba et al. (2019), authors have used FPGA to implement an Izhikevich neuron of fractional order. The Izhikevich model has been represented by the following expression. In the FO domain, it is generalized with fractional orders α and β.

$$\frac{\partial^{\alpha} V}{\partial t^{\alpha}} = 0.04V^2 + 5V + 140 - U + I, \tag{17}$$

$$\frac{\partial^{\beta} U}{\partial t^{\beta}} = a(bV - U), \tag{18}$$

where

V: the membrane potential; U: the recovery variable which is used to provide negative feedback to V; I: external current stimulus; a, b: input parameters.

An algorithm for the approximation of FO differential equation solution:

1. Input to the approximation algorithm is the FO differential equation.
2. Uniform and non-uniform segmentation.
3. Grouping.
4. Address mapping.
5. Each segment has been approximated by (17).

Input has been partitioned into segments, and the following expression has approximated each segment in the following form.

$$Y = C_2X^2 + C_1X + C_o, \tag{19}$$

where, C_o, C_1, C_1: constant coefficients.

X: input segment.

Y: second-degree polynomial.

The hardware circuit is illustrated in Tolba et al. (2019). This model has been implemented using the Xilinx Virtex 5 FPGA kit. Here synchronization of the neurons for various FO has been studied. In Malik and Mir (2020), FPGA has been used to implement a Hindmarsh Rose (HR) neuron of fractional order. The dynamic properties of FO and integer HR neurons differ in a variety of ways.

In Malik and Mir (2020), FPGA has been used to implement a model based on Hindmarsh Rose (HR) neuron of fractional order. The dynamic properties of FO and integer HR neurons differ in a variety of ways. For the particular set of parameters, integer order models have the same neuron's firing characteristics as that of the FO model, however, FO models have shown different dynamical behaviors. The firing frequency of neurons is dependent on the fractional order of FC. This model is implemented in two steps, in the first step, the digital realization of various FO operator approximations is performed. The second step involves the use of a fractional integrator to obtain a low-power and low-cost hardware implementation. This model has been employed on a low-voltage, low-power circuit and has been compared with the equivalent integer order circuit. Various dynamical behaviors of HR neurons of fractional order have been demonstrated with the proposed hardware model.

In Dar et al. (2021), FO spiking NN has been implemented for application purposes after studying their dynamics. Both software and hardware approaches have been proposed for implementation purposes. Since the behavior of biological neurons has been governed by FO dynamics, several FO models and implementations have been proposed Dar et al. (2021). In this work, the technical aspects of designing the FO FitzHugh-Nagumo neuron model in analog and digital domains have been considered. Hence, this model provides enhanced dynamics and design flexibility.

8 Applications of FANN

From the analysis, it is found ANN combined with FC has been used in the approximation of functions, description of chaos, estimation, global dissipativity, periodicity, and modeling heat transfer process. Some research has been applied to the different areas of science and engineering, such as medicine, image processing and encryption, robotics, and many more. The most important applications of FANN are described in this section. There are two main applications of FC in NNs.

- FANNs.
- Use of NN methods to solve fractional differential equations.

8.1 FANNs

8.1.1 Digital image processing

A digital image can be displayed in two dimensions. It is a collection of digital picture elements. Digital image processing is required to improve the quality of pictorial information and process this digital data for various applications. Fig. 6 depicts the steps involved in image processing (Gonzalez and Woods 2009).

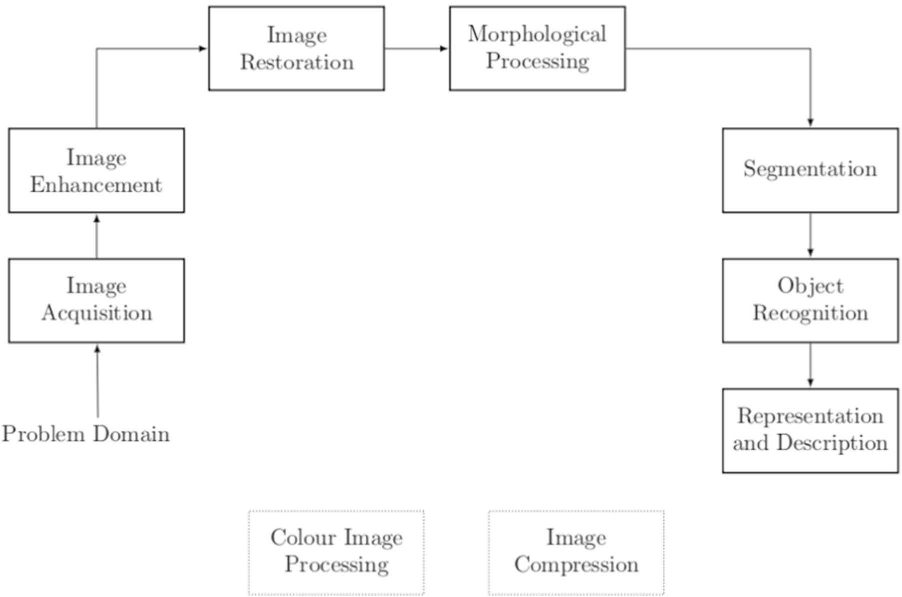

Fig. 6 Image processing steps Gonzalez and Woods (2009)

An additional degree of freedom can be obtained by adding FC with extra parameters. These additional parameters increase the flexibility of optimization. Because of this, FC is used in various stages of digital image processing. Following steps are used in image processing.

Algorithm of FO image processing:

1. Input is an image. For this image proper model/equation/operator is finalized by using a differential/integral equation. The obtained form is in integer.
2. Implementation of FC (by using a suitable equation like R-L, Caputo, G-L, etc) which will convert integer order to fractional order.
3. An approximated numerical solution is obtained by using a suitable discretization method.

In literature, FANN is used for various image processing applications like noise removal, image encryption, and image enhancement using different fractional derivatives. A summary is presented in Table 5.

8.1.2 Biomedical engineering

In Anem et al. (2020), a deep-convolution long-short term memory a (deep-ConvLSTM) network has been used to remove artifacts from an EEG signal. For weight selection and updation, the Optimum algorithm cat swarm fractional calculus optimization (CSFCO)

Table 5 Summary of work done for various stages of digital image processing

Sr.No.	Name of stage	Type of NN	Type of FC	Observations
1	Image enhancement Krouma et al. (2018)	MLP	G-L FO differential	Using FC, the fractional differential mask is designed. AFDA is applied to MLP. Input to the network will be medical images, after training output will be in FO. Put this value in the mask. The value of FO is more for pixels of edge and smaller for details area.
2	Image enhancement Xue (2021)	RBFNN	Caputo FO differential	Input to NN is a low illumination image. The desired target is an illuminance enhancement image. Error is calculated and based on error. NN is trained. It implements FO gradient descent with a momentum algorithm
3	Image encryption Chen et al. (2020)	FODHNN	Left Caputo discrete	The proposed NN is implemented for the image encryption algorithm and compared with the logistic map method. Proposed FANN improves security. Control systems are designed based on stability to achieve synchronization of the system.
4	Noise Removal Couceiro et al. (2011)	DCNN	G-L	FOC Net is designed for image denoising which is having long memory in both passes forward as well as backward. FOCNet obtains better PSNR values compared to other methods, having limited speed.

and FC have been employed. The algorithm is a hybrid of FC and cat swarm optimization (CSO).

The following steps have been followed to remove artifacts from an EEG signal,

1. Detection of EEG signals from various patients.
2. Pre-processing of signals to improve the quality of the signal.
3. Feature extraction using the wavelet transform.
4. Application of the CSFCO algorithm, which is a combination of the cat swarm optimization algorithm and FC.
5. Deep-Conv LSTM has been used to remove artifacts from EEG signals.

A fractional derivative order-based model is proposed for identifying the human immunodeficiency virus (HIV) in Sharafian et al. (2020). Generally, uncertainties and disturbances have an impact on the model. Here, uncertainties have been solved by a neural network, and disturbances have been removed using a sliding mode observer. The network has been stabilized using Mittag-Leffler stability. When compared to other models, this fractional model is the most accurate. In Zuñiga-Aguilar et al. (2020), a recurrent neural network based on the G-L derivative has been implemented to predict glucose levels from a blood sample. The outcomes have been compared with methods like LSTM-CNN, DRNN-LSTM, and LSTM in terms of the root mean squared value. Muscle Artifacts (MA) cause noise to be added during the recording process. In Nagar and Kumar (2022), CNN combined with the FO gradient descent algorithm has been employed to remove noise from an EEG signal. The best denoisation has been obtained with Caputo fractional derivative at $\alpha = 1.2$.

8.1.3 Finance

In Bohner and Stamova (2018), NN has been proposed to train a financial system model. The non-local property and flexibility of fractional order differences suits to create the financial model and indicators. A stochastic term has been included to aid in determining the effects of noise disturbances on financial assets. The dynamic behavior of the model has been properly controlled by including impulsive perturbations and the time delay parameter. The fractional Lyapunov method has been used to determine network stability.

Traditional networks are incapable of forecasting stock market prices. In Bukhari et al. (2020), authors have proposed a LSTM network with Fc to predict unconfirmed variation in financial markets. Auto-regressive fractional integrated moving average (ARFIMA) model is a compatible tool used for models requiring a large amount of memory. The combined ARFIMA-LSTM model is a recurrent network that reduces volatility and avoids the problem of neural network overfitting. The model's performance is assessed in terms of mean absolute error (MAE), root mean squared error (RMSE), and mean absolute percentage error (MAPE) and has been compared to other models such as ARIMA, autoregressive fractional integrated moving average (ARFIMA), and generalized regression radial basis neural network (GRNN) and shows an improvement in accuracy of 80 percent when compared to other conventional methods.

Authors have proposed a design phase and antiphase synchronization between two-fractional order CNNs (3-cell) and the financial system in Yaghoubi and Zarabadipour (2012). Three-cell CNN is used as a driving system. In Wang (2019), the authors combine two methods: Principal component analysis (PCA) and ANN (BPNN), and propose a model based on fractional calculus that is capable of predicting and forecasting stock market indexes and foreign exchange rates. ANN is used to reduce prediction time. The wavelet transform can also be used to maintain a constant window for varying frequencies.

8.1.4 Controller

A controller is required to achieve stability and synchronization in FANN. Non-linear systems also necessitate the use of a controller. The design of a controller for a fractional order system is a difficult task. In literature, various types of controllers are cited. Here, we review the important controller such as SMC.

Sliding mode controller (SMC)

SMC is the most commonly used controller for nonlinear systems because it addresses two major challenges, stability and robustness (Slotine and Li 1991; Utkin 1992; Iordanou and Surgenor 1997). SMC has been used in applications such as robotic manipulators, process control, defense, derive control for power electronics, and so on Gambhire et al. (2021). SMC's structure and implementation have been simplified depending on the system dynamics so that it is less sensitive to disturbances. Several designs have been proposed consisting the integration of neural networks and sliding mode controllers (Lee and Choi 2000; Visioli and Legnani 2002; Lin 2006; Huh and Bien 2007; Chen 2008; Xu et al. 2009; Ak and Cansever 2009; Xiaojiang and Yangzhou 2008; Chaoui et al. 2007; Ak and Cansever 2006; Sadati et al. 2005). Many researchers have used fractional order-based NN to implement SMCs. The addition of FC for such controllers has improved controller design freedom and accuracy as compared to integer order. BPNN forming manipulator inversion system has been proposed in Xu et al. (2015). This system has been cascaded with an industrial robot to form a linear system that improves robot control. BPNN design is based on fractional order sliding mode control. The Lyapunov function has been used to ensure network stability.

A fractional order SMC and adaptation laws have been proposed to synchronize FANN using the theory of the Lyapunov theorem in Liu et al. (2018). FO adaptation laws have been intended for online system parameter estimation (unknown). In Zhang and Yang (2019), authors have created a stable FANN of the hopfield type using optimal discontinuous control. To achieve system stability, properties, and theorems, the Mittag Leffler function has been used. FC has been defined using the Caputo definition. In Wang (2017), Ding and Shen (2016), He et al. (2021), Song et al. (2018), He et al. (2019), SMCs have been employed to obtain projective synchronization of FANN.

To achieve Global Projective synchronization of two non-identical FANNs in finite-time, authors have employed a sliding mode controller using FC. Here the properties of global asymptotic stability over finite-time have been demonstrated and the amount of time required for this synchronization has been calculated in Wu et al. (2017).

8.1.5 System identification

In Yin et al. (2020), Aguilar et al. (2020), the combination of FC and NN has been described as a system identification tool. Specifically, in Yin et al. (2020) an algorithm that is a combination of least mean square (LMS) and newly developed LMS to improve the accuracy of a nonlinear system (such as Van der Pol-Dung oscillator) identification has been proposed. For the generation of input samples with varying time shifts, a functional link ANN (FLANN) (a single-layered ANN) filter has been used and the computational timing of different algorithms has been investigated.

The NN learning algorithm in Aguilar et al. (2020) is based on G-L as a fractional operator and has been used for system identification. The Gradient descent algorithm has been modified and used to identify three different systems. The performance of this model has been compared to other system identification models in terms of RMSE and goodness of fit. This model requires fewer parameters and has is more accurate.

In Rahmani and Farrokhi (2020), authors have described a hybrid of RBFNN and the fractional-order system (FOS) fractional order model for identifying nonlinear dynamic systems. In this model, the system identification process has been divided into two stages. The first stage identifies the structural parameters of FOS in the frequency domain and the second stage calculates the RBFNN weights and FOS parameters. To achieve system stability, the Lyapunov stability theory has been used. In Zúñiga-Aguilar et al. (2022), a new NN methodology for system identification has been proposed using optimal parameters to eliminate uncertainties that exist between the real system and the proposed model. A three-layered neuronal compensation has been used to discover the relationship between derivative fractality and Caputo.

8.1.6 Heat transfer process

Discrete FANN has been used in Sierociuk and Petráš (2011) to model the heat transfer process. A model for the heater has been created using G-L-based fractional derivatives. Simulations demonstrated that FANN can accurately describe unknown dynamics of the process.

8.1.7 Sustainable energy

Variable order FANN with a single layer and multiple layers has been implemented in Aslipour and Yazdizadeh (2019) for the identification of nonlinear systems in wind turbines applications. Aslipour and Yazdizadeh (2019), Aslipour and Yazdizadeh (2020) employ FANN based on Caputo derivatives and PSO to identify the system accurately.

8.1.8 Deep FANN and various applications

Deep FANN structures consist of more layers compared to FANN. The number of layers is based on the complexity of the application. However, as the number of layers increases, so does the computational complexity of algorithms involving FC and more layers. Due to computational complexity, very little work is available on Deep FANN as in Bao et al. (2018), Wang et al. (2017), Anem et al. (2020), Fractional (2020), Sheng et al. (2020), Chen et al. (2022), Admon et al. (2023), Saneifard et al. (2022).

Specifically, in Bao et al. (2018) and Wang et al. (2017), deep FO BPNN has been implemented for digit recognition based on Caputo fractional derivatives. In Anem et al. (2020), deep FOConvLSTM has been implemented to remove artifacts from EEG signals. In Fractional (2020), deep FO LSTM has been proposed to predict unconfirmed variation in financial markets. Authors have proposed FOCNN for digit recognition using Caputo fractional derivatives in Sheng et al. (2020), Chen et al. (2022).

8.1.9 Formation and dynamical analysis

For discrete data, fractional discrete-time models are proposed in Huang et al. (2020). This model is employed in NNs and enables short memory effects. This NN is found to be stable by using the Banach fixed point technique. A class of semilinear fractional difference equations is introduced in Wu et al. (2019). The fixed-point theorem and the discrete Mittag-Leffler function are used to demonstrate that NN is stable. Since the discrete FC, defined as a finite sum, has been used, memory effects are exact, allowing for the application of big data and long-term models.

8.2 Use of NN methods for solving fractional differential equations

An ANN architecture has been proposed for approximating fractional order derivative operators using GL and Caputo in Kadam et al. (2019). Training is performed using the LM algorithm. Changing the net size and the type of mathematical function based on the FO derivative yields different MSE values. Network efficiency is tested using DSP processors and discovered to be stable and fast in computation.

The NN method for solving the time-fractional Fokker–Planck equation is suggested in Wei et al. (2022). A gradient descent algorithm is implemented for weight updating. The L_1 numerical scheme is used to approximate the Caputo derivative. Any activation function can be used to solve equations. In Qu et al. (2022), NN method is presented to simplify the spatiotemporal variable-order fractional advection-diffusion equation. The network is built with shifted Legendre orthogonal polynomials with variable coefficients. The loss function of NN is theoretically deduced using the properties of variable fractional derivative. The NN method is used in Biswas et al. (2023) to solve the spatiotemporal FO nonlinear reaction-advection-diffusion equation. The properties of a FO derivative are used to calculate the loss function of a NN. In Jafarian et al. (2022), a suitable three-layered feed-forward neural architecture has been implemented to approximate the solution of an ordinary linear FO integrodifferential equation . The gradient descent algorithm is used for the weight updating of NN. In Biswas et al. (2023), the NN method is employed to solve an equation like spatio-temporal FO nonlinear reaction-advection-diffusion. The properties of a FO derivative are applied to calculate the loss function of a NN.

A deep feedforward NN is recommended to solve fractional differential equations using a vectorized algorithm in Admon et al. (2023). To solve Caputo FDEs, a new scheme based on deep feedforward NN with vectorized algorithm and selected first-order optimization techniques such as gradient descent, momentum method, and adaptive moment estimation method has been proposed in Admon et al. (2023).It is observed that feedforward NN with one or two hidden layers performed better. In Saneifard et al. (2022), four-layered feed forward NN architecture is proposed to approximate the solution of FO linear Volterra-type integrodifferential equations. Here steepest descent algorithm is employed for learning NN.

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9 Discussion

From the aforementioned review, it is observed that integration of the concepts of ANN and fractional calculus has tremendous research and application potential, albeit with many challenges. From this review, some important facts can be stated as follows:

1. Optimized fractional algorithm
   - A review of the literature reveals that FANN is used in a variety of applications. Weights are updated by fractional values due to the addition of fractional terms in the weight-updating process. More time is required for training and testing. Researchers face a challenge in developing a new algorithm for weight updation using FC with improved speed. Thus speed is still a constraint. Some work states that the implementation of PSO and IPA algorithms can be used for the optimization of the training process.
   - The literature survey reveals the implementation of algorithms like PSO and IPA algorithms for optimization.
2. FC Definition in the weight update process In many works, FANN is employed by using fractional derivatives like R-L, Caputo, and G-L. However, very little work is available using new fractional derivatives such as the Antagana-Baleanu and the Caputo-Fabrizio. Thus FANN based on Antagana-Baleanu and the Caputo-Fabrizio can be explored for various applications.
3. Fractional activation Function
   In NN architecture, fractional order activation functions are employed to improve accuracy, however, it is observed that a few of the activation functions cannot be utilized by a specific NN architecture. For example, BPNN cannot use the step function. ReLU can destroy neurons most of the time during training, so it will not activate the network for any data. To solve this problem, ELUs, Leaky ReLUs, the PReLU, and the Swish activation functions have been proposed. Manually, various activation functions are selected and implemented for FANN structures. But by using fractional derivatives, different types of activation functions can be combined by changing the order of fraction. There have been few attempts to describe a technique for changing the activation of a neural network. Designers face a significant challenge in implementing various adaptive activation functions.
4. Rate of convergence
   The network faces convergence issues due to the inclusion of fractional order-based derivatives. Improvement in the rate of convergence remains a challenge for designers.
5. The training time of FANN
   From the review of FANN networks, it is observed that FANN requires more time for training as compared to integer-order NN in most of the applications. Improving training time is a real challenge for designers.
6. Accuracy of FANN
   Results have proved that FANN structures have slightly more accuracy as compared to integer order NN in various applications at the cost of computational complexity. Faster training algorithms with optimum parameters are a real challenge in FANN implementation in the real world.

7. Stability of FANN
   A review of various methods for determining the stability of various types of FANN structures illustrates that considerable efforts have been made using various inequality theorems, LMI-based conditions and the fractional Lyapunov method.
   The results have demonstrated that the order of the fractional derivative (α) greatly effects convergence and finite stability of FANN structures. By increasing the value of (α), the values of convergence and finite stability have been improved. It is also observed that the LMI-based conditions of stability necessitate more calculations and a greater number of neurons. Researchers proposed methods to create matrices that would reduce the number of calculations.
8. Synchronization of FANN
   A review of the various methods for synchronization between systems of the same dynamical behavior reveals the use of SMC, state feedback control, adaptive controls, and period intermittent control methods available in the literature to achieve FANN synchronization.
   The results illustrate that fractional order (α) affects on the synchronization of FANN structures. Complete synchronization and anti-synchronization has been obtained by choosing $\alpha = 1$ and $\alpha = -1$ respectively. For larger values of α, the convergence speed of matrix-projective synchronization error is faster. Settling time and adaptive control is also affected by α.
9. Hardware implementation of FANN
   There have been fewer reports on the implementation of hardware for FANN structures. This is a promising area for investigation.
10. Deep FANN
   There have been fewer reports on implementing deep FANN structures due to the complexity of deep FANN. This is a promising area for investigation with a need for optimized algorithms.

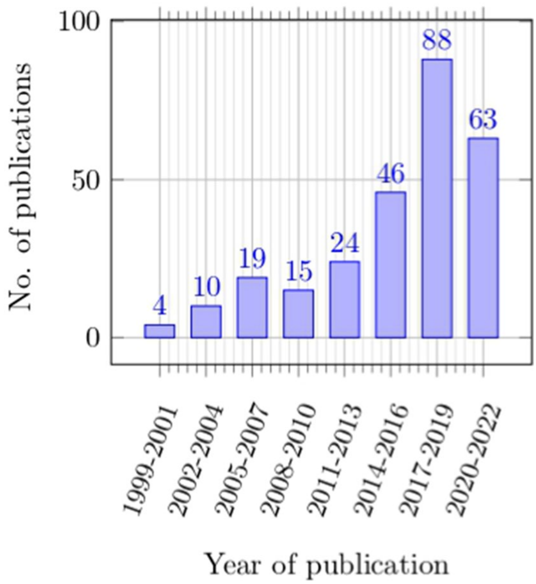

Fig. 7 Research contributions towards FANN in chronological order

11. Number of publications in last 2 decades
    Figure 7 summarizes the contributions from various researchers in the field of FANN in the last two decades. According to the review, it is observed that more papers are published in 2017-2019 than in previous years. In this list, major contributions in the FANN area are considered. The number of papers is rigorous but not exhaustive.

10 Conclusion

This paper provides an up-to-date survey of modeling systems that involve fractional artificial neural networks (FANNs), including their challenges, issues, and applications across various fields. The article summarizes various equations used to calculate fractional derivatives in different FANN networks. Fractional derivatives are observed to be an effective tool for describing the memory and hereditary properties of various processes, making systems or processes employing FANNs more accurate than their integer-order counterparts.

The study examines learning and optimization algorithms for FANN structures, including the gradient descent algorithm, PSO, LM, and IPA. It also outlines fractional activation functions that provide the adaptive nature of activation functions. Regularization is suggested as a means of removing over-fitting of FANN networks. However, due to fractional weights, training a FANN can take some time, and convergence issues may arise.

To determine FANN stability criteria, a rigorous review of various techniques, including the fixed-point theorem, fractional-order differential equations theory, the Lyapunov direct method, linear matrix inequality approach, is presented. The work also summarizes several synchronization criteria, such as the fractional Lyapunov method, Mittag-Leffler function, matrix eigenvalue theory, some inequality techniques such as Young's inequality, fractional order Razumikhin theorem, and various controllers like sliding mode control and many more.

The paper also presents reviews on hardware implementations of FANN, various FANN application areas, and work related to deep FANN. Overall, the research community's focus on fractional calculus theory in ANN is helping develop efficient and accurate systems using optimal resources.

Acknowledgements The authors would like to thank Dr Prasiddh Trivedi and Dr Gajanan Birajdar for fruitful discussions on the subject matter of this work.

Author contributions All authors contributed equally for the work presented in this manuscript.

Declarations

Conflict of interest The authors declare no competing interests.

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