Analysis of constraint equations and their singularities

Rangaprasad Arun Srivatsan and Sandipan Bandyopadhyay

Abstract The identification of singularities is an important aspect of research in parallel manipulators, which has received a great deal of attention in the past few decades. Yet, even in many well-studied manipulators, very few reported results are of complete or analytical nature. This paper tries to address this issue from a slightly different perspective than the standard method of Jacobian analysis. Using the condition for existence of repeated roots of the univariate equation representing the forward kinematic problem of the manipulator, it shows that it is possible to gain some more analytical insight into such problems. The proposed notions are illustrated by means of applications to a spatial 3-R$\underline{\text{P}}$S manipulator, leading to the closed-form expressions for the singularity manifold of the 3-R$\underline{\text{P}}$S in the actuator space.

Key words: Singularity; Parallel manipulator; Univariate equation

1 Introduction

This paper attempts to revisit the relationship between the constraint equations inherent to a parallel manipulator, and the singularities thereof. It is well-established that singularities in a physical manipulator can be characterised in terms of a corresponding degeneracy in the mathematical equations defining its motion. In particular, in all the cases of singularities in *direct kinematics* (which has also been termed as a "singularity of the sec-

Rangaprasad Arun Srivatsan
Robotics Institute, Carnegie Mellon University,
Pittsburgh-15213, USA. e-mail: arangapr@andrew.cmu.edu
Sandipan Bandyopadhyay
Department of Engineering Design, Indian Institute of Technology Madras,
Chennai - 600036, India. e-mail: sandipan@iitm.ac.in

ond type" [7], "gain-type" singularity [1], "constraint singularity", etc.), it is known that one or more pairs of branches of forward kinematics merge – leading to the mathematical condition that the *loop-closure* equations defining the forward kinematics admit repeated roots. There exists a considerable variety in the development of the constraint equations, and the nature of variables included in them. However, notwithstanding the differences in the implementation details, many of these formulations are connected via a fundamental logical thread – namely, the application of *implicit function theorem* to a set of non-linear equations, leading to the analysis of the rank degeneracy of certain resulting Jacobian matrices. While a comprehensive discussion of the Jacobian-based formulations is out of scope of this paper, the mention of two typical examples may help motivate the point better. In [7], the constraint equation is also the *input-output* equation of the manipulator, i.e., the equations are formed to connect the task-/output-space (i.e., dependent/unknown) variables directly to the input (i.e., independent) variables by eliminating the other unknown variables associated with the passive joints. On the other hand, in other works such as [6, 1], the passive variables are in focus, and they, along with the input variables, define the *configuration space* of the manipulator. Degeneracy of the Jacobian of the constraint equations with respect to the passive variables define the condition for the *gain-type* singularities in the configuration space. It is also understood, that these methods lead to similar results, since a gain of degree-of-freedom in the configuration space typically results in a corresponding gain in the task-space degree-of-freedom.

This paper follows the same basic approach for the analysis of singularities. However, it differs in the fact that instead of the vanishing of the determinant of a certain Jacobian to identify singularities, it uses the derivative of a special scalar equation in conjunction with the equation itself. This equation is designated as the "forward kinematic univariate", or FKU for brevity. Typically, an FKU is derived in the process of solving the forward kinematic problem itself, by the process of systematic elimination of all the unknowns but one, which is then solved from the FKU itself. This observation motivates a very simple procedure/algorithm for deriving singularity conditions:

1. Derive the FKU depicting the forward kinematics;
2. Set its derivative w.r.t. the lone remaining unknown variable to zero;
3. Solve the above two equations simultaneously.

There are several advantages to this procedure in comparison to the standard Jacobian-based formulations. Firstly, in the case of a number of spatial manipulators of practical importance, the FKU can be derived in closed form – see, e.g., [5, 3]. This opens up the opportunity of deriving the scalar singularity condition also in the closed-form. Secondly, sometimes, it is possible to decompose the FKU into factors. In such cases, the algorithm can be applied separately to the individual factors, which further simplifies the task of the analytical computation. This aspect would be demonstrated in Section 3.

Finally, often the FKU is either in the form of a polynomial in the unknown variable, or can be converted into one. In such cases, the problem reduces to the computation of the *resultant* of the FKU and its derivative w.r.t. the unknown, which can be accomplished easily provided the degree of the FKU (or its individual factors) is not too high in this variable. There are, however, several limitations of the proposed formulation as well. Obviously, it cannot be applied to situations, where the FKU is not available in closed form. Also, since the FKU is typically the result of the elimination of a number of variables from the original set of constraint equations, it may accrue one or more *spurious solutions*. The zeros of the discriminant is only guaranteed to be a *super-set* of the singularities in the original system (see, e.g., [4]).

The remaining of the paper is organised as follows: in Section 2, the mathematical formulation for the proposed method is described. In Section 3, the same is illustrated with the example of a spatial 3-R$\underline{\text{P}}$S manipulator, and the conclusions are presented in Section 4.

2 Formulation

Let, the loop-closure/kinematic constraint equations be defined as:

$$\boldsymbol{\eta}(\boldsymbol{\theta}, \boldsymbol{\phi}) = \mathbf{0}, \tag{1}$$

where $\boldsymbol{\theta}$ represents the set of *active* or known variables, and $\boldsymbol{\phi}$ the set of unknown variables, which could consist of any combination of the *passive* joint/configuration variables, and Cartesian-/task-space variables. At a *regular* point, these equations yield *unique* solutions for $\boldsymbol{\phi}$. At a singularity, however, one or more pairs of solutions merge. Applying the *implicit function theorem* to this situation, the corresponding condition emerges as:

$$\det(\boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}}) = 0, \text{ where } \boldsymbol{J}_{\boldsymbol{\eta}\boldsymbol{\phi}} = \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\phi}}. \tag{2}$$

The above observation forms the basis for the identification of the "*gain-type* singularities in the configuration space" [6, 1], as well as the "singularities of the *second type* in the task space" [7], albeit with different meanings for the variable $\boldsymbol{\phi}$. Thus, the said singularities can be identified as the set of points in the workspace, where Eqs. (1,2) are satisfied simultaneously[1]. In practice, however, it is difficult to solve these equations together – particularly since

[1] If constraint equations are written in, or, converted to, their algebraic (i.e., polynomial) forms, then an equivalent condition for such singularities would be that the *singularity condition* in Eq. (2) in its algebraic form, belongs to the *constraint ideal* generated by the algebraic form of Eq. (1). Thus, the determination of singularities in the constraint equations can also be posed as an *ideal membership* (see, e.g., [4]) problem.

the singularity condition involves a determinant. This observation motivates the following alternative approach.

Consider, that $\boldsymbol{\theta} \in \mathbb{R}^n$, and $\boldsymbol{\phi} \in \mathbb{R}^m$ (locally at least), where m, n are positive integers, such that n equals the degrees-of-freedom of the mechanism, and $\boldsymbol{\eta} \in \mathbb{R}^m$. The *forward kinematic* problem refers to the finding of solutions for $\boldsymbol{\phi}$ in terms of $\boldsymbol{\theta}$. To achieve this, it is fairly standard to eliminate the variables ϕ_i $(i = 1, \dots, m-1)$, i.e., the components of $\boldsymbol{\phi}$ from Eq. (1), till a single variable (which is ϕ_m in this context) remains in a single equation – which encapsulates into it *all* the kinematic characteristics captured by the original system (1). This final *univariate*, (defined as the FKU in Section 1) can be written as:

$$f(\boldsymbol{\theta}, \phi_m) = 0. \tag{3}$$

In the algebraic context, Eq. (3) would be the *resultant* of the algebraic form of Eq. (1). To find the singularities in Eq. (3), the implicit function theorem can be invoked again, leading to the new singularity condition:

$$\frac{\partial f(\boldsymbol{\theta}, \phi_m)}{\partial \phi_m} = 0. \tag{4}$$

Elimination of ϕ_m between Eq. (3,4) results in the singularity condition in terms of the actuator variables and the geometric parameters alone.

3 Illustrative example: the 3-R$\underline{\text{P}}$S manipulator

The 3-R$\underline{\text{P}}$S parallel manipulator was introduced by Lee and Shah in 1988, and has since been studied extensively by several researchers. Some of the important works on the singularity of this manipulator include [2, 8]. The manipulator, as shown in Fig. 1, consists of a fixed and a moving platform. The two platforms are connected by means of three "legs", each of which has a rotary, a prismatic, and a spherical joint. The prismatic joints are actuated, and all the other joints are passive. This gives rise to three-degrees-of-freedom at the moving platform. The coordinates of the (centres of the) spherical joints on the top platform are obtained in the fixed base reference frame as:

$$\begin{aligned}
{}^0\boldsymbol{p}_1 &= (b - l_1 \cos\phi_1, 0, l_1 \sin\phi_1)^T, \\
{}^0\boldsymbol{p}_2 &= \boldsymbol{R}_Z\,(2\pi/3)\,(b - l_2 \cos\phi_2, 0, l_2 \sin\phi_2)^T, \\
{}^0\boldsymbol{p}_3 &= \boldsymbol{R}_Z\,(4\pi/3)\,(b - l_3 \cos\phi_3, 0, l_3 \sin\phi_3)^T,
\end{aligned}$$

where $\boldsymbol{R}_Z(\alpha)$ denotes the rotation matrix for CCW rotation about axis $\boldsymbol{Z}$ through an angle α. Without any loss of generality, the base dimension, b,

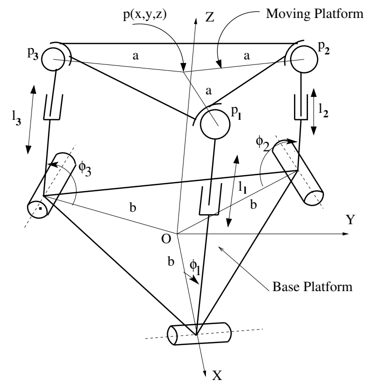


Fig. 1 The 3-R$\underline{\text{P}}$S manipulator

is scaled to *unity* in the following, which renders all the linear dimensions unit-less. All angles are expressed in radians.

3.1 Derivation of the FKU

Given the input variables $\boldsymbol{\theta}=(l_1, l_2, l_3)^T$, there are three passive joint variables $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$, which are to be solved from the loop-closure equations denoted by $\boldsymbol{\eta} = \mathbf{0}$, where $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)^T$, and:

$$\eta_1 \stackrel{\Delta}{=} ({}^0\boldsymbol{p}_2 - {}^0\boldsymbol{p}_1) \cdot ({}^0\boldsymbol{p}_2 - {}^0\boldsymbol{p}_1) - 3a^2 = 0,$$
$$\eta_2 \stackrel{\Delta}{=} ({}^0\boldsymbol{p}_3 - {}^0\boldsymbol{p}_2) \cdot ({}^0\boldsymbol{p}_3 - {}^0\boldsymbol{p}_2) - 3a^2 = 0,$$
$$\eta_3 \stackrel{\Delta}{=} ({}^0\boldsymbol{p}_1 - {}^0\boldsymbol{p}_3) \cdot ({}^0\boldsymbol{p}_1 - {}^0\boldsymbol{p}_3) - 3a^2 = 0.$$

The functions η_1, η_2, η_3 are first converted into polynomials in the variables $t_i = \tan(\phi_i/2)$, $(i = 1, 2, 3)$ using the standard *tangent half-angle* substitutions (see, e.g., [5]). After some manipulations, Eq. (1) transforms into a set of three simultaneous quadratic equations of the form $f_1(t_1, t_2) = 0$, $f_2(t_2, t_3) = 0$, $f_3(t_3, t_1) = 0$. The variable t_1 is then eliminated between $f_1 = 0$ and $f_3 = 0$, thereby leading to a new equation of the form $f_4(t_2, t_3) = 0$, which is quartic in t_2, t_3. The second unknown, t_2, is eliminated between $f_4 = 0$ and $f_3 = 0$, yielding the FKU $f(t_3) = 0$, which turns out to be of degree 8 in t_3^2 (see [9] for further details of the elimination scheme). On further analysis, it is found that is it possible to decompose $f(t_3)$ into two quartic factors, i.e., $f(t_3) = g_1(s_3)g_2(s_3)$, where both g_1, g_2 are of degree 4 in $s_3 = t_3^2$. The coefficients of s_3 in g_1, g_2 are functions of the platform dimension a and the inputs l_i only, and these have been obtained in closed-form. The coefficients reveal that g_2 becomes identical to g_1 when a is replaced by $-a$. The actual expressions of the coefficients are too big to be included in this paper; for the sake of illustration, the coefficient of s_3 in g_1 is given below:

$$(9a^4 + 12a^3(l_3 + 3) - 3a^2(l_1^2 + l_2^2 - l_3^2 - 10l_3 - 15) - 2a(l_3 + 3)(l_1^2 + l_2^2 + l_3^2 - 3) \\ - l_1^2(-l_2^2 + 2l_3 + 3) - 2l_2^2 l_3 - 3l_2^2 - l_3^4 - 8l_3^3 - 18l_3^2 - 12l_3)^2.$$

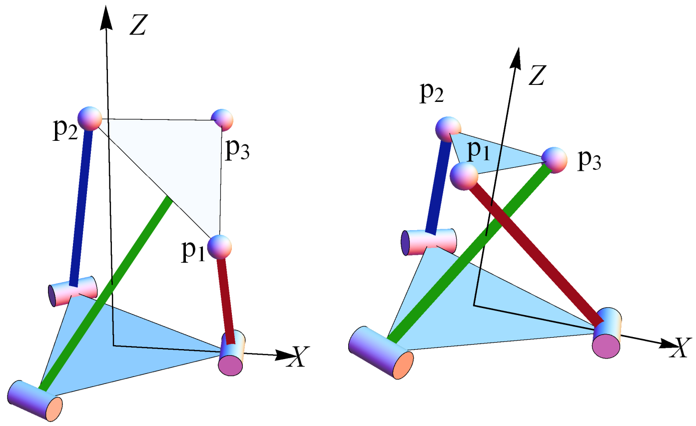
3.2 Analysis of singularities using the FKU

Singularities in forward kinematics occur when the FKU $f(t_3) = 0$ has repeated roots. Taking advantage of the factorisation, and using the formulation presented in Section 2, it can be seen easily that singularities can occur in one of two possible manners, and/or their combinations: $g_1(s_3)$ or $g_2(s_3) = 0$ has one or more repeated root(s); $g_1(s_3) = 0$ and $g_2(s_3) = 0$ share one or more common root(s). Both the cases are described below.

- **Case 1:** $g_1(s_3) = 0$ **has a repeated root**
  For the numerical values $l_1 = 1, l_2 = 2, l_3 = 3, a = 0.851$, the equation $g_1(s_3) = 0$, written as a *monic polynomial*, becomes:
  $$s_3^4 + 1.83886s_3^3 - 0.23294s_3^2 + 0.00243s_3 + 0.00032 = 0.$$
  The roots of this equation are: $(-1.95839, -0.02950, 0.07452, 0.07452)$, and the passive variables corresponding to the repeated real root are: $\phi_1 = 0.68488$, $\phi_2 = 1.52654$, $\phi_3 = 0.52303$. The corresponding pose of the manipulator is shown in Fig. 2(a).
- **Case 2:** $g_1(s_3) = 0$ **and** $g_2(s_3) = 0$ **share a common root**
  For the inputs $l_1 = 1.57928, l_2 = 1, l_3 = 2$, and $a = 1/2$, the roots of $g_1(s_3) = 0$ are: $(-0.95589, -0.00028, 0.05276, 0.07163)$, and the roots of $g_2(s_3) = 0$ are: $(0.07163, 0.09818, 0.46876 \pm 0.05965i)$. Thus, the root $s_3 = 0.07163$ is shared between the two factors. The corresponding values of the passive variables are: $\phi_1 = 1.45197$, $\phi_2 = 1.27655$, $\phi_3 = 0.53296$. The manipulator is shown in this pose in Fig. 2(b).

3.3 Special cases

The loop-closure equations (1) suffer another type of degeneracy for "special" combinations of leg inputs, e.g., when two or more of the leg inputs are identical. For instance, consider the case when $l_2 = l_3$: obviously, in this case, $\phi_2 = \phi_3$, and hence, η_2 becomes identical to η_3. Proceeding as before with the equations $\eta_1 = 0$ and $\eta_2 = 0$, the FKU is obtained in terms of t_1 in this case, which turns out to be a quadratic in t_1^2. Once again, the coefficients



(a) Case 1: $g_1(s_3) = 0$ has a repeated root

(b) Case 2: $g_1(s_3) = 0$ and $g_2(s_3) = 0$ share a common root

Fig. 2 Singular poses of the 3-R$\underline{\text{P}}$S

of the polynomial are obtained in close-form; e.g., the coefficient of t_1^2 is:

$$2\left(9a^4 - 18a^3 + a^2(9 + l_1^2 - 6l_2^2) + a(-14l_1^2 + 6l_2^2) + 4l_1^2 + l_1^4 - 6l_1^2l_2^2 + l_2^4\right).$$

In the following, the case of gain of one-degree-of-freedom derived in [9] following geometric reasonings, is studied again, albeit in the framework of analysis proposed in this paper. For $a = 1/2$ and $l_2 = l_3 = 1$, the above-mentioned quadratic equation has a double root when[2] $l_1 = (\sqrt{37} - 3)/4$. The solutions for t_1 are obtained as: $\left(\pm\sqrt{\left(31 - 5\sqrt{37}\right)/6}, \pm\sqrt{\left(31 - 5\sqrt{37}\right)/6}\right)$, i.e., both the positive and negative solutions of t_1 are repeated (as they should, since they correspond to the poses *mirrored* at the base plane). The pose corresponding to the positive solutions is the same as in Fig. 7 of [9].

In the case where $l_1 = l_2 = l_3$, only one of the constraint equations, say, $\eta_1 = 0$, matters. Since $\phi_2 = \phi_1$ in this situation, this equation becomes a quadratic in $\cos\phi_1$. Setting the discriminant of this equation to zero, one obtains the final condition for singularity as $a^2 l_1^4 = 0$ – which can occur only if the top platform shrinks to a point, or coincides with the base platform.

---

[2] Note that numerically, $l_1 \simeq 0.770$, as noted in Section 5.5.1 of [9]. However, thanks to the proposed algorithm, it is now possible to compute this value *exactly*.

4 Conclusions

A new method for deriving the singularity condition of a parallel manipulator is presented in this paper. The method depends upon the solution of the forward kinematic problem of the manipulator through a single univariate equation, which is a fairly common practice. The proposed computational scheme involves the elimination of a single variable between two equations, for any manipulator. Special structures in the final univariate equation, e.g., its decomposition into factors, or polynomial nature etc. can also be taken advantage of to reduce the computational complexity, while still obtaining analytical results. This is demonstrated by means of the spatial 3-R$\underline{\text{P}}$S manipulator, leading to the description of its singularity manifold in terms of closed-form expressions in the general case, perhaps for the first time.

References

[1] Bandyopadhyay, S., Ghosal, A.: Analysis of configuration space singularities of closed-loop mechanisms and parallel manipulators. Mechanism and Machine Theory **39**(5), 519–544 (2004)

[2] Basu, D., Ghosal, A.: Singularity analysis of platform-type multi-loop spatial mechanisms. Mechanism and Machine Theory **32**(3), 375–389 (1997). DOI 10.1016/S0094-114X(96)00033-X

[3] Bonev, I.A.: Direct kinematics of zero-torsion parallel mechanisms. In: Proceedings of the 2008 IEEE International Conference on Robotics and Automation, May 19-23, Pasadena, CA, USA, pp. 3851–3856 (2008)

[4] Cox, D., Little, J., O'Shea, D.: Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra. Springer-Verlag, New York (1991)

[5] Ghosal, A.: Robotics: Fundamental Concepts and Analysis. Oxford University Press, New Delhi (2006)

[6] Ghosal, A., Ravani, B.: A differential-geometric analysis of singularities of point trajectories of serial and parallel manipulators. Trans. of ASME, Journal of Mechanical Design **123**(1), 80–89 (2001)

[7] Gosselin, C., Angeles, J.: Singularity analysis of closed-loop kinematic chains. IEEE Transactions on Robotics and Automation **6**, 281–290 (1990). DOI 10.1109/70.56660

[8] Schadlbauer, J., Walter, D., Husty, M.: The 3-RPS parallel manipulator from an algebraic viewpoint. Mechanism and Machine Theory **75**, 161–176 (2014). DOI 10.1016/j.mechmachtheory.2013.12.007. URL http://www.sciencedirect.com/science/article/pii/S0094114X13002504

[9] Srivatsan, R.A., Bandyopadhyay, S., Ghosal, A.: Analysis of the degrees-of-freedom of spatial parallel manipulators in regular and singular configurations. Mechanism and Machine Theory **69**, 127–141 (2013)