An Existence Theorem of Solutions for the System of Generalized Vector Quasi-Variational-Like Inequalities

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ABSTRACT

In this paper, we introduce and study the system of generalized vector quasi-variational-like inequalities in Hausdorff topological vector spaces, which include the system of vector quasi-variational-like inequalities, the system of vector variational-like inequalities, the system of vector quasi-variational inequalities, and several other systems as special cases. Moreover, a number of C-diagonal quasiconvexity properties are proposed for set-valued maps, which are natural generalizations of the g-diagonal quasiconvexity for real functions. Together with an application of continuous selection and fixed-point theorems, these conditions enable us to prove unified existence results of solutions for the system of generalized vector quasi-variational-like inequalities. The results of this paper can be seen as extensions and generalizations of several known results in the literature.

Keywords: The System of Generalized Vector Quasi-Variational-Like Inequalities; Fixed Point Theorem; Open Lower Section; Upper Semicontinuous; C-Diagonal Quasiconvexity

1. Introduction and Formulation

In recent years, the system of generalized vector quasi-variational-like inequality, which is a unified model for the system of vector quasi-variational-like inequalities, the system of vector variational-like inequalities, the system of vector variational inequalities, the system of vector equilibrium problems and the system of variational inequalities etc., has been studied (see [1-18] and references therein).

In this paper, we consider the systems of four kinds of generalized vector quasi-variational-like inequalities with set-valued mappings and discuss the existence of its solutions in locally convex topological vector space (l.c.s. in short), motivated and inspired by the recent works of Peng [1] and Ansari et al. [2].

Throughout this paper, unless otherwise specified, assume that I be an index set. For each i ∈ I, let Z_i be a locally convex topological vector space (l.c.s., in short) and K_i be a nonempty convex subset of Hausdorff topological vector space (t.v.s., in short) E_i. Let Y_i be a subset of continuous function space L(E_i, Z_i) from E_i into Z_i, where L(E_i, Z_i) is equipped with a σ-

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topology. Let int A and coA denote the interior and convex hull of a set A respectively. Let C_i : K → 2^{E_i} be a set-valued mapping such that int C_i(x) ≠ ∅ for each x ∈ K. Denote that K = ⋂_{i∈I} K_i and E = ⋂_{i∈I} E_i.

For each i ∈ I, let η_i : K_i × K → E_i be a vector-valued mapping, G_i : L(E, Z) → 2^{(E_i, Z_i)}

S_i : K × K → 2^{E_i}, T_i : K → 2^{Y_i} and D_i : K → 2^{Y_i} be four set-valued mappings. Then,

1) Strong type I system of generalized vector quasi-variational-like inequalities which is to find

(\bar{x}, \bar{r}) ∈ K × Y such that \bar{x} ∈ D_1(\bar{x}), \bar{r} ∈ T_1(\bar{x}) and

\{G_i(\bar{r}, \eta_i(y, \bar{x})) + S_i(\bar{x}, y)\} \subset C_i(\bar{x}), \forall y ∈ D_i(\bar{x}), (1.1)

2) Strong type II system of generalized vector quasi-variational-like inequalities which is to find

(\bar{x}, \bar{r}) ∈ K × Y such that \bar{x} ∈ D_1(\bar{x}), \bar{r} ∈ T_1(\bar{x}) and

\{G_i(\bar{r}, \eta_i(y, \bar{x})) + S_i(\bar{x}, y)\} \cap C_i(\bar{x}) ≠ ∅, \forall y ∈ D_i(\bar{x}), (1.2)

3) Weak type I system of generalized vector quasi-variational-like inequalities which is to find

(\bar{x}, \bar{r}) ∈ K × Y such that \bar{x} ∈ D_1(\bar{x}), \bar{r} ∈ T_1(\bar{x}) and

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where \( l(x) \) denotes the evaluation of \( l(x, E, z) \) at \( x \in E \). By the corollary of the Schaefer [3], \( L(E, z) \) becomes a l.c.s. By Ding and Tarafdar [4], the bilinear map \( \langle \cdot, \cdot \rangle: L(K, z) \times K \to \mathbb{R} \) is continuous.

The following problems are the special cases of above four kinds of systems of generalized vector quasi-variational-like inequalities.

The above system of generalized vector quasi-variational-like inequalities encompass many models of system of variational inequalities. The following problems are the special cases of (1.4).

1) If for each \( i \in I \), let \( G_i \) be an identity mapping, \( S_i = 0 \), problem (1.4) reduces to the system of generalized quasi-variational-like inequalities of finding \( x \in K \) such that for each \( i \in I \), \( x \in T_i(x) \) and

\[
\forall y_i \in D_i(x), \exists T_i \in T_i(x) : \langle T_i, \eta_i(y_i, x) \rangle \notin \text{int} C_i(x),
\]

which was introduced and studied by Peng [1].

2) If for each \( i \in I \), let \( G_i \) be an identity mapping, \( S_i = 0 \) and \( D_i(x) = K_i \), problem (1.5) reduces to the system of generalized quasi-variational-like inequalities of finding \( x \in K \) such that for each \( i \in I \), \( x \in K_i \) and

\[
\forall y_i \in K_i, \exists T_i \in T_i(x) : \langle T_i, \eta_i(y_i, x) \rangle \notin \text{int} C_i(x).
\]

In addition, let \( Z_1 = \mathbb{R}^n \) and let \( C_i(x) = \mathbb{R}^n = \{ r \in \mathbb{R} | r \geq 0 \} \) for all \( x \in K \), then problem (1.5) reduces to the system of generalized vector quasi-variational inequalities studied by Ansari and Yao [5].

3) If for each \( i \in I \), \( G_i \) be an identity mapping, \( S_i = 0 \), \( \eta_i(y_i, x) = y_i - x \), and \( D_i(x) = K_i \), then problem (1.5) reduces to the system of generalized vector quasi-variational inequalities of finding \( x \in K \) such that for each \( i \in I \), \( x \in K_i \) and

\[
\forall y_i \in K_i, \exists T_i \in T_i(x) : \langle T_i, y_i - x \rangle \notin \text{int} C_i(x).
\]

4) If \( I = \{1\} \), problem (1.4) reduces to generalized vector quasi-variational-like inequalities of finding \( x \in K \) such that \( x \in D(x) \) and

\[
\langle G \eta(y, x), x \rangle + S(x, y) \notin \text{int} C(x), \forall y_i \in K,
\]

such type of problem studied in [6-10].

5) If \( I = \{1\} \) and \( \eta(y, x) = y - x \), \( T \) is single valued mapping, \( G \) be an identity mapping, \( S = 0 \), and \( C(x) = \mathbb{R}^n \) for all \( x \in K \), then problem (1.4) reduces to classical variational inequality problem of finding \( x \in K \) such that \( x \in D(x) \) and

\[
\forall y \in D(x), \exists T \in T(x) : \langle T(x), y - x \rangle \notin \text{int} C(x),
\]

which was introduced and studied by Hartman and Stampacchia [11].

2. Preliminaries

Definition 2.1. [12] Let \( E \) and \( Z \) be two t.v.s. and \( K \) be a convex subset of t.v.s. \( E \). Let \( C: K \to 2^Z \) and \( \theta: K \times K \to 2^Z \) be two set-valued mappings. Assume given any finite subset \( \Lambda = \{x_1, x_2, \ldots, x_n\} \) in \( K \), any \( x = \sum_{i=1}^{n} \alpha_i x_i \), with \( \alpha_i \geq 0 \) for \( i = 1, \ldots, n \), and \( \sum_{i=1}^{n} \alpha_i = 1 \). Then, 1) \( \theta \) is said to be strong Type I C-diagonally quasiconvex (SIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \subseteq C(x);
\]

2) \( \theta \) is said to be strong Type II C-diagonally quasiconvex (SIIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \cap C(x) \neq \emptyset;
\]

3) \( \theta \) is said to be weak Type I C-diagonally quasiconvex (WIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \cap \text{int} C(x) \neq \emptyset;
\]

4) \( \theta \) is said to be weak Type II C-diagonally quasiconvex (WIIC-DQC, in short) in the second argument if for some \( x_i \in \Lambda \),

\[
\theta(x, x_i) \notin \text{int} C(x).
\]

It is easy to verify that the following proposition, 1) SIC-DQC implies SIIC-DQC; 2) SIIC-DQC implies WIC-DQC; 3) WIC-DQC implies WIIC-DQC. The converse is not true. Following example shows that the converse is not true.

Example 2.1. Let \( E = Z = \mathbb{R} \) and \( \varphi(x, x_i) = \cos \{x, x_i\} \).

1) If \( C(x) = [x + \epsilon, +\infty) \). Then \( \varphi \) is SIC-DQC, but it is not SIIC-DQC.

2) If \( -\text{int} C(x) = (-\infty, x + \epsilon) \). Then \( \varphi \) is WIC-DQC, but it is not WIIC-DQC.

Definition 2.2. [13] Let \( E \) and \( Z \) be two t.v.s. and \( K \) be a convex subset of t.v.s. \( E \). A mapping \( \theta: K \times K \to 2^Z \) is called (generalized) vector 0-
diagonally convex if for any finite subset 
\[ \Lambda = \{x_1, x_2, \ldots, x_n\} \] of \( K \) and any \( x = \sum_{i=1}^{n} \alpha_i x_i \) with 
\[ \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n, \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1, \]

\[ \sum_{i=1}^{n} \alpha_i \theta(x_i, x_i) \notin \text{int} \ C(x). \]

**Definition 2.3.** [14] Let \( X \) and \( Y \) be two topological spaces and \( T : X \rightarrow 2^Y \) be a set-valued mapping. Then,

1) \( T \) is said to have open lower sections if the set 
\[ T^{-1}(y) = \{x \in X : y \in T(x)\} \]

is open in \( X \) for every \( y \in Y \);

2) \( T \) is said to be upper semicontinuous (u.s.c., in short) if for each \( x_o \in X \) and each open set \( U \) in \( Y \) with \( T(x_o) \subseteq U \), there exists an open neighborhood \( V \) of \( x_o \) in \( X \) such that \( T(x) \subseteq U \) for each \( x \in V \);

3) \( T \) is said to be lower semicontinuous (l.s.c., in short) if for each \( x_o \in X \) and each open set \( U \) in \( Y \) with \( T(x_o) \cap U \neq \emptyset \), there exists an open neighborhood \( V \) of \( x_o \) in \( X \) such that \( T(x) \cap U \neq \emptyset \) for each \( x \in V \);

4) \( T \) is said to be continuous if it is both upper and lower semicontinuous;

5) \( T \) is said to be closed if for any net \( \{x^n\} \) in \( X \) such that \( x^n \rightarrow x^o \) and any net \( \{y^n\} \) in \( B \) such that \( y^n \rightarrow y^o \) \( \text{and} \quad y^n \in T(x^n) \) for any \( o \), we have \( y^o \in T(x^o) \).

**Lemma 2.1.** [15] Let \( X \) and \( Y \) be two topological spaces. If \( T : X \rightarrow 2^Y \) is u.s.c. set-valued mapping with closed values, then \( T \) is closed.

**Lemma 2.2.** [16] Let \( X \) and \( Y \) be two topological spaces and \( T : X \rightarrow 2^Y \) is u.s.c. mapping with compact values. Suppose \( \{x^n\} \) is a net in \( X \) such that \( x^n \rightarrow x^o \). If \( y^n \in T(x^n) \) for each \( o \), then there are a \( y^o \in T(x^o) \) and a subnet \( \{y^{o_n}\} \) of \( \{y^n\} \) such that \( y^o \rightarrow y^o \).

**Lemma 2.3.** [17] Let \( X \) and \( Y \) be two topological spaces. Suppose that \( T : X \rightarrow 2^Y \) and \( K : X \rightarrow 2^Y \) are set-valued mappings having open lower sections, then

1) A set-valued mapping \( F : X \rightarrow 2^Y \) defined by, for each \( x \in X \), \( F(x) = \text{co}T(x) \) has open lower sections; and

2) A set-valued mapping \( J : X \rightarrow 2^Y \) defined by, for each \( x \in X \), \( J(x) = T(x) \cap K(x) \) has open lower sections.

For each \( i \in I \), \( E_i \) a Hausdorff t.v.s. Let \( \{K_i\} \) be a family of nonempty compact convex subsets with each \( K_i \) in \( E_i \). Let \( K = \prod_{i \in I} K_i \) and \( E = \prod_{i \in I} E_i \). The following system of fixed-point theorem is needed in this paper.

**Lemma 2.4.** [18] For each \( i \in I \), let \( T_i : K \rightarrow 2^{K_i} \) be a set-valued mapping. Assume that the following conditions hold.

1) For each \( i \in I \), \( T_i \) is convex set-valued mapping;
2) \( K = \bigcup \{ \text{int} T_i^{-1}(x) : x \in K_i \} \).

Then there exist \( \overline{x} \in K \) such that \( \overline{x} \in T(\overline{x}) = \bigcap_{i \in I} T_i(\overline{x}) \), that is, \( \overline{x} \in T_i(\overline{x}) \) for each \( i \in I \), where \( \overline{x}_i \) is the projection of \( \overline{x} \) onto \( K_i \).

### 3. Main Results

**Theorem 3.1.** For each \( i \in I \), let \( Z_i \) be a l.c.s., \( K_i \) a nonempty compact convex subset of Hausdorff t.v.s. \( E_i \), \( Y_i \) a nonempty compact convex subset of \( L(E_i, Z_i) \), which is equipped with a \( \sigma \)-topology. For each \( i \in I \), assume that the following conditions are satisfied.

1) \( D_i : K \rightarrow 2^{K_i} \) and \( T_i : K \rightarrow 2^{Y_i} \) are two nonempty convex set-valued mappings and have open lower sections;

2) For each \( t_i \in Y_i \) and \( x_i \in \text{co} A_i \), the mapping \( \{G(t_i, \eta((., x_i))) + S_i(x_i, x_i) : K \rightarrow 2^{K_i} \} \) is WIIC-DQC;

3) For each \( y_i \in K_i \), the set \( \{x(t, x) \in X \times Y : \{G(t_i, \eta((., x_i))) + S_i(x_i, x_i) \} \subseteq \text{int} C_i(x) \} \) is open.

Then there exist \( \overline{x} \in D_i(\overline{x}) \) and \( \overline{y}_i \in T_i(\overline{x}) \) such that \( \{G(\overline{y}_i, \eta((., \overline{x})) + S(\overline{x}, y_i) \} \subseteq \text{int} C_i(\overline{x}) \), \( \forall y_i \in D_i(\overline{x}) \).

**Proof.** Define a set-valued mapping \( P_i : K \times Y \rightarrow 2^{K_i} \) by

\[ P_i(x, t) = \{ y_i \in K_i : \{G(t_i, \eta((., x_i))) + S_i(x_i, x_i) \} \subseteq \text{int} C_i(x) \}, \]

\( \forall (x, t) \in K \times Y \).

We first prove that \( x_i \not\in \text{co}(P_i(x, t)) \) for all \( (x, t) \in K \times Y \). To see this, suppose, by way of contradiction, that there exist some \( i \in I \) and some point \( (\overline{x}, \overline{t}) \in K \times Y \) such that \( \overline{x}_i \in \text{co}(P_i(\overline{x}, \overline{t})) \). Then, there exist finite points \( y_{i_1}, y_{i_2}, \ldots, y_{i_n} \in K_i \) and \( \alpha_j \geq 0 \) with \( \sum_{j=1}^{n} \alpha_j = 1 \) such that \( \overline{x}_i = \sum_{j=1}^{n} \alpha_j y_{i_j} \) and \( y_{i_j} \in P_i(\overline{x}, \overline{t}) \) for all \( j = 1, \ldots, n \) such that \( \{G(\overline{y}_{i_j}, \eta((., \overline{x})) + S(\overline{x}, y_{i_j}) \} \subseteq \text{int} C_i(\overline{x}) \), \( j = 1, \ldots, n \), which contradicts the hypothesis 2). Hence, \( x_i \not\in \text{co}(P_i(x, t)) \).

By hypothesis 3), for each \( i \in I \) and each \( y_i \in K_i \), we known that
\[ Q^{-1}(y) = \{ (x,t) \in K \times Y : \langle G_i, \eta_i(y,x) \rangle + S_i(x,y) \subseteq - \text{int } C_i(x) \} \]

is open and so \( P_i \) has open lower sections.

For each \( i \in I \), consider a set-valued mapping \( Q : K \times Y \to 2^{K} \) defined by

\[ Q_i(x,t) = \text{co}(P_i(x,t)) \cap D_i(x), \quad \forall (x,t) \in K \times Y. \]

Since \( D_i \) has open lower sections by hypothesis 1), we may apply Lemma 2.3 to assert that the set-valued mapping \( Q_i \) has also open lower sections. Let

\[ W_i = \{ (x,t) \in K \times Y : Q_i(x,t) \neq \emptyset \} \subset K \times Y. \]

There are two cases to consider. In the case \( W_i = \emptyset \), we have

\[ \text{co}(P_i(x,t)) \cap D_i(x) = \emptyset, \quad \forall (x,t) \in K \times Y. \]

This implies that, \( \forall (x,t) \in K \times Y \),

\[ P_i(x,t) \cap D_i(x) = \emptyset. \]

On the other hand, by condition 1), and the fact \( K_i \) is a compact convex subset of \( E_i \), we can apply Lemma 2.4 to assert the existence of a fixed point \( x_i^* \in D_i(x_i^*) \). Since \( T_i(x_i^*) \neq \emptyset \), picking \( t_i^* \in T_i(x_i^*) \), we have

\[ P_i(x_i^*, t_i^*) \cap D_i(x_i^*) = \emptyset. \]

This implies \( \forall y_i \in D_i(x_i^*), y_i \notin P_i(x_i^*, t_i^*) \). Hence, in this particular case, the assertion of the theorem holds.

We now consider the case \( W_i \neq \emptyset \). Define a set-valued mapping \( S_i : K \times Y \to 2^{K} \) by

\[ S_i(x,t) = \begin{cases} Q_i(x,t), & (x,t) \in W_i \\ D_i(x), & (x,t) \in K \times Y \setminus W_i. \end{cases} \]

Then, \( S_i(x,t) \) is a convex set-valued mapping and for each \( u \in K \), \( S_i(u) = Q^{-1}(u) \cup (D^{-1}(u) \setminus Y_i) \) is open. For each \( i \in I \), consider the set-valued mapping \( H_i : K \times Y \to 2^{K \times Y} \) where \( H_i = \Pi_{i=1}^{n} H_i \) defined by

\[ H_i(x,t) = (S_i(x,t), T_i(x,t)). \]

By condition 1) and the properties of \( S_i(x,t), H_i \) satisfies all the conditions of Lemma 2.4. Therefore, there exists \( (x^*, t^*) \in K \times Y \) such that

\[ (x^*, t^*) \in H_i(x^*, t^*). \]

Suppose that \( (x^*, t^*) \in W_i \), then

\[ x_i^* \in \text{co}(P_i(x_i^*, t_i^*)) \cap D_i(x_i^*), \]

so that \( x_i^* \in \text{co}(P_i(x_i^*, t_i^*)) \). This is a contradiction.

Hence, \( (x^*, t^*) \notin W_i \). Therefore,

\[ (x^*, t^*) \in D_i(x_i^*, t_i^*), \quad \text{and } Q_i(x_i^*, t_i^*) = \emptyset. \]

Thus

\[ x_i^* \in D_i(x_i^*), t_i^* \in T_i(x_i^*), \quad \text{co}(P_i(x_i^*, t_i^*)) \cap D_i(x_i^*) = \emptyset. \]

This implies

\[ P_i(x_i^*, t_i^*) \cap D_i(x_i^*) = \emptyset. \]

Consequently, the assertion of the theorem holds in this case.

**Corollary 3.2.** For each \( i \in I \), let \( Z_i \) be a l.c.s., \( K_i \) a nonempty compact convex subset of Hausdorff t.v.s. \( E_i \), \( Y_i \) a nonempty compact convex subset of \( L(E_i, Z_i) \), which is equipped with a \( \sigma \)-topology. For each \( i \in I \), assume that the following conditions are satisfied.

1) \( D_i : K \to 2^K \) and \( T_i : K \to 2^Y \) are two nonempty convex set-valued mappings and have open lower sections;

2) For all \( y_i \in K_i \), the mapping \( \{ G_i, \eta_i(y_i, \cdot) \} + S_i(\cdot, y_i) : K \times Y \to 2^K \) is an u.s.c. set-valued mapping;

3) \( C_i : K \to 2^Z_i \) is a convex set-valued mapping with \( \text{int } C_i(x) \neq \emptyset \) for all \( x \in K \);

4) \( \eta_i : K \times K_i \to E_i \) is affine in the first argument and for all \( i \in K_i \), \( \eta_i(x_i, x_i) = 0 \);

5) \( S_i : K \times K \to 2^Z_i \) is a generalized vector 0-diagonally convex set-valued mapping;

6) For a given \( x_i \in K_i \), and a neighborhood \( U_i \) of \( x_i \) for all \( u \in U_i \), \( \text{int } C_i(x) = \text{int } C_i(u) \).

Then there exists \( x_i \in D_i(x_i) \) and \( t_i \in T_i(x_i) \) such that

\[ \{ G_i, \eta_i(y_i, x_i) \} + S_i(x_i, y_i) \subseteq \text{int } C_i(x_i), \quad \forall y_i \in D_i(x_i). \]

**Proof.** Define a set-valued mapping \( P_i : K \times Y \to 2^K \) by

\[ P_i(x,t) = \{ y_i \in K_i : \{ G_i, \eta_i(y_i, x_i) \} + S_i(x_i, y_i) \subseteq \text{int } C_i(x_i) \}, \]

\[ \forall (x,t) \in K \times Y. \]

We first prove that \( x_i \notin \text{co}(P_i(x_i,t_i)) \) for all \( (x_i,t_i) \in K \times Y \). By contradiction, for each \( i \in I \), suppose there exists some point \( (\overline{x}_i, \overline{t}_i) \in K \times Y \) such that

\[ x_i \in \text{co}(P_i(\overline{x}_i, \overline{t}_i)). \]

Then, there exist finite points \( y_1, y_2, \ldots, y_n \in K_i \) such that

\[ \{ G_i, \eta_i(y_1, x_i) \} + S_i(x_i, y_i) \subseteq \text{int } C_i(\overline{x}_i), \quad i = 1, 2, \ldots, n. \]

Since \( \eta_i \) is affine and int \( C_i(\overline{x}_i) \) is convex, for \( \alpha_j \geq 0 \) with \( \sum_{j=1}^{n} \alpha_j = 1 \) such that \( \overline{x}_i = \sum_{j=1}^{n} \alpha_j y_{ij} \) and \( y_{ij} \in P_i(\overline{x}_i, \overline{t}_i) \) for all \( j = 1, \ldots, n \) such that

\[ \{ G_i, \eta_i \left( \sum_{j=1}^{n} \alpha_j y_{ij}, \overline{t}_i \right) \} + \sum_{j=1}^{n} \alpha_j S_i(\overline{x}_i, y_{ij}) \subseteq \text{int } C_i(\overline{x}_i), \quad j = 1, \ldots, n. \]
Since $\eta_i(x_i, x_j) = 0$ for all $x_i \in K_j$
\[ \sum_{j=1}^{n} \alpha_j S_j(\bar{x}_i, y_j) \subseteq -\text{int } C_i(\bar{x}) \]
which contradicts the hypothesis 5). Therefore $x_i \not\in \text{co}(P_i(x, t))$.

We now prove that for each $y_i \in K_i, P_i^{-1}(y_i)$
\[ \{(x, t) \in K \times Y : \langle G_i t, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \] 
\[ \subseteq -\text{int } C_i(\bar{x}) \}\]
is open. Indeed, let $(\bar{x}, \bar{t}) \in P_i^{-1}(y_i)$, that is
\[ \langle G_i \bar{t}, \eta_i(y_i, \bar{x}) \rangle + S_i(\bar{x}, y_i) \subseteq -\text{int } C_i(\bar{x}) \].
Since \[ \langle G_i \bar{t}, \eta_i(y_i, \bar{x}) \rangle + S_i(\bar{x}, y_i) : K \times Y \to 2^E \] is an u.s.c. set-valued mapping, there exists a neighborhood $U_i$ of $(\bar{x}, \bar{t})$ such that
\[ \langle G_i \bar{t}, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(\bar{x}), \forall (x, t) \in U_i. \]

By 6),
\[ \langle G_i \bar{t}, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x), \forall (x, t) \in U_i. \]

Hence, $U_i \subseteq P_i^{-1}(y_i)$. This implies, $P_i^{-1}(y_i)$ is open for each $y_i \in K_i$, and so $P_i$ has open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.1. This completes the proof.

**Corollary 3.3.** For each $i \in I$, let $Z_i$ be a l.c.s., $K_i$ a nonempty compact convex subset of Hausdorff t.v.s. $E_i$, $Y_i$ a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a $\sigma$-topology. For each $i \in I$, assume that $S_i$ and $G_i$ are single valued mappings and the following conditions are satisfied.

1) $D_i : K \to 2^{\varepsilon_i}$ and $T_i : K \to 2^{\varepsilon_i}$ are two nonempty convex set-valued mappings and have open lower sections;
2) For all $y_i \in K_i$, the mapping
\[ \{G_i \bar{t}, \eta_i(y_i, \bar{x})\} + S_i(\bar{x}, y_i) : K \times Y \to Z_i \] is a convex set-valued mapping with $\text{int } C_i(x) \neq \emptyset$ for all $x \in K$;
4) $\eta_i : K_i \times K_i \to E_i$ is affine in the first argument and for all $x_i \in K_i, \eta_i(x_i, x_i) = 0$;
5) $S_i : K \times K \to 2^{\varepsilon_i}$ is a generalized vector 0-diagonally convex set-valued mapping;
6) For a given $x_i \in K_i$, and a neighborhood $U_i$ of $x_i$, for all $u \in U_i$, $\text{int } C_i(x) = \text{int } C_i(u)$.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that
\[ \langle G_i \bar{t}_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x), \forall y_i \in D_i(\bar{x}). \]

**Proof.** By hypothesis 3), the condition 4) in Corollary 3.2 is satisfied. Hence, all the conditions are satisfied as in Corollary 3.2.

**Corollary 3.4.** For each $i \in I$, let $Z_i$ be a l.c.s., $K_i$ a nonempty compact convex subset of Hausdorff t.v.s. $E_i$, $Y_i$ a nonempty compact convex subset of $L(E_i, Z_i)$, which is equipped with a $\sigma$-topology. For each $i \in I$, assume that $S_i$ and $G_i$ are single valued mappings and the following conditions are satisfied.

1) $D_i : K \to 2^{\varepsilon_i}$ and $T_i : K \to 2^{\varepsilon_i}$ are two nonempty convex set-valued mappings and have open lower sections;
2) For all $y_i \in K_i$, the mapping
\[ \{G_i \bar{t}, \eta_i(y_i, \bar{x})\} + S_i(\bar{x}, y_i) : K \times Y \to Z_i \] is continuous;
3) $C_i : K \to 2^{\varepsilon_i}$ is a convex set-valued mapping with $\text{int } C_i(x) \neq \emptyset$ for all $x \in K$;
5) $S_i : K \times K \to Z_i$ is a vector 0-diagonally convex mapping;
6) $Z_i \setminus \{-\text{int } C_i(x)\}$ is an u.s.c. set-valued mapping.

Then there exist $\bar{x}_i \in D_i(\bar{x})$ and $\bar{t}_i \in T_i(\bar{x})$ such that
\[ \langle G_i \bar{t}_i, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \subseteq -\text{int } C_i(x), \forall y_i \in D_i(\bar{x}). \]

**Proof.** Define a set-valued mapping $P_i : K \times Y \to 2^{\varepsilon_i}$ by
\[ P_i(x, t) = \{y_i \in K_i : \langle G_i \bar{t}, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \] 
\[ \subseteq -\text{int } C_i(x)\}, \forall (x, t) \in K \times Y. \]

We now prove that for each
\[ y_i \in K_i, P_i^{-1}(y_i) \]
\[ = \{(x, t) \in K \times Y : \langle G_i t, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \] 
\[ \subseteq -\text{int } C_i(x)\} \]
is open, that is, the set
\[ \{(x, t) \in K \times Y : \langle G_i t, \eta_i(y_i, x_i) \rangle + S_i(x_i, y_i) \] 
\[ \subseteq -\text{int } C_i(x)\} \]
is closed. Indeed, let $\{(x^0, t^0)\}$ be a net in $K \times Y$ such that
\[ (x^0, t^0) \to (x^*, t^*) \]
and
\[ \{G_i t^*, \eta_i(y_i, x^*)\} \cap S_i(x^*, y_i) \subseteq Z_i \setminus \{-\text{int } C_i(x^*)\}. \]

Since $\{G_i t^*, \eta_i(y_i, x^*)\} + S_i(x^*, y_i) : K \times Y \to 2^{\varepsilon_i}$ is continuous, hence
\[ \{G_i t^*, \eta_i(y_i, x^*)\} \cap S_i(x^*, y_i) \] 
\[ \to \{G_i t^*, \eta_i(y_i, x^*)\} + S_i(x^*, y_i). \]
Since \(|Z \setminus \{\text{int } C_i(x)\}|\) is an u.s.c. set-valued mapping with closed values, by Lemma 2.1, we have
\[
\{G_t^\ast, \eta \left( y, x_t \right) \} + S_i \left( x_t, y \right) \in Z \setminus \{\text{int } C_i(x')\},
\]
and hence \(\{x', y\}^\ast\) in the set
\[
\{(x, t) \in K \times Y :\{G_t, \eta \left( y, x_t \right) \} + S_i \left( x_t, y \right) \in \{\text{int } C_i(x')\}\}
\]
This implies \(P_t^{-1}(y, i)\) is open for each \(y, i\) and so \(P_t^{-1}\) has open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.1 and Corollary 3.2. This completes the proof.

**Theorem 3.5.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i, Z_i)\), which is equipped with a \(\sigma\) - topology. For each \(i \in I\), assume that the following conditions are satisfied.

1) \(D_i : K \rightarrow 2^{E_i}\) and \(T_i : K \rightarrow 2^{E_i}\) are two nonempty convex set-valued mappings and have open lower sections;

2) For each \(t \in Y_i\) and \(x \in \text{co} \Lambda_i\), the mapping \(\{G_t, \eta \left( y, x \right) \} + S_i \left( x, y \right) \rightarrow \{\text{int } C_i(x')\}\) is WIC-DQC;

3) \(Z_i \setminus \{\text{int } C_i(x)\}\) is WIC-DQC.

This implies \(P_t^{-1}(y, i)\) holds for each \(y, i\) and \(P_t^{-1}\) has open lower sections.

**Proof.** Define a set-valued mapping \(P_t : K \times Y \rightarrow 2^{E_i}\) by
\[
P_t(x, t) = \{y \in K_i : \{G_t, \eta \left( y, x_t \right) \} + S_i \left( x_t, y \right) \}
\]
\[
\quad \cap \{\text{int } C_i(x')\} \neq \emptyset,
\]
\[
\forall (x, t) \in K \times Y.
\]
For the remainder proof, we just follow that of Theorem 3.1.

**Corollary 3.6.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i, Z_i)\), which is equipped with a \(\sigma\) - topology. For each \(i \in I\), assume that the following conditions are satisfied.

1) \(D_i : K \rightarrow 2^{E_i}\) and \(T_i : K \rightarrow 2^{E_i}\) are two nonempty convex set-valued mappings and have open lower sections;

2) For each \(t \in Y_i\) and \(x \in \text{co} \Lambda_i\), the mapping \(\{G_t, \eta \left( y, x \right) \} + S_i \left( x, y \right) \rightarrow \{\text{int } C_i(x')\}\) is WIC-DQC.
Hence, for all \((x',t') \in U_i(x',y') \cap \{U_i(x') \times Y_i\}\), there exists \(w' \in \left\{(G_{ti},\eta_{i}(y_i,\bar{x}_i)) + S_i(x_i,y_i)\right\}\) such that \(w' \notin Z_i \setminus \{-\inf C_i(x')\}\), which is contradiction. Therefore, the set 
\[
\{(x,t) \in K \times Y : \left\{(G_{ti},\eta_{i}(y_i,\bar{x}_i)) + S_i(x_i,y_i)\right\} \cap C_i(x) = \emptyset\}
\]
is closed. Hence, all the conditions of Theorem 3.5 satisfied. Consequently, the assertion of the theorem holds.

**Theorem 3.7.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i,Z_i)\), which is equipped with a \(\sigma\) -topology. For each \(i \in I\), assume that the following conditions are satisfied.
1) \(D_i : K \to 2^{K_i}\) and \(T_i : K \to 2^{K_i}\) are two nonempty convex set-valued mappings and have open lower sections;
2) For each \(t_i \in Y_i\) and \(x_i \in \co \Lambda_i\), the mapping 
\[
\{G_{ti},\eta_{i}(\cdot,x_i)\} + S_i(x_i,y_i) : K \to 2^{K_i}
\]
is SIIC-DQC;
3) For each \(y_i \in K_i\), the set 
\[
\{(x,t) \in K \times Y : \left\{(G_{ti},\eta_{i}(y_i,\bar{x}_i)) + S_i(x_i,y_i)\right\} \cap C_i(x) = \emptyset\}
\]
is open. Then there exist \(\bar{x}_i \in D_i(\bar{x})\) and \(\bar{t}_i \in T_i(\bar{x})\) such that 
\[
\left\{(G_{ti},\eta_{i}(y_i,\bar{x}_i)) + S_i(x_i,y_i)\right\} \cap C_i(x) \neq \emptyset, \forall y_i \in D_i(x).
\]

**Proof.** Define a set-valued mapping \(P_i : K \times Y \to 2^{K_i}\) by 
\[
P_i(x,t) = \left\{y_i \in K_i : \left\{(G_{ti},\eta_{i}(y_i,\bar{x}_i)) + S_i(x_i,y_i)\right\} \cap C_i(x) = \emptyset\right\},
\]
\(\forall (x,t) \in K \times Y\).

For the remainder proof, we just follow that of Theorem 3.1.

**Corollary 3.8.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i,Z_i)\), which is equipped with a \(\sigma\) -topology. For each \(i \in I\), assume that the following conditions are satisfied.
1) \(D_i : K \to 2^{K_i}\) and \(T_i : K \to 2^{K_i}\) are two nonempty convex set-valued mappings and have open lower sections;
2) For each \(t_i \in Y_i\) and \(x_i \in \co \Lambda_i\), the mapping 
\[
\{G_{ti},\eta_{i}(\cdot,x_i)\} + S_i(x_i,y_i) : K \to 2^{K_i}
\]
is SIIC-DQC;
3) For all \(x \in K\), \(C_i(x)\) is closed convex cone \(C_i\). Then there exist \(\bar{x}_i \in D_i(\bar{x})\) and \(\bar{t}_i \in T_i(\bar{x})\) such that 
\[
\left\{(G_{ti},\eta_{i}(y_i,\bar{x}_i)) + S_i(x_i,y_i)\right\} \cap C_i(x) = \emptyset, \forall y_i \in D_i(x).
\]

**Proof.** Define a set-valued mapping \(P_i : K \times Y \to 2^{K_i}\) by 
\[
P_i(x,t) = \left\{y_i \in K_i : \left\{(G_{ti},\eta_{i}(y_i,\bar{x}_i)) + S_i(x_i,y_i)\right\} \cap C_i(x) = \emptyset\right\},
\]
\(\forall (x,t) \in K \times Y\).

The rest of the proof is similar to that of Theorem 3.1.

**Corollary 3.10.** For each \(i \in I\), let \(Z_i\) be a l.c.s., \(K_i\) a nonempty compact convex subset of Hausdorff t.v.s. \(E_i\), \(Y_i\) a nonempty compact convex subset of \(L(E_i,Z_i)\), which is equipped with a \(\sigma\) -topology. For each \(i \in I\), assume that the following conditions are satisfied.
1) \(D_i : K \to 2^{K_i}\) and \(T_i : K \to 2^{K_i}\) are two nonempty convex set-valued mappings and have open lower sections;
2) For each \(t_i \in Y_i\) and \(x_i \in \co \Lambda_i\), the mapping 
\[
\{G_{ti},\eta_{i}(\cdot,x_i)\} + S_i(x_i,y_i) : K \to 2^{K_i}
\]
\[ \{G_t, \eta_s (x, y)\} + S_s (x, y) : K \to 2^Z \] is SIC-DQC;
3) \( C_G (x) \) is an u.s.c. mapping with closed values.

Then there exist \( x \in D_G (x) \) and \( \bar{x} \in T_G (x) \) such that

\[ \{G_t, \eta_s (y, \bar{x})\} + S_s (y, \bar{x}) \subseteq C_G (\bar{x}), \forall y \in D_G (x). \]

**Proof.** Let \( P : K \times Y \to 2^{K_i} \) a set-valued mapping defined in Theorem 3.9. We prove that for each \( y_i \in K_i \), the set

\[ \{(x, t) \in K \times Y : \{G_t, \eta_s (y, x_i)\} \cap S_s (x, y_i) \nsubseteq C_s (x) \} \]

is open, that is, the set

\[ \{(x, t) \in K \times Y : \{G_t, \eta_s (y, x_i)\} \cap S_s (x, y_i) \subseteq C_s (x) \} \]

is closed. Indeed, let \( \{(x^o, t^o) \} \) be a net in \( K \times Y \) such that

\[ (x^o, t^o) \to (x^t, t^t) \]

and

\[ \{G_t, \eta_s (y, x^o_i)\} \cap S_s (x^o, y_i) \subseteq C_s (x^o). \]

We claim that

\[ \{G_t, \eta_s (y, x^t_i)\} \cap S_s (x^t, y_i) \subseteq C_s (x^t). \]

To prove this assertion, we can just follow that of Corollary 3.6. Hence, the set

\[ \{(x, t) \in K \times Y : \{G_t, \eta_s (y, x_i)\} \cap S_s (x, y_i) \nsubseteq C_s (x) \} \]

is open. Therefore, all the conditions of Theorem 3.9 are satisfied. Consequently, the assertion of the corollary hold.

**REFERENCES**


