Qualitative analysis of discrete nonlinear delay survival red blood cells model

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Abstract

The objective of this paper is to systematically study the qualitative properties of the solutions of the discrete nonlinear delay survival red blood cells model

\[ x(n + 1) - x(n) = -\delta(n)x(n) + p(n)e^{-q(n)x(n-\omega)}, \quad n = 1, 2, \ldots, \]

where \(\delta(n), p(n)\) and \(q(n)\) are positive periodic sequences of period \(\omega\). First, by using the continuation theorem in coincidence degree theory, we prove that the equation has a positive periodic solution \(x(n)\) with strictly positive components. Second, we prove that the solutions are permanent and establish some sufficient conditions for oscillation of the positive solutions about \(x(n)\). Finally, we give an estimation of the lower and upper bounds of the oscillatory solution and establish some sufficient conditions for global attractivity of \(x(n)\).

From applications point of view permanence guarantees the long term survival of mature cells, oscillation implies the prevalence of the mature cells around the periodic solution and the convergence implies the absence of any dynamical diseases in the population. Our results in the special case when the coefficients are positive constants involve and improve the oscillation and global attractivity results on the literature.

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1. Introduction

In order to describe the survival of red blood cells in an animal Wazewska-Czyzewska and Lasota [39] proposed the nonlinear delay differential equation

\[ x'(t) = -\delta x(t) + p \exp(-q x(t - \omega)), \]

where \(\delta, p, q, \omega \in (0, \infty)\). In Eq. (1.1) \(x(t)\) denotes the number of red blood cells at time \(t\), \(\delta\) is the probability of death of a red blood cell, \(p\) and \(q\) are positive constants related to the production of red blood cells per unit of time and \(\omega\) is

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the time needed to produce a red blood cell. The oscillation and global attractivity of Eq. (1.1) and its generalization with several delays has been studied in Györi and Ladas [10], Kulenović et al. [19], Li and Cheng [24] and Xu and Li [40] and the bifurcation and the direction of the stability studied recently by Song et al. [33]. For global stability of survival blood cells model with several delay and piecewise constant argument we refer the reader to the paper by Yu-Ji [43].

Many dynamical systems that model biological or ecological phenomena contain several parameters. Biologists are tasked to determine the exact parameter values in order to use the model for prediction purpose. Unfortunately, in the real world the parameters are not fixed constants and the parameters are estimated using statistical methods and at each stage in time the estimate will be improved. Thus the assumption of existence of convergent functions that are convergent to constant parameters values as time goes to infinity (in a way) incorporates this case.

On the other hand, the variation of the environment plays an important role in behavior of the biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as selective forces in systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In fact, it has been suggested by Nicholson [28] that any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes.

In view of this it is realistic to assume that the parameters in the model are:

1. actually convergent functions converge to constant parameters values as time goes to infinity;
2. periodic functions of constant period \( \tau = m \omega \) for some positive integer \( m \) (the delay itself periodic function \( \omega(t) \)); or
3. almost periodic functions.

According to the second case, when the delay is a periodic function, Graef et al. [9] studied the oscillation and the global attractivity of the nonlinear delay differential equation

\[
x'(t) = -\delta(t)x(t) + P(t) \exp(-x(t - \omega(t))),
\]

where \( \delta(t), P(t), \) and \( \omega(t) \) are positive periodic functions.

Jiang and Wei [14] considered Eq. (1.2) when (1.3) holds and studied the existence of positive periodic solutions.

Saker and Agarwal [30] considered the equation

\[
x'(t) = -\delta(t)x(t) + P(t) \exp(-qx(t - m\omega)),
\]

where \( q > 0, \delta(t), P(t) \) are positive periodic functions of period \( \omega \) and \( m \) is a positive integer and proved that the equation has a unique positive periodic solution and established some sufficient conditions for oscillations and global attractivity.

According to the third case, Gopalsamy and Trofimchuk [8] investigated the existence of a global attractive almost periodic solution of a single species model given by the nonautonomous delay differential equation of Lasota–Wazewska-type

\[
x'(t) = -\delta(t)x(t) + P(t) f(x(t - \omega)),
\]

where \( \delta(t), P(t) \) are continuous almost periodic functions and \( f \) is a positive decreasing function. For existence and global attractivity results for impulsive survival red blood cells model with several coefficients, we refer the reader to the paper by Yan [41].

It remains an open problem to study the asymptotic behavior of Eq. (1.2) when the coefficients are functions which converge to fixed parameters, i.e., it remains to prove that: if \( \lim_{t \to \infty} P(t) = \overline{P} \), \( \lim_{t \to \infty} \omega(t) = \omega \) and \( \lim_{t \to \infty} \delta(t) = \overline{\delta} \), then the solution \( x(t) \) oscillates about the equilibrium point that satisfies the equation \( -\overline{\delta}x + \overline{P} \exp(-q\overline{\tau}) = 0 \), and prove that \( \lim_{t \to \infty} x(t) = \overline{\tau} \). One can think from the first sight that any continuous dynamical system when the parameters are replaced with convergent functions have the same asymptotic behavior. To show that this is not true and explains the
difficulty to study such question and show also that the dynamics can change, we present the following counterexample. Consider the equations

\begin{align}
  x'(t) + e^{-1}x(t-1) &= 0, \quad t > 0, \\
  x'(t) + (e^{-1} + t^{-1})x(t-1) &= 0, \quad t > 0.
\end{align}

According to the oscillation theory of first order delay differential equations, see [10], Eq. (1.6) has a nonoscillatory solution, while the results in [2] implies that all solution of Eq. (1.7) are oscillatory.

Researching the asymptotic behavior of the solutions of the discrete analogue of Eq. (1.1), i.e., for the equation

\begin{equation}
  x(n+1) - x(n) = -\delta x(n) + P \exp(-q x(n-\omega)), \quad n \geq 1,
\end{equation}

where

\begin{equation}
  P, q \in (0, \infty), \quad \omega \geq 1, \quad \delta \in (0, 1)
\end{equation}

was posed as an open problem by Kocic and Ladas [16, open problem 4.6.1]. In (1.8) the state variable \( x(n) \) represents the number of the mature red blood cells in cycle \( n \) as a closed system of the mature cells surviving from previous cycles plus the cells which have survived from the previous \( \omega \) cycle. Specially \( P e^{-q x(n-\omega)} \) represents the number of mature cells that were produced in the \( (n-\omega) \)th cycle and survived to maturity in the \( n \)th cycle.

Kubiaczyk and Saker [18] and Li and Cheng [25] investigated the oscillation of Eq. (1.8) about positive equilibrium point \( \overline{x} \), where \( \overline{x} \) is the unique solution of the equation

\begin{equation}
  \delta x = P e^{-q x},
\end{equation}

and showed that every positive solution of (1.8) oscillates about \( \overline{x} \) if

\begin{equation}
  P q e^{-q \overline{x}} > \left( 1 - \delta \right)^{\omega+1} \frac{\omega^{\omega}}{(\omega + 1)^{\omega+1}}.
\end{equation}

Karakostas et al. [15], Ivanov [12], Zheng et al. [45] and Li and Cheng [25] investigated the global attractivity of the positive equilibrium point \( \overline{x} \) and showed that \( \overline{x} \) is a global attractor of all positive solutions of Eq. (1.8) if

\begin{equation}
  \frac{P q}{\delta} < e.
\end{equation}

Meng and Y an [27] established a different condition for global attractivity of \( \overline{x} \) and showed that, if

\begin{equation}
  \frac{P^2 q^2}{\delta^2} e^{q(Q_1 + \overline{x})} < 1 \quad \text{where} \quad Q_1 = \frac{P}{\delta} e^{-P q / \delta},
\end{equation}

then \( \overline{x} \) is a global attractor of all positive solutions of Eq. (1.8).

Kubiaczyk and Saker [18] also considered Eq. (1.8) when \( \omega = 1 \), and proved that \( \overline{x} \) is a global attractor of all positive solutions of Eq. (1.8) provided that

\begin{equation}
  P q e^{-q \overline{x}} < \delta.
\end{equation}

By (1.10), we get

\begin{equation}
  \overline{x} e^{q \overline{x}} = \frac{P}{\delta}.
\end{equation}

Then from (1.12) and (1.15), we have \( q \overline{x} e^{q \overline{x}} = q P / \delta < e \), which obviously leads to

\begin{equation}
  q \overline{x} < 1.
\end{equation}

Also from (1.14) and (1.15), we have \( q \delta \overline{x} = P q e^{-q \overline{x}} < \delta \), which obviously leads to (1.16).

So that the conditions (1.12) and (1.14) that has been established by Karakostas et al. [15], Ivanov [12], Zhou and Zhans [46], Li and Cheng [25] and Kubiaczyk and Saker [18] are equivalent to the condition (1.16).
Ma and Yu [26] proved that \( \bar{x} \) is a global attractor of all solutions of Eq. (1.8), if
\[
q \bar{x} (1 - (1 - \delta)^{\omega + 1}) \leq 1.
\] (1.17)
Also one can easily see that the condition (1.17) is equivalent to
\[
q \bar{x} \delta < 1.
\] (1.18)
Recently, El-Morshedy and Liz [3] proved that \( \bar{x} \) is a global attractor of all solutions of Eq. (1.8), if
\[
(1 - \delta)^{\omega + 1} \geq P q e^{-q \beta},
\] (1.19)
or
\[
(1 - \delta)^{\omega + 1} \geq 2P q e^{-q \beta} (\omega + 1)^2 / (3\omega + 4),
\] (1.20)
where \( \beta \) is the unique solution in \((0, \bar{x})\) of the equation
\[
x = \frac{P}{\delta} \exp \left[ - \left( \frac{P q x}{\delta} e^{-q \bar{x}} \right) \right].
\]

We are inspired to study the qualitative properties of the solution of the nonlinear equation
\[
x(n + 1) - x(n) = -\delta(n)x(n) + P(n)e^{-q(n)x(n-\omega)},
\] (1.21)
where the coefficients are positive periodic sequences from the fact that: if the parameters are periodic sequences of a common period and if the time domain is discrete, the resulting sequences in (1.21) may be periodic sequences.

To the best of our knowledge nothing is known regarding the qualitative behavior of Eq. (1.21) with periodic coefficients so our results are essentially new. It would be interesting also to study the behavior of Eq. (1.21) when the coefficients are positive convergent sequences.

Also, one can think from the first glance that any discrete dynamical system where the parameters are replaced with convergent sequences have the same asymptotic behavior. To show that the dynamics can change, we present the following counterexample. Consider the equation
\[
\Delta x(n) + (1/(n + 1))x(n) = 0, \quad n = 1, 2, \ldots
\] (1.22)

It is clear that the solution of this equation is given by \( x(n) = (1/n)x_1 \), so that \( \lim_{n \to \infty} x(n) = 0 \). But one can also easily see that the solution of the limiting equation \( \Delta x(n) = 0 \) is given by \( x(n) = x_1 \). That is, every initial condition in \( \mathbb{R} \) is a fixed point.

We remark that some different discrete models with periodic coefficients has been studied recently by some authors, we refer the reader to the papers [6,29,31,32,44] and the references cited therein.

By the biological interpretation, we assume that Eq. (1.21) has the initial conditions
\[
x(-\omega), x(-\omega + 1), x(-\omega + 2), \ldots, x(1) \in [0, \infty) \text{ and } x(0) > 0.
\] (1.23)

**Definition 1.1.** By a solution of Eq. (1.21) we mean a sequence \( x(n) \) which is defined for \( n \geq -\omega \) and satisfies (1.21) for \( n \geq 0 \). Then, it is easy to see that the initial value problem (1.21) and (1.23) has a unique positive solution \( x(n) \).

**Definition 1.2.** A solution \( x(n) \) of Eq. (1.21) is said to be periodic of prime period \( \omega \), or of minimal period \( \omega \), if \( \omega \) is the least positive integer for which \( x(\omega + n) = x(n) \) for \( n = 0, 1, \ldots \).

**Definition 1.3.** A solution \( x(n) \) of Eq. (1.21) is said to be permanent if there exist positive constants \( C \) and \( D \) with \( 0 < C \leq D < \infty \) such that for any initial conditions satisfy (1.23) there exists a positive integer \( n_1 \) which depends on the initial conditions such that
\[
C \leq x(n) \leq D \quad \text{for } n \geq n_1.
\]

**Definition 1.4.** A solution \( x(n) \) of Eq. (1.21) is said to oscillate about the sequence \( \bar{x}(n) \) if the terms \( x(n) - \bar{x}(n) \) of the sequence \( \{x(n) - \bar{x}(n)\} \) are neither eventually positive nor eventually negative.
Definition 1.5. The solution $x(n)$ is said to be asymptotically attractive to $x(n)$ provided $\lim_{n \to \infty} [x(n) - x(n)] = 0$. Further, $x(n)$ is called globally attractive if $x(n)$ is asymptotically attractive to all positive solutions of (1.21).

This paper is organized as follows: in Section 2, by using the continuation theorem in coincidence degree theory due to Gaines and Mawhin [7], we prove that Eq. (1.21) has a positive $\omega$-periodic solution $x(n)$ with strictly positive components. The technique we will use in Section 2 is different from the technique used in [30] and gives the lower and upper bounds of the periodic solutions. In Section 3, first, we prove that the solutions of Eq. (1.21) are permanent and prove that every nonoscillatory solution tends to $x(n)$. Second, we establish some sufficient conditions for oscillation of the positive solutions about $x(n)$ and give an estimation of the lower and upper bounds of the oscillatory solutions. Third, we establish some sufficient conditions for the global attractivity of $x(n)$. Finally, we prove that our results include and improve some oscillation and global attractivity results for Eq. (1.8) that has been established by Karakostas et al. [15], Ivanov [12], Meng and Yan [27], Zheng et al. [45], Kubiaczyk and Saker [18], Li and Cheng [25], Ma and Yu [26] and El-Morshedy and Liz [3].

2. Existence of positive periodic solutions

A very basic and important ecological problem in the study of dynamics of population in a periodic environment is the global existence of a positive periodic solution, which plays a similar role played by the equilibrium of the autonomous models. Thus, it is reasonable to ask for a condition under which the resulting periodic nonautonomous equation have a positive periodic solution. We begin with the nondelay case of (1.21) with $x(0) > 0$ and first prove that there exists a positive periodic solution $x(n)$ of period $\omega$. Clearly, in the delay case also $x(n)$ is a periodic positive solution of (1.21).

Consider the nondelay case, i.e.,

$$x(n + 1) - x(n) = -\delta(n)x(n) + P(n)e^{-q(n)x(n)}, \quad n \geq 0,$$

(2.1)

where

$$\delta(n), P(n) \text{ and } q(n) \text{ are periodic sequences of period } \omega \text{ and } 0 < \delta(n) < 1.$$  

(2.2)

For the reader’s convenience, we now recall some basic tools in the frame of Mawhin’s coincidence degree theorem that will be used to prove the existence of the periodic solutions of (2.1), borrowing notations and terminology from [7].

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, let $L : Dom L \subset \mathbb{X} \to \mathbb{Y}$ be a linear mapping. The linear mapping $L$ will be called a Fredholm mapping of index zero if the following three conditions hold:

(i) $\text{Ker} L$ has a finite dimension.
(ii) $\text{Im} L$ is closed in $\mathbb{Y}$ and has a finite codimension.
(iii) $\dim \text{Ker} L = co \dim \text{Im} L < \infty$.

If $L$ is a Fredholm mapping of index zero then there exist continuous projections

$$P : \mathbb{X} \to \mathbb{X} \quad \text{and} \quad Q : \mathbb{Y} \to \mathbb{Y},$$

such that $\text{Im} P = \text{Ker} L$, $\text{Im} L = \text{Ker} Q = \text{Im}(I - Q)$. It follows that the mapping

$$L|_{Dom L \cap \text{Ker} P} : Dom L \cap (I - P)\mathbb{X} \to \text{Im} L$$

is invertible. We denote the inverse of that map by $K_p$.

Let $\Omega$ is an open bounded subset of $\mathbb{X}$, and $N : \overline{\Omega} \to \mathbb{Y}$ be continuous mapping. The mapping $N$ is called $L$-compact on $\overline{\Omega}$, if the mapping $QN : \overline{\Omega} \to \mathbb{Y}$ is continuous, $QN(\overline{\Omega})$ is bounded, and $K_p(I - Q)N : \overline{\Omega} \to \mathbb{Y}$ is compact, i.e., it is continuous and $K_p(I - Q)N(\overline{\Omega})$ is relatively compact, where $K_p : \text{Im} L \to Dom L \cap \text{Ker} P$ is the inverse of the restriction $L_p$ of $L$ to $Dom L \cap \text{Ker} P$, so that $LK_p = I$ and $K_pL = I - P$. Since $\dim \text{Im} Q = \dim \text{Ker} L$, there exists an isomorphism $J : \text{Im} Q \to \text{Ker} L$. 

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ and $x \in \mathbb{R}^n \setminus f(\partial \Omega \cup S_f)$, i.e., $x$ is regular value of $f$. Hence, $S_f = \{x \in \Omega : J_f(x) = 0\}$, the critical set of $f$, and $J_f$ is the Jacobian of $f$ at $x$. The degree $\deg\{f, \Omega, x\}$ is defined by

$$\deg\{f, \Omega, x\} = \sum_{x \in f^{-1}(x)} \text{sgn} \ J_f(x)$$

with the argument that $\sum \phi = 0$.

In fact, we will only encounter a differentiable function from the one-dimensional Banach space (which is identified with its equivalent Banach space $\mathbb{R}$) to itself and its regular value $0$. In fact the degree will be Brouwer degree and is determined by the sign of the derivative of the function $f$. For more details about the degree theory, we refer the reader to the book [20].

In the proof of our main result in this section, we will use the following continuation theorem of Gaines and Mawhin [7, p. 40].

**Lemma 2.1 (Continuation Theorem).** Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and $L$ be a Fredholm mapping of index zero. Assume that $N : \overline{\Omega} \to \mathcal{X}$ is $L$-compact on $\overline{\Omega}$ with $\Omega$ is open and bounded in $\mathcal{X}$. Furthermore assume:

(a) for each $\lambda \in (0, 1)$, every solution of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$,

(b) $QNx \neq 0$ for each $x \in \partial \Omega \cap \text{Ker } L$, and

$$\deg\{QN, \Omega \cap \text{Ker } L, 0\} \neq 0.$$

Then the operator equation $Lx = Nx$ has at least one solution in $\text{Dom } L \cap \overline{\Omega}$.

Let $\mathbb{Z}$, $\mathbb{Z}^+, \mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ denote the sets of all integers, nonnegative integers, natural numbers, real numbers and nonnegative real numbers, respectively. For convenience in what follows we shall let

$$I_\omega = \{0, 1, 2, \ldots, \omega - 1\}, \quad \overline{f} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} f(n), \quad f^* = \max_{n \in I_\omega} f(n) \quad \text{and} \quad f_* = \min_{n \in I_\omega} f(n),$$

where $f(n)$ is an $\omega$-periodic sequence of real numbers defined for all $n \in \mathbb{Z}$.

In order to prove the prior estimation in the case of difference equation, we need the following lemma which is extracted from [6].

**Lemma 2.2.** Let $f : \mathbb{Z} \to \mathbb{R}$ be periodic of period $\omega$, i.e., $f(n + \omega) = f(n)$. Then for any fixed $n_1, n_2 \in I_\omega$ and for any $n \in \mathbb{Z}$, one has

$$f(n) \leq f(n_1) + \sum_{s=0}^{\omega-1} |f(s + 1) - f(s)| \quad \text{and} \quad f(n) \geq f(n_2) - \sum_{s=0}^{\omega-1} |f(s + 1) - f(s)|.$$

The following result proves that (2.1) has a positive periodic solution $\overline{x}(n)$.

**Theorem 2.1.** Assume that (1.23) and (2.2) hold. Then Eq. (2.1) has at least one $\omega$-periodic bounded solution $\overline{x}(n)$.

**Proof.** In order to embed our problem into the framework of continuation theorem, we define

$$l_1 = \{x = x(n) : x(n) \in \mathbb{R}, n \in \mathbb{Z}\},$$

and let $l_\omega \subseteq l_1$ denotes the subspace of all $\omega$-periodic sequences equipped with the usual norm $\| \cdot \|$, i.e.,

$$\| x \| = \max_{n \in I_\omega} |x(n)| \quad \text{for} \quad x = \{x(n) : n \in \mathbb{Z}\} \in l_\omega.
It is easy to see that $l^\omega$ is a finite-dimensional Banach space with the norm defined above. Define the linear operator $S : l^\omega \to \mathbb{R}$ by

$$S(x) := \frac{1}{\omega} \sum_{n=0}^{\omega-1} x(n), \quad x = \{x(n)\} \in l^\omega,$$

and define the two subspaces $l^\omega_0$ and $l^\omega_c$ from $l^\omega$ by

$$l^\omega_0 := \{ x = x(n) \in l^\omega : S(x) = 0 \},$$

and

$$l^\omega_c := \{ x = x(n) \in l^\omega : x(n) = \beta, \text{ for some } \beta \in \mathbb{R} \text{ and for all } n \in \mathbb{Z} \}.$$

Then from Lemma 2.1 in [44], we see that the two spaces $l^\omega_0$ and $l^\omega_c$ are closed linear subspaces of $l^\omega$, $l^\omega = l^\omega_0 \oplus l^\omega_c$ and $\dim l^\omega_c = 1$.

Eq. (2.1) can be written as

$$\Delta x(n) = -\delta(n)x(n) + P(n) \exp(-q(n)x(n)). \tag{2.3}$$

Define $\mathcal{X} = \mathcal{Y} = l^\omega$, and the operators $L$ and $N$ by

$$L(x(n)) = \Delta x(n),$$

and

$$N(x(n)) = -\delta(n)x(n) + P(n) \exp(-q(n)x(n)) \quad \text{for } x \in \mathcal{X} \text{ and } n \in \mathbb{Z}.$$ 

From Lemma 2.1 in [44], we see that $L$ is a bounded linear operator with

$$\text{Ker } L = l^\omega_c, \quad \text{Im } L = l^\omega_0 \quad \text{and} \quad \dim \text{Ker } L = 1 = co \dim (\text{Im } L),$$

which implies that the mapping $L$ is a Fredholm mapping of index zero, and there exist two continuous projections $P(x)$ and $Q(z)$ defined by

$$P(x) = \frac{1}{\omega} \sum_{n=0}^{\omega-1} x(n), \quad x \in \mathcal{X}, \quad Q(z) = \frac{1}{\omega} \sum_{n=0}^{\omega-1} z(n), \quad n \in \mathbb{Z},$$

such that

$$\text{Im } P = \text{Ker } L \quad \text{and} \quad \text{Ker } Q = \text{Im } L = \text{Im}(I - Q).$$

Furthermore, the generalized inverse (of $L$) $K_P : \text{Im } L \to \text{Ker } L \cap \text{Dom } L$ exists and is given by

$$K_P(z) = \sum_{s=0}^{\omega-1} z(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s)z(s).$$

Then $QN : \mathcal{X} \to \mathcal{Y}$ and $K_P(I - Q)N : \mathcal{X} \to \mathcal{X}$, read

$$QN(x) = \frac{1}{\omega} \sum_{s=0}^{\omega-1} [ -\delta(s)x(s) + P(s) \exp(-q(s)x(s))] \cdot$$
\[ K_P(I - Q)N(x) = \sum_{s=0}^{n-1} \left[ -\delta(s)x(s) + P(s)\exp(-q(s)x(s)) \right] - \frac{1}{\omega} \sum_{n=0}^{\omega-1} \sum_{s=0}^{n-1} \left[ -\delta(s)x(s) + P(s)\exp(-q(s)x(s)) \right] - \left( \frac{n}{\omega} - \frac{1}{2} \right) \sum_{n=0}^{\omega-1} \left[ -\delta(n)x(n) + P(n)\exp(-q(n)x(n)) \right]. \]

Obviously, \(QN\) and \(K_P(I - Q)N\) are continuous with respect to \(s\) and they map bounded continuous functions to bounded continuous functions. Since \(X\) is a finite-dimensional Banach space, using the Ascoli–Arzela Theorem, we see that \(QN(\Omega)\) and \(K_P(I - Q)N(\Omega)\) are relatively compact for any open bounded set \(\Omega \subset X\). Moreover, \(QN(\Omega)\) is bounded. Thus, \(N\) is \(L\)-compact on \(\Omega\) for any open bounded set \(\Omega \subset X\). Now we reach the position to search for an appropriate open, bounded subset \(\Omega\) for the application of Lemma 2.1. Corresponding to the operator equation \(Lx = \lambda Nx, \lambda \in (0, 1)\), we have

\[ x(n + 1) - x(n) = \lambda [ -\delta(n)x(n) + P(n)\exp(-q(n)x(n))], \]  

(2.4)

Suppose that \(x = x(n) \in X\) is an arbitrary solution of (2.4) for a certain \(\lambda \in (0, 1)\). Summing on both sides of (2.4) from 0 to \(\omega - 1\), we have

\[ 0 = \sum_{n=0}^{\omega-1} (x(n + 1) - x(n)) = \sum_{n=0}^{\omega-1} \lambda [ -\delta(n)x(n) + P(n)\exp(-q(n)x(n))], \]  

(2.5)

that is,

\[ \sum_{n=0}^{\omega-1} \delta(n)x(n) = \sum_{n=0}^{\omega-1} P(n)\exp(-q(n)x(n)) \leq \omega \overline{P}. \]  

(2.6)

Since \(x \in X\), there exist \(\zeta, \eta \in I_\omega\) such that

\[ x(\zeta) = \min_{n \in I_\omega} x(n) \quad \text{and} \quad x(\eta) = \max_{n \in I_\omega} x(n). \]  

(2.7)

From (2.6) and (2.7), we see that

\[ \omega \overline{\delta} x(\zeta) \leq \sum_{n=0}^{\omega-1} P(n)\exp(-q(n)x(n)) \leq \omega \overline{P}, \]  

which implies that

\[ x(\zeta) \leq \overline{P}/\overline{\delta}. \]  

(2.8)

Also from (2.4) and (2.6), we find that

\[ \sum_{n=0}^{\omega-1} |x(n + 1) - x(n)| = \lambda \sum_{n=0}^{\omega-1} | -\delta(n)x(n) + P(n)\exp(-q(n)x(n))| \]  

\[ \leq \sum_{n=0}^{\omega-1} \delta(n)x(n) + \sum_{n=0}^{\omega-1} P(n) = 2\omega \overline{P}, \]  

that is,

\[ \sum_{n=0}^{\omega-1} |x(n + 1) - x(n)| \leq 2\omega \overline{P}. \]  

(2.9)
Hence from Lemma 2.2, (2.8) and (2.9), we get
\[
x(n) \leq x(\zeta) + \sum_{n=0}^{\omega-1} |x(n+1) - x(n)| \leq \overline{\theta} + 2\omega \overline{P} := M_1.
\] (2.10)

Also, from (2.6), we obtain
\[
\omega \overline{\theta} x(\eta) \geq \sum_{n=0}^{\omega-1} \delta(n) x(n) = \sum_{n=0}^{\omega-1} P(n) \exp(-q(n) x(\eta)) \geq \sum_{n=0}^{\omega-1} P(n) \exp(-q(n) x(\eta)) \geq \omega \overline{P} \exp(-q^* x(\eta)),
\]
so that by (2.10), we have
\[
\overline{\theta} x(\eta) \geq \overline{P} \exp(-q^* M_1),
\]
and this leads to
\[
x(\eta) \geq \overline{P} \exp(-q^* M_1).
\] (2.11)

Hence from Lemma 2.2, (2.9) and (2.11), we have
\[
x(n) \geq x(\eta) - \sum_{n=0}^{\omega-1} |x(n+1) - x(n)| \geq \overline{P} \exp(-q^* M_1) - 2\omega \overline{P} := M_2.
\] (2.12)

It is clear that the solution of $Lx = \lambda Nx$ for $\lambda \in (0, 1)$ is bounded and the lower and the upper bound $M_1$ and $M_2$ are independent of $\lambda$. Then $\|x\| = \max_{n \in I_0} |x(n)| \leq \max\{|M_1|, |M_2|\} := B_1$, and $B_1$ is independent of the choice of $\lambda$. Take $M = B_1 + M_3$, where $M_3$ is chosen sufficiently large such that $u^* < M_3$, where $u^*$ is the unique solution of the equation $-\overline{\theta} x + \overline{P} e^{-q^* x} = 0$. Then $\|x\| < B$, and we define
\[
\Omega := \{x \in \mathbb{X} : \|x\| < M\}.
\]

It is clear that $\Omega$ verifies requirement (a) of Lemma 2.1 and (2.10) and (2.12) imply that there is no $\lambda \in (0, 1)$ and $x \in \partial \Omega$ such that $Lx = \lambda Nx$. We remark that
\[
\text{Ker } L = \{x \in \text{Dom } L : x \in \mathbb{R}\}, \quad \partial \Omega = \{x \in \mathbb{X} : x = -M \text{ or } x = M\},
\]
\[
\text{Ker } L \cap \overline{\Omega} = \{x \in \mathbb{R} : M_1 \leq x \leq M_2\}.
\]

For $x \in \partial \Omega \cap \text{Ker } L = \partial \Omega \cap \mathbb{R}$, then $x$ is a constant with $\|x\| = M$. Thus
\[
QN x = \frac{1}{\omega} \sum_{n=0}^{\omega-1} \left[ -\delta(n) x(n) + P(n) \exp(-q(n) x(n - \omega)) \right] = -\overline{\theta} x + \overline{P} e^{-q x}.
\]

This implies that
\[
QN(\pm M) = -\overline{\theta}(\pm M) + \overline{P} e^{q(\pm M)} \neq 0,
\]
which implies that $QN x \neq 0$ for $x \in \partial \Omega \cap \text{Ker } L$. Furthermore, it is easy to see that
\[
\text{deg} \{J \overline{QN} z, \Omega \cap \text{Ker } L, 0\} = \text{deg} \{J \overline{QN} z, \Omega \cap \mathbb{R}, 0\} = \text{sign}(Q'(x)) = -1 \neq 0,
\]
where the degree is the Brouwer degree and the isomorphism $J$ can be chosen to be the identity mapping, since $\text{Im } P = \text{Ker } L$. Thus, we have proved that $\Omega$ verifies all the requirements of Lemma 2.1. Hence (2.3), i.e., (2.1) has at least one $\omega$-periodic bounded solution $\overline{x}(n)$ in $\overline{\Omega}$. The proof is complete. □
**Theorem 2.2.** Assume that \( (1.21) \) holds. Then Eq. \((1.21)\) has at least one \( \omega \)-periodic solution \( x(n) \) which satisfies \((2.13)\).

**3. Permanence, oscillation and attractivity**

In Section 2, we proved that Eq. \((1.21)\) has a positive bounded \( \omega \)-periodic solution. In this section, first we prove that all the solutions of Eq. \((1.21)\) with nonnegative initial conditions are permanent, the permanence is one of the most important questions from a biological point of view, which guarantees the long term survival of the mature cells. Second, we prove that, every positive solution of \((1.21)\) which does not oscillate about \( x(n) \) converges to \( x(n) \). Third, we establish sufficient conditions which guarantee that all positive solutions of \((1.21)\) oscillate about \( x(n) \). This will be followed by estimation of the lower and upper bounds for positive solutions of \((1.21)\) which oscillate about \( x(n) \). Finally, we establish sufficient conditions for \( x(n) \) to be a global attractor of all other positive solutions of \((1.21)\). From applications point of view such convergence is very significant which implies the absence of any dynamical diseases.

Now, we show that if \((1.23)\) holds, then the corresponding solution \( x(n) \) of \((1.21)\) is positive and permanent.

**Theorem 3.1.** Assume that \((2.2)\) holds. Then each solution \( x(n) \) of \((1.21)\) is positive and permanent. Furthermore

\[
\lim_{n \to \infty} \sup_{n} x(n) \leq \frac{P^*}{\delta_x}.
\]  

**Proof.** From \((1.21)\) and \((1.23)\), we see that

\[
x(1) = (1 - \delta(0))x(0) + P(0)e^{-q(0)x(-\omega)} \geq P(0)e^{-q(0)x(-\omega)} > 0,
\]

which proves that \( x(n) > 0 \) \( (n = 1, 2, \ldots) \) by induction. Next, we prove that the solutions are permanent. Assume that \( M \geq \max\{x(-\omega), x(-\omega + 1), x(-\omega + 2), \ldots, x(0)\} \), \( P^* / \delta_x \). Then from \((1.21)\), we see that

\[
x(1) \leq (1 - \delta_x)x(0) + P^* \leq (1 - \delta_x)M + \delta_x M = M.
\]

It follows by induction that \( x(n) \leq M \) for \( n = 1, 2, \ldots \). On the other hand from \((1.21)\), we see that

\[
x(n + 1) = (1 - \delta(n))x(n) + P(n)e^{-q(n)x(n-\omega)} \geq (1 - \delta(n))x(n) + P(n)e^{-q(n)M} \geq P_xe^{-q^*M}.
\]

Then, we have

\[
P_xe^{-q^*M} \leq x(n) \leq M, \quad n \geq 1,
\]

i.e., every positive solution of Eq. \((1.21)\) is permanent. From \((1.21)\), we can easily see that

\[
x(n + 1) \leq (1 - \delta_x)x(n) + P^* \quad \text{for } n = 0, 1, 2, \ldots.
\]

Define a sequence \( \{y(n)\} \), by

\[
y(n + 1) = (1 - \delta_x)y(n) + P^*, \quad y(0) = x(0).
\]

Clearly, \( x(n) \leq y(n) = (1 - \delta_x)^n x(0) + (P^*/\delta_x)[1 - (1 - \delta_x)^n]. \) Letting \( n \to \infty \) yields the desired result. The proof is complete. \( \square \)
The following theorem proves that every nonoscillatory solution of (1.21) converges to \( \bar{x}(n) \).

**Theorem 3.2.** Assume that (2.2) holds. Then every nonoscillatory solution \( x(n) \) of (1.21) satisfies

\[
\lim_{n \to \infty} [x(n) - \bar{x}(n)] = 0. \tag{3.2}
\]

**Proof.** Let \( x(n) \) be a nonoscillatory solution of (1.21) about \( \bar{x}(n) \). Then there exists a sufficiently large integer \( n_1 > 0 \) such that \( x(n) > \bar{x}(n) \) or \( x(n) < \bar{x}(n) \) for \( n \geq n_1 \). Assume without loss of generality that \( x(n) > \bar{x}(n) \) for \( n \geq n_1 \). The proof when \( x(n) < \bar{x}(n) \) is similar and will be omitted since \( ug(u) > 0 \) for \( u \neq 0 \), where \( g(u) \) is defined below. Set

\[
z(n) = x(n) - \bar{x}(n). \tag{3.3}
\]

Then \( z(n) > 0 \) and satisfies the difference equation

\[
\Delta z_{n+1} + \delta(n)z_n + P(n)e^{-q(n)\bar{x}(n)}g(z(n - \omega)) = 0, \tag{3.4}
\]

where \( g(u) := (1 - e^{-q(n)z(u)}) \). To prove that (3.2) holds it is sufficient to prove that \( \lim_{n \to \infty} z(n) = 0 \). It is clear that \( g(0) = 0 \) and \( ug(u) > 0 \) for \( u \neq 0 \). Since \( z(n) \) is positive, we have from (3.4) that

\[
z(n+1) \leq (1 - \delta(n))z(n) < z(n) \quad \text{for} \quad n \geq n_1. \tag{3.5}
\]

Hence, \( z(n) \) is decreasing and there exists a nonnegative real number \( z \geq 0 \) such that \( \lim_{n \to \infty} z(n) = z \). If \( z > 0 \), then there exists a positive integer \( n_2 > n_1 \) such that \( z/2 \leq z(n) \leq 3z/2 \) for \( n > n_2 \). This implies from (3.5) that

\[
z(n+1) - z(n) \leq -\frac{\gamma}{2}\delta(n) \leq -\frac{\gamma}{2}\delta \quad \text{for} \quad n > n_2. \tag{3.6}
\]

Thus summing up the last inequality from \( n_2 \) to \( n - 1 \), we obtain

\[
z(n) \leq z(n_2) - \frac{\gamma}{2}\delta(n - n_2) \to -\infty \quad \text{as} \quad n \to \infty.
\]

This contradicts the fact that \( z(n) \) is positive. Then \( z = 0 \), i.e., \( \lim_{n \to \infty} z(n) = 0 \) and hence (3.2) holds. The proof is complete. \( \square \)

Next, we establish some sufficient conditions for oscillation of Eq. (1.21) about the positive periodic solution \( \bar{x}(n) \). We first give the following theorem which states that the oscillation of (1.21) about the periodic solution \( \bar{x}(n) \) is equivalent to the oscillation of a first order delay difference equation about zero.

**Theorem 3.3.** Assume that (2.2) holds. Then every solution \( x(n) \) of Eq. (1.21) oscillates about \( \bar{x}(n) \) if every solution of the linear equation

\[
y(n+1) - y(n) + (1 - \varepsilon)P(n)q(n)e^{-q(n)\bar{x}(n)} \left[ \prod_{i=n-\omega}^{n} (1 - \delta(i)) \right]^{-1} y(n - \omega) = 0 \tag{3.7}
\]

oscillates (where \( \varepsilon > 0 \) is sufficiently small).

**Proof.** Assume that (1.21) has a nonoscillatory solution \( x(n) \). Without loss of generality, we assume that \( x(n) > \bar{x}(n) \) for \( n \geq n_1 \). The proof when \( x(n) < \bar{x}(n) \) is similar and will be omitted since \( uf(u) > 0 \) for \( u \neq 0 \), where \( f(u) \) is defined below. Let \( z(n) \) be as defined by (3.3). From the transformation (3.3) it is clear that \( x(n) \) oscillates about \( \bar{x}(n) \) if and only if \( z(n) \) oscillates about zero. The substitution (3.3) transforms Eq. (1.21) to the equation

\[
z(n+1) - z(n) + \delta(n)z(n) + P(n)q(n)e^{-q(n)\bar{x}(n)} f(z(n - \omega)) = 0, \tag{3.8}
\]

where

\[
f(u) := \frac{1 - e^{-q(n)u}}{q(n)}.
\]
Note that \( f(0) = 0 \), and
\[
uf(u) > 0 \text{ for } u \neq 0 \quad \text{and} \quad \lim_{u \to 0} \frac{f(u)}{u} = 1. \tag{3.9}
\]

From (3.9) it follows that for any given arbitrarily small \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( 0 < u < \delta \), \( f(u) \geq (1 - \varepsilon)u \) (for all \( -\delta < u < 0 \), \( f(u) \leq (1 - \varepsilon)u \)). Since in view of Theorem 3.2, as \( z(n) \to 0 \), for sufficiently large \( n \), we can use this estimate in (3.8) to conclude that \( z(n) \) is a positive solution of the differential inequality
\[
z(n + 1) - z(n) + \delta(n)z(n) + (1 - \varepsilon)P(n)q(n)e^{-q(n)\pi(n)}z(n - \omega) \leq 0. \tag{3.10}
\]

Setting \( z(n) = y(n) \prod_{i=n-\omega}^{n-1} (1 - \delta(i)) \), then \( y(n) > 0 \) and satisfies the difference inequality
\[
y(n + 1) - y(n) + (1 - \varepsilon)P(n)q(n)e^{-q(n)\pi(n)} \left[ \prod_{i=n-\omega}^{n-1} (1 - \delta(i)) \right]^{-1} y(n - \omega) \leq 0.
\]
But, then by the comparison oscillation results due to Ladas and Qian [22] the delay difference equation (3.7) have an eventually positive solution also, which contradicts the assumption that every solution of Eq. (3.7) oscillates. Thus every positive solution of (1.21) oscillates about \( \overline{x}(n) \). The proof is complete. \( \square \)

For the oscillation of the first order delay difference equation (3.7), several known criteria can be employed. For example, the results given in Erbe and Zhang [5] and Ladas et al. [21] lead to the following:

**Claim 1.** Assume that (2.2) holds. Then
\[
\lim_{n \to \infty} \sup \sum_{i=n-\omega}^{n-1} \Theta^\varepsilon(i) > 1 \tag{3.11}
\]
or
\[
\lim_{n \to \infty} \inf \sum_{i=n-\omega}^{n-1} \Theta^\varepsilon(i) \geq \left( \frac{\omega}{\omega + 1} \right)^{\omega+1}, \tag{3.12}
\]
where \( \Theta^\varepsilon(n) = (1 - \varepsilon)P(n)\Theta(n)e^{-\Theta(n)\pi(n)} \left[ \prod_{i=n-\omega}^{n-1} (1 - \delta(i)) \right]^{-1} \), implies that every solution of Eq. (3.7) oscillates.

Clearly, if the strict inequalities in (3.11) and (3.12) hold for \( \varepsilon = 0 \), then the same must be true for all sufficiently small \( \varepsilon > 0 \) also. Thus, we can restate Claim 1 as follows:

**Claim 2.** Assume that (2.2) holds. Then
\[
\lim_{n \to \infty} \sup \sum_{i=n-\omega}^{n-1} \Theta^0(i) > 1 \tag{3.13}
\]
or
\[
\lim_{n \to \infty} \inf \sum_{i=n-\omega}^{n-1} \Theta^0(i) \geq \left( \frac{\omega}{\omega + 1} \right)^{\omega+1}, \tag{3.14}
\]
where \( \Theta^0(n) = P(n)\Theta(n)e^{-\Theta(n)\pi(n)} \left[ \prod_{i=n-\omega}^{n-1} (1 - \delta(i)) \right]^{-1} \), implies that every solution of Eq. (3.7) oscillates.

From Theorem 3.2 and Claim 2, the following oscillation result of Eq. (1.21) is immediate.

**Theorem 3.4.** Assume that (2.2) holds. Then (3.13) or (3.14) implies that every positive solution of (1.21) oscillates about the positive periodic solution \( \overline{x}(n) \).
For the oscillation of Eq. (3.7), it is clear that there is a gap between the conditions (3.11) and (3.12), when the limit
\[
\lim_{n \to \infty} \sum_{i=n-\omega}^{n-1} \Theta^i(i)
\]
does not exist. To fill this gap partially, we employ some known results from the literature. For example, the criteria by Stavroulakis [34], when applied to the first order delay difference equation (3.7) guarantee that every solution of (1.21) oscillates about \( \bar{x}(n) \) provided
\[
\lambda := \liminf_{n \to \infty} \sum_{i=n-\omega}^{n-1} \Theta^i(i) \leq \frac{\omega^{\alpha+1}}{(\omega+1)^{\alpha+1}} > 0
\]
and
\[
\limsup_{n \to \infty} \sum_{i=n-\omega}^{n-1} \Theta^0(i) > 1 - \frac{\alpha^2}{4}.
\] (3.15)
We also note that in view of the criteria of [1], condition (3.15) can be improved by
\[
\limsup_{n \to \infty} \sum_{i=n-\omega}^{n-1} \Theta^0(i) > 1 - \frac{1 - \sqrt{1 - 2\alpha - \alpha^2}}{2}.
\] (3.16)
We remark that condition (3.16) can be improved further by employing the recent results presented in Tang and Yu [36] and Zhou and Zhang [46]. Also, we remark that some oscillation criteria for Eq. (3.7) has been established when neither (3.11) nor (3.12) is satisfied. In this direction by employing the results in Tang [35], we see that if
\[
\sum_{i=n-\omega}^{n-1} \Theta^i(i) \geq \frac{\omega^{\alpha+1}}{(\omega+1)^{\alpha+1}},
\] (3.17)
and
\[
\sum_{n=0}^{\infty} \Theta^0(n) \left[ \sum_{i=n-\omega}^{n-1} \Theta^0(i) - \left( \frac{\omega}{\omega+1} \right)^{\omega+1} \right] = \infty,
\] (3.18)
then every solution of (1.21) oscillates about \( \bar{x}(n) \), whereas the result of Tang and Yu [37] gives an improvement over (3.18) to
\[
\sum_{n=0}^{\infty} \Theta^0(n) \left[ \left( \sum_{i=n-\omega}^{n-1} \Theta^0(i) \right)^{1/(1+\omega)} \left( 1 + \omega \right) - \omega \right] = \infty.
\]
Finally, we remark that the last condition can be improved further by employing the recent results presented in Jiang and Tang [13] and Tang and Yu [38].

To prove the main global attractivity results for Eq. (1.21), we need to estimate the lower and the upper bounds of the oscillatory solutions about \( \bar{x}(n) \). To do this we define the two functions \( h_1(x) \) and \( h_2(x) \) by
\[
h_1(x) := -\delta^* x + P^* e^{-q^* x} \quad \text{and} \quad h_2(x) := -\delta^* x + P^* e^{-q^* x}.
\]
Observe that \( h_1(0) = P^* > 0 \) and \( h_1(\infty) = -\infty \), so that there exists \( x_1 > 0 \) such that \( h_1(x_1) = 0 \). Now since \( h'_1(x) = -q^* P^* e^{-q^* x} - \delta^* < 0 \) for all \( x > 0 \) then \( h'_1(x_1) < 0 \), from which it follows that \( h_1(x) = 0 \) has exactly one solution \( x_1 \). Similarly, we can prove that \( h_2(x) = 0 \) has exactly one solution \( x_2 > 0 \). Furthermore, one can easily see that \( x_2 > x_1 \) since \( P^* > P^*_s \).
**Theorem 3.5.** Assume that (2.2) holds, and let $x(n)$ be a positive solution of (1.21) which oscillates about $\bar{x}(n)$. Then, there exists $n_0 > 0$ such that for all $n \geq n_0$

\[
X_1 := x_1 (1 - \delta^*)^{\alpha} \leq x(n) \leq x_2 + P^* \omega := X_2, \tag{3.19}
\]

where $x_1$ and $x_2$ are the positive roots of the functions $h_1(x)$ and $h_2(x)$, respectively.

**Proof.** First, we shall show the upper bound in (3.19). The sequence $x(n)$ is oscillatory about the positive periodic solution $\bar{x}(n)$ in the sense that: there exists a sequence of positive integers $\{n_l\}$ for $l = 1, 2, \ldots$ such that $\omega \leq n_1 < n_2 < \cdots < n_l < \cdots$ with $\lim_{l \to \infty} n_l = \infty$, and

\[
x(n_l) < \bar{x}(n_l) \quad \text{and} \quad x(n_l) > \bar{x}(n_l) \quad \text{for} \quad l = 1, 2, \ldots,
\]

and for each $l = 1, 2, \ldots$, some of the terms $x_j$ with $n_l < j \leq n_{l+1}$ are greater than $\bar{x}(j)$ and some are less than $\bar{x}(j)$. Our strategy is to show that the upper bound holds in each interval $[n_l, n_{l+1}]$. For each $l = 1, 2, \ldots$, let $\zeta_l$ be the integer in the interval $[n_l, n_{l+1}]$ such that

\[
x(\zeta_{l+1}) = \max\{x(j) : n_l < j \leq n_{l+1}\}.
\]

Then for each $l = 1, 2, \ldots$,

\[
x(\zeta_{l+1}) > \bar{x}(\zeta_l), \quad x(\zeta_{l+1}) \geq x(\zeta_l) \quad \text{which implies that} \quad \Delta x(\zeta_l) \geq 0.
\]

To show the upper bound on (3.19), it suffices to show that

\[
x(\zeta_l) \leq x_2 + P^* \omega = X_2. \tag{3.20}
\]

We assume that $x(\zeta_l) > x_2$, otherwise there is nothing to prove. Now, since $\Delta x(\zeta_l) \geq 0$, it follows from (1.21) that

\[0 \leq \Delta x(\zeta_l) = -\delta(\zeta_l)x(\zeta_l) + P(\zeta_l)e^{-q(\zeta_l)x(\zeta_l-\omega)} \leq -\delta x_2 + P^* e^{-q x_2},\]

and hence

\[
\delta x_2 + P^* e^{-q x_2} \geq -\delta x_2 + P^* e^{-q x_2} = 0 \quad \text{which implies that} \quad x(\zeta_l - \omega) < x_2.
\]

Now, since $x(\zeta_l) > x_2$ and $x(\zeta_l - \omega) < x_2$, there exists an integer $\bar{\zeta}_l$ in the interval $[\zeta_l - \omega, \zeta_l]$, such that $x(\bar{\zeta}_l) \leq x_2$ and $x(j) > x_2$ for $j = \bar{\zeta}_l + 1, \ldots, \zeta_l$. Summing (1.21) from $\bar{\zeta}_l$ to $\zeta_l - 1$, we get

\[
\sum_{\bar{\zeta}_l}^{\zeta_l-1} \Delta x(n) = \sum_{\bar{\zeta}_l}^{\zeta_l-1} \{-\delta(\zeta_l)x(\zeta_l) + P(\zeta_l)e^{-q(\zeta_l)x(\zeta_l-\omega)}\} \leq P^* \omega,
\]

which immediately gives (3.20). Hence, there exists an $n_1$ such that $x(n) \leq X_2$ for all $n \geq n_1$. Now, we show the lower bound in (3.19) for $n \geq n_1 + \omega$. For this, let $\mu_l$ be the integer in the interval $[n_l, n_{l+1}]$ such that

\[
x(\mu_{l+1}) = \min\{x(j) : n_l < j \leq n_{l+1}\}.
\]

Then for each $l = 1, 2, \ldots$,

\[
x(\mu_{l+1}) < \bar{x}(\mu_l), \quad \bar{x}(\mu_l) \leq x(\mu_l) \quad \text{which implies that} \quad \Delta x(\mu_l) \leq 0.
\]

We can assume that $x(\mu_l) < x_1$, otherwise there is nothing to prove. Then, it suffices to show that

\[P_1 = x_1 (1 - \delta^*)^{\alpha} \leq x(\mu_l). \tag{3.21}
\]

Since $\Delta x(\mu_l) \leq 0$, we have

\[0 \geq \Delta x(\mu_l) = -\delta(\mu_l)x(\mu_l) + P(\mu_l)e^{-q(\mu_l)x(\mu_l-\omega)} \geq -\delta x_1 + P_1 e^{-q x_1},\]
which implies that
\[-\delta^* x_1 + P_x e^{-q^*(t)} < 0 = -\delta^* x_1 + P_x e^{-q^* x_1},\]
and this leads to \(x(\mu_l - \omega) > x_1\). Therefore, there exists a \(\bar{\mu}_l \in [\mu_l - \omega, \mu_l]\) such that \(x(\bar{\mu}_l) \geq x_1\) and \(x(j) < x_1\) for \(j = \bar{\mu}_l + 1, \ldots, \mu_l\). From (1.21), we see that
\[x(n + 1) - x(n) \geq -\delta(n)x(n) \geq -\delta^* x(n),\]
and this implies that
\[\frac{x(n + 1)}{x(n)} \geq (1 - \delta^*).\]
Multiplying the last inequality from \(\bar{\mu}_l\) to \(\mu_l - 1\), we have
\[\frac{x(\mu_l)}{x(\bar{\mu}_l)} \geq (1 - \delta^* \mu_l + \bar{\mu}_l - 1 \geq (1 - \delta^* \omega),\]
which immediately leads to (3.21). The proof is complete. □

The following theorem provides a sufficient condition for the global attractivity of \(\bar{x}(n)\).

**Theorem 3.6.** Assume that (2.2) holds and let \(x(n)\) be a positive solution of (1.21). If
\[\lim_{n \to \infty} \sup_{n, n+\omega} \sum_{i=n}^{n+\omega} P(n)q(n)e^{-q(n)X_1} \left[ \prod_{i=n-\omega}^{n} (1 - \delta(i)) \right]^{-1} < 1, \tag{3.22}\]
where \(X_1\), as in Theorem 3.5, then (3.2) holds.

**Proof.** In Theorem 3.2, we proved that every nonoscillatory solution \(x(n)\), of (1.21) converges to \(\bar{x}(n)\). So to complete the global attractivity result it remains to prove \(\lim_{n \to \infty} \{x(n) - \bar{x}(n)\} = 0\) holds for the positive solutions of (1.21) which oscillate about \(\bar{x}(n)\). Let \(x(n)\) be an oscillatory solution of Eq. (1.21) about \(\bar{x}(n)\). From the transformation (3.3) it is clear that \(z(n)\) satisfies the delay difference equation
\[z(n + 1) - z(n) + \delta(n)z(n) + G(n, z(n - \omega)) = 0, \tag{3.23}\]
where
\[G(n, u) := P(n) \left( e^{-q(n)\bar{x}(n)} - e^{-q(n)(u + \bar{x}(n))} \right).\]
So to prove that \(\lim_{n \to \infty} \{x(n) - \bar{x}(n)\} = 0\), it suffices to prove that \(\lim_{n \to \infty} z(n) = 0\). Since \(\bar{x}(n)\) is a positive \(\omega\)-periodic solution of Eq. (1.21), Eq. (3.23) can be rewritten as
\[z(n + 1) - z(n) + \delta(n)z(n) + G(n, z(n - \omega)) - G(n, 0) = 0. \tag{3.24}\]
By the mean value theorem Eq. (3.24) can be written as
\[z(n + 1) - z(n) + \delta(n)z(n) + F(n)z(n - \omega) = 0, \tag{3.25}\]
where
\[F(n) := \frac{\partial G(n, u)}{\partial u} \bigg|_{u = \bar{x}_n} = q(n)P(n)e^{-q(n)(\bar{x}_n + \bar{x}(n))} = q(n)e^{-q(n)\eta_n},\]
and \(\eta_n\) lies between \(\bar{x}(n)\) and \(x(n - \omega)\). By setting \(z(n) = y(n)\prod_{i=0}^{n-1} (1 - \delta(i))\) in (3.25), we find that
\[y(n + 1) - y(n) + g(n)y(n - \omega) = 0, \tag{3.26}\]
where
\[ g(n) := P(n)q(n)e^{-q(n)\eta_n} \left[ \prod_{i=n-\omega}^{n} (1 - \delta(i)) \right]^{-1}. \]

Since the solution is bounded above by \( X_2 \), we see that
\[ g(n) = P(n)q(n)e^{-q(n)\eta_n} \prod_{i=n-\omega}^{n} (1 - \delta(i)) \geq P_* q_* e^{-q_* X_2} > 0. \]

Therefore, we have
\[ \sum_{n=0}^{\infty} g(n) = \infty. \]

Also, since the solution is bounded below, then we have
\[ g(n) \leq P(n)q(n)e^{-q(n)X_1} \left[ \prod_{i=n-\omega}^{n} (1 - \delta(i)) \right]^{-1}. \]

Thus, in view of (3.22), we find
\[ \lim_{n \to \infty} \sup_{n+\omega} \sum_{i=n}^{n+\omega} g(n) \leq \lim_{n \to \infty} \sup_{n+\omega} \sum_{i=n}^{n+\omega} P(n)q(n)e^{-q(n)X_1} \left[ \prod_{i=n-\omega}^{n} (1 - \delta(i)) \right]^{-1} < 1. \]

But, now by a known result by Ladas et al. [23] and the last inequality, we see that the zero solution of Eq. (3.26) is global attractor, i.e., \( \lim_{n \to \infty} \bar{z}(n) = 0 \), which implies that \( \lim_{n \to \infty} [x(n) - \bar{x}(n)] = 0 \). The proof is complete. \( \square \)

From Theorem 3.6, it is clear that the global attractivity of \( \bar{x}(n) \) is equivalent to the global attractivity of zero solution of the linear difference equation (3.26). By employing the results by Ladas et al. [23], we showed that if (3.22) holds then (3.2) is satisfied. Now, we apply different results which improve the condition (3.22) based on the improvement of the global attractivity results of Ladas et al. [23] for difference equation (3.26).

For example, the criterion in Győri and Pituk [11], when applied to the first order delay difference equation (3.26) guarantees that every solution of (1.21) converges to \( \bar{x}(n) \) provided that
\[ \lim_{n \to \infty} \sup_{n+\omega} \sum_{i=n}^{n+\omega} P(n)q(n)e^{-q(n)X_1} \left[ \prod_{i=n-\omega}^{n} (1 - \delta(i)) \right]^{-1} < 1, \]  
(3.27)

whereas the results by Erbe et al. [4] gives an improvement over (3.27) to
\[ \lim_{n \to \infty} \sup_{n+\omega} \sum_{i=n}^{n+\omega} P(n)q(n)e^{-q(n)X_1} \left[ \prod_{i=n-\omega}^{n} (1 - \delta(i)) \right]^{-1} < \frac{3}{2} + \frac{1}{2(\omega + 1)}. \]  
(3.28)

Also by employing the results in Kovacsölgy [17], we see that if
\[ \lim_{n \to \infty} \sup_{n+\omega} \sum_{i=n}^{n+\omega} P(n)q(n)e^{-q(n)X_1} \left[ \prod_{i=n-2\omega}^{n} (1 - \delta(i)) \right]^{-1} < \frac{7}{4}, \]  
(3.29)

then every solution of (1.21) converges to \( \bar{x}(n) \) which improve (3.28), whereas the result by Yu and Cheng [42] gives an improvement over (3.29) to
\[ \lim_{n \to \infty} \sup_{n+2\omega} \sum_{i=n}^{n+2\omega} P(n)q(n)e^{-q(n)X_1} \left[ \prod_{i=n-2\omega}^{n} (1 - \delta(i)) \right]^{-1} < 2. \]  
(3.30)
**Remark 3.1.** When $P(n) = P$, $q(n) = q$ and $\delta(n) = \delta$, then Eq. (1.21) becomes (1.8). In this case Eq. (1.8) has a positive equilibrium point $\bar{x}$ satisfies the equation

$$\delta \bar{x} = P e^{-q \bar{x}},$$

(3.31)

and $x_1 = x_2 = \bar{x}$.

**Remark 3.2.** Note that when $P(n) = P$, $q(n) = q$ and $\delta(n) = \delta$, the condition (3.13) of Theorem 3.3, becomes

$$P q e^{-q \bar{x}} > (1 - \delta) e^{-1} \frac{e^{q \bar{x}}}{e^{(\omega + 1) \bar{x} + 1}},$$

(3.32)

which is the same as the oscillation condition that has been established by Kubiaczyk and Saker [18] and Li and Cheng [25] for Eq. (1.8). Thus Theorem 3.3 extends the well-known oscillation criterion of Kubiaczyk and Saker [18] and Li and Cheng [25] for Eq. (1.8).

The following global attractivity result for Eq. (1.8) is an immediate consequence of Theorem 3.6.

**Theorem 3.7.** Assume that $P$, $q$ and $\delta$ are positive constants and $\delta < 1$. Then each one of the following conditions is sufficient to ensure that all solutions of Eq. (1.8) converge to $\bar{x}$:

(a) $P q e^{-q \bar{x}}(1 - \delta) e^{-1} (\omega + 1) < 1 - \delta e^{1}$,

(b) $P q e^{-q \bar{x}}(1 - \delta) e^{-1} \omega < 1 - \delta e^{1}$,

(c) $2 P q e^{-q \bar{x}}(1 - \delta) e^{-1} (\omega + 1)^2 < 1 - \delta e^{1} (3 \omega + 4)$,

(d) $8 P q e^{-q \bar{x}}(1 - \delta) e^{-1} \omega < 7(1 - \delta) e^{1}$,

(e) $P q e^{-q \bar{x}}(1 - \delta) e^{-1} (2 \omega + 1) < 2(1 - \delta) e^{1}$.

**Remark 3.3.** (1) From Theorem 3.7, it is clear that if the condition (a) holds, then the condition (1.16) is already satisfied. In fact, the condition (a) can be reduced to

$$P q < e^{q \bar{x}}.$$  

(3.33)

From (3.31), we have $e^{q \bar{x}} = P / \delta \bar{x} = P / \bar{x}$, that along with (3.33) imply that $q \bar{x} < 1$, which is exactly the same as the condition (1.16) that has been established by Karakostas et al. [15] and Ivanov [12], Zhou and Zhang [46], Kubiaczyk and Saker [18] and Li and Cheng [25]. So the condition (a) improve and involve the global attractivity results that has been established in [12,15,18,25,27,46].

(2) Also from Theorem 3.7, when the condition (e) is satisfied, we see that the condition (1.17) holds. In fact the condition (e) can be reduced to

$$P q e^{-q \bar{x}}(1 - \delta)(2 \omega + 1) \leq P q e^{-q \bar{x}}(1 - \delta) e^{1} (2 \omega + 1) < 2(1 - \delta) e^{1},$$

which leads to

$$P q e^{\delta q \bar{x}} \leq \frac{2(1 - \delta) e^{1} e^{q \bar{x}}}{(2 \omega + 1)} = \frac{P}{\delta \bar{x}} \frac{2(1 - \delta) e^{1} (2 \omega + 1)}{\delta \bar{x}} \leq \frac{P}{\delta \bar{x}},$$

and $q \bar{x} \delta < 1$, which is the same as the condition (1.18) that is equivalent to the condition (1.17) given by Meng and Yu [27]. So that the condition (e) improves the condition (1.17) established by Meng and Yan [27].

(3) The conditions (a) and (c) of Theorem 3.7 improve the conditions (1.19) and (1.20) established by El-Morshedy and Liz [3] for global attractivity of Eq. (1.8). In fact our conditions are sharp and do not require any additional constant $\beta$ by solving the equation

$$x = \frac{P}{\delta} \exp \left[ - \left( \frac{P q}{\delta} e^{-q \bar{x}} \right) \right],$$

which is not easy to handle.
One can use the estimation
\[
\lim_{n \to \infty} \sup_{i=n}^{n+\omega} g(n) \leq \lim_{n \to \infty} \sup_{i=n}^{n+\omega} P(n)q(n)e^{-q(n)}X_1 \left[ \prod_{i=n+\omega}^{n} (1 - \delta(i)) \right]^{-1} 
\]
and establish new global attractivity results for Eq. (1.21). Due to the limited space the details are left to the interested reader.

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References