Existence of positive periodic solutions of nonlinear discrete model exhibiting the Allee effect

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Abstract

In this paper we shall consider the discrete nonlinear delay population model with Allee effect

\[ x(n + 1) = x(n) \exp(a(n) + b(n)x^p(n - \omega) - c(n)x^q(n - \omega)), \]

where \( a(n), b(n) \) and \( c(n) \) are positive sequences of period \( \omega \) and \( p \) and \( q \) are positive integers. By using the coincidence degree theorem as well as some priori estimates we will establish a sufficient condition for the existence of positive periodic solution \( \{x_n^*\} \).

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1. Introduction

Most dynamic population evolution models are described by the delay differential equations. Analytically, differential delay equations are difficult to manage and therefore many articles have examined the models such as the difference equations in [4,5,8,9,11]. In practice, one can formulate a discrete model directly from experiments and observations. Sometimes, for numerical purpose one wants to purpose a finite-difference scheme to numerically solve a given differential model, especially when the differential equation cannot be solved explicitly.

The so-called Allee effect refers to a population that has a maximal per capita growth rate at intermediate density. This occurs when the per capita growth rate increases as density increases and decreases after the density passes a certain value. This is certainly not the case in the delayed logistic equation,

\[ N'(t) = rN(t) \left(1 - \frac{N(t - \tau)}{K}\right), \]

where the per capita growth rate is a decreasing function of the density. As known, in nature some species often cooperate among themselves in their search for food or to escape from predators. A number of social species, such as ants, termites, bees, humans, etc., have developed complex cooperative behavior involving division of labor, altruism, etc. Processes such as these provide individuals with a great chance to survive and reproduce as density increases. In sexual populations cooperation among individuals is necessary for mating, nest building, rearing the young, etc. Aggregation and associated cooperative and social characteristics among members of species had been extensively studied in animal population by Allee [1,2]. When the density of population becomes too large, the positive feedback effect of aggregation and cooperation may be dominated by density dependent stabilizing negative feedback effect due to intraspecific competition due to excessive crowding and the ensuing shortage of resources. To study these processes, Gopalsamy and Ladas [7] introduced the following delay Lotka–Volterra type single species population growth model:

\[ x'(t) = x(t)[a + bx(t - \tau) - cx^2(t - \tau)], \tag{1.1} \]

where

\[ a, b, c, \quad \text{and} \quad \tau \in (0, \infty), \quad \text{with} \quad c > b. \tag{1.2} \]

When \( \tau = 0 \) the per capita growth is \( g(x) = a + bx - cx^2 \). Then \( g(0) = b > 0 \) and \( g(x) \) achieves its maximum at \( x = \frac{b}{2c} \), thus exhibiting the Allee effect. When \( b < 0, \) \( g(x) \) is a decreasing function for \( x \in [0, \frac{b}{2c}] \) and thus there is no Allee effect. The following equation is more general than Eq. (1.1):
\[ x'(t) = x(t)[a + bx^p(t - \tau) - cx^q(t - \tau)], \tag{1.3} \]

where

\[ a, b, c, \quad \text{and} \quad \tau \in (0, \infty), \quad \text{with} \quad c > b \quad \text{and} \quad q > p. \tag{1.4} \]

We consider only solutions of (1.1) and (1.3) that correspond to initial functions of the form

\[ x(t) = \phi(t), \quad -\tau \leq t \leq 0, \quad \phi(0) > 0, \quad \phi \in C([-\tau, 0], \mathbb{R}). \tag{1.5} \]

It is clear that in the interval \( 0 \leq t \leq \tau \), Eqs. (1.3) with (1.5) has a solution

\[ x(t) = \phi(0) \exp \left[ \int_0^t \{ a + b\phi^p(s - \tau) - c\phi^q(s - \tau) \} \, ds \right]. \]

Then \( x(t) \) exists and is positive throughout the interval \( 0 \leq t \leq \tau \), and \( x(t) \) is positive for all \( t \geq 0 \) by induction, where the solution is given by the method of steps. In [3], Elabbasy, Saker and Saif considered Eq. (1.3) and as the application of nonlinear delay differential equations with positive and negative coefficients, proved that every nonoscillatory solution tends to the positive steady state and established some sufficient conditions for the oscillation about the positive steady state. Clearly Eq. (1.1) has a unique positive equilibrium

\[ x^* = \frac{1}{2c} \left[ b + \sqrt{b^2 + 4ac} \right]. \]

For the equilibrium point of (1.3) Ladas and Qian [12] proved the following lemma:

**Lemma 1.1.** Assume that (1.4) holds, and set

\[ F(x) = a + bx^p - cx^q. \]

Then there exists a unique positive number \( k \) such that \( F(k) = 0 \). Furthermore,

\[ F(x) > 0, \quad \text{for} \quad 0 < x < k, \quad F(x) < 0 \quad \text{for} \quad k < x < \infty \]

and \( F(x) \) is increasing for

\[ 0 < x < \left( \frac{bp}{cq} \right)^{\frac{1}{p}} \]

and \( F(x) \) is decreasing for

\[ \left( \frac{bp}{cq} \right)^{\frac{1}{p}} < x < \infty. \]
For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points. But unless we can explicitly solve both equations, it is impossible to satisfy this requirement. Most of the time, it is desirable that a difference equation, when derived from a differential equation, preserves the dynamical features of the corresponding continuous time model such as equilibria, oscillation, their local and global stability characteristics and bifurcation behaviors. If such discrete models can be derived from continuous time delay models, then the discrete time models can be used without loss of any functional similarity to the continuous-time models and it will preserve the considered realities; such discrete time models can be called "Dynamically consistent" with the continuous time models.

The variation of the environment plays an important role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In fact, It has been suggested by Nicholson [15] that any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. In view of this it is realistic to assume that the parameters in the models are periodic functions of period $\omega$. Thus, the modification of (1.3) according to the environmental variation is the nonautonomous delay differential equation

$$x'(t) = x(t)[a(t) + b(t)x^p(t - \omega) - c(t)x^q(t - \omega)],$$

where $m$ is a positive integer and

$$a(t), b(t) \text{ and } c(t) \text{ are positive periodic functions of period } \omega. \tag{1.7}$$

With Eq. (1.6) we can assume that there is an initial condition

$$\begin{cases} x(t) = \phi(t) \quad \text{for} \quad -\omega \leq t \leq 0, \\ \phi \in C([-\omega, 0], [0, \infty)) \text{ and } \phi(0) > 0. \end{cases} \tag{1.8}$$

By the method of steps it is clear that the initial value problem (1.6) and (1.8) has a unique positive solution $x(t)$ which exists for all $t \geq 0$. We remark that in recent years periodic population dynamics models have become a very popular subject. In fact, several different periodic models have been studied by several authors, for example we refer the reader to the results in [5,10,13,16–22] and the references cited therein.
There is no unique way of deriving discrete time version of dynamical systems corresponding to continuous time formulations. One of the ways of deriving difference equations modelling from the dynamic of populations with nonoverlapping generations is based on appropriate modifications of models with overlapping generations. In this approach, differential equations with piecewise constant arguments have been useful, see for example [14]. Things to differential equations with piecewise constant arguments, we can go on with the discrete analogy of Eq. (1.6). Let us assume that the average growth rate in (1.6) changes at regular intervals of time, then we can incorporate this aspect in (1.6) and obtain the following modified equation:

\[
\frac{1}{x(t)} \frac{dx(t)}{dt} = a([t]) + b([t])x^p([t - \omega]) - c([t])x^q([t - \omega]),
\]

where \([t]\) denotes the integer part of \(t, t \in (0, \infty)\). Equation of type (1.9) is known as differential equation with piecewise constant argument and this equation occupies a position midway between differential and difference equation. By a solution of (1.9), we mean a function \(x(t)\), which is defined for \(t \in [0, 1)\), and satisfies the properties:

(a) \(x\) is continuous on \([0, \infty)\).

(b) The derivative \(\frac{dx(t)}{dt}\) exists at each point \(t \in [0, \infty)\) with the possible exception of the points \(t \in \{0, 1, 2, \ldots\}\), where left side derivative exists.

(c) Eq. (1.9) is satisfied on each interval \([n, n + 1)\) with \(n = 0, 1, 2, \ldots\).

By integrating Eq. (1.9) on any interval of the form \([n, n + 1), n = 0, 1, 2, \ldots\) we obtain

\[
x(t) = x(n) \exp([a(n) + b(n)x^p(n - \omega) - c(n)x^q(n - \omega)](t - n)).
\]

Letting \(t \to n + 1\), we obtain that

\[
x(n + 1) = x(n) \exp(a(n) + b(n)x^p(n - \omega) - c(n)x^q(n - \omega)), \tag{1.10}
\]

which is a discrete time analogy of (1.9). We note that in the case when the coefficients are constants, the equilibrium point of (1.10) is the same as the equilibrium point of the delay differential equation (1.3). So the derived discrete analogy preserves the equilibria.

By a solution of Eq. (1.10), we mean a sequence \(\{x(n)\}\) which is defined for \(n \geq -\omega\) and which satisfies (1.11) for \(n \geq 0\). Together with (1.10) we consider the initial condition

\[
x(i) = x_i > 0 \quad \text{for} \quad i = -\omega, \ldots, 0. \tag{1.11}
\]

The exponential form of Eq. (1.10) assures that the solution \(\{x(n)\}\) with respect to any initial condition (1.11) remains positive.
We are inspired to study (1.10) from the fact that if the parameters in (1.10) are periodic of some common period and if the time domain is discretised, the resulting sequences in (1.10) may be periodic sequences. It is our belief that while nonlinear delay models with periodic coefficients have wide ranging applications, it is reasonable to investigate the periodic solutions of difference equations with periodic coefficients, though in some cases, discretisations of periodic differential equations may lead to almost periodic difference equations.

In Section 2, we prove that Eq. (1.10) has a periodic positive solution \( \{x_n^*\} \). The approach is based on the coincidence degree theorem due to Gaines and Mawhin [6] and the related continuation theorem, as well as some priori estimates.

2. Existence of positive periodic solutions

In this section, we will assume that the parameters in (1.10) are positive periodic sequences of period \( \omega \), i.e.,

\[
a(n + \omega) = a(n), \quad b(n + \omega) = b(n) \quad \text{and} \quad c(n + \omega) = c(n). \quad (2.1)
\]

A very basic and important ecological problem in the study of dynamics of population in a periodic environment is the global existence of a positive periodic solution, which plays a similar role played by the equilibrium of the autonomous models. Thus, it is reasonable to ask for a condition under which the resulting periodic nonautonomous equation has a positive periodic solution.

For the reader’s convenience, we now recall some basic tools in the frame of Mawhin’s coincidence degree theorem that will be used to prove the existence of periodic solution of (1.10), borrowing notations and terminology from [6].

Let \( \mathbb{X} \) and \( \mathbb{Y} \) be two Banach spaces, let \( L : \text{Dom}L \subset \mathbb{X} \to \mathbb{Y} \) be a linear mapping, and let \( N : \mathbb{X} \to \mathbb{Y} \) be a continuous mapping.

The mapping \( L \) will be called a Fredholm mapping of index zero if the following three conditions hold:

(i) \( \text{Ker} L \) has a finite dimension.
(ii) \( \text{Im} L \) is closed and has a finite codimension.
(iii) \( \text{dim} \text{Ker} L = \text{codim} \text{Im} L < \infty \).

If \( L \) is a Fredholm mapping of index zero and there exist continuous projectors

\[
P : \mathbb{X} \to \mathbb{X} \quad \text{and} \quad Q : \mathbb{Y} \to \mathbb{Y},
\]

such that \( \text{Im} P = \text{Ker} L, \ \text{Im} L = \text{Ker} Q = \text{Im}(I - Q) \), it follows that:

\[
L|_{\text{Dom}L \cap \text{Ker} P} : (I - P)\mathbb{X} \to \text{Im} L
\]

is invertible. We denote the inverse of that map by \( K_P \).
If $\Omega$ is an open bounded subset of $\mathbb{X}$, the mapping $N$ will be called $L$-compact on $\overline{\Omega}$ if the mapping $QN : \overline{\Omega} \to \mathbb{Y}$ is continuous, $QN(\overline{\Omega})$ is bounded, and $K_p(I - Q)N : \overline{\Omega} \to \mathbb{X}$ is compact, i.e., it is continuous and $K_p(I - Q)N(\overline{\Omega})$ is relatively compact, where $K_p : \text{Im} L \to \text{Dom} L \cap \text{Ker} P$ is the inverse of the restriction $L_p$ of $L$ to $\text{Dom} L \cap \text{Ker} P$, so that $LK_p = I$ and $K_pL = I - P$. Since $Q$ is isomorphic to $\text{Ker} L$, there exists an isomorphic $J : \text{Im} Q \to \text{Ker} L$.

Now we are ready to cite the continuous theorem (Gaines and Mawhin [6, p. 40].

**Lemma 2.1** (Continuation theorem). Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces and $L$ be a Fredholm mapping of index zero. Assume that $N : \overline{\Omega} \to \mathbb{Y}$ is $L$-compact on $\overline{\Omega}$ with $\Omega$ open and bounded in $\mathbb{X}$. Furthermore assume:

(a) for each $\lambda \in (0, 1)$, every solution of $Lx = \lambda Nx$ is such that $x \not\in \partial \Omega$,

(b) $QNx \neq 0$ for each $x \in \partial \Omega \cap \text{Ker} L$, and

$$\text{deg}\{QNx, \Omega \cap \text{Ker} L, 0\} \neq 0.$$

Then the operator equation $Lx = Nx$ has at least one solution in $\text{Dom} L \cap \overline{\Omega}$.

Let $\mathbb{Z}, \mathbb{Z}^+, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ denote the sets of all integers, nonnegative integers, natural numbers, real numbers and nonnegative real numbers, respectively. For convenience in what follows we shall let:

$$I_\omega = \{0, 1, 2, \ldots, \omega - 1\}, \quad \bar{f} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} f(n), \quad F = \frac{1}{\omega} \sum_{n=0}^{\omega-1} |f(n)|,$$

where $f(n)$ is an $\omega$-periodic sequence of real numbers defined for all $n \in \mathbb{Z}$.

We need the following lemma in the proof of our main results.

**Lemma 2.2** [5]. Let $f : \mathbb{Z} \to \mathbb{R}$ be periodic, i.e., $f(n + \omega) = f(n)$. Then for any fixed $n_1, n_2 \in I_\omega$ and for any $n \in \mathbb{Z}$, one has

$$f(n) \leq f(n_1) + \sum_{s=0}^{\omega-1} |f(s + 1) - f(s)|,$$

$$f(n) \geq f(n_2) - \sum_{s=0}^{\omega-1} |f(s + 1) - f(s)|.$$

**Theorem 2.1.** Assume that (2.1) holds and

$$\frac{1}{q} \ln \left(\frac{\bar{a}}{\bar{c}}\right) > 2\omega \bar{a} \quad \text{and} \quad \bar{a} + \bar{b}e^{\omega M_1} > \bar{c}. \quad (2.2)$$
Then Eq. (1.10) has at least one $\omega$-periodic solution $x_n^\omega$, and there exist positive constants $x_1$ and $x_2$ such that $x_1 \leq x_n^\omega \leq x_2$, for $n \in \mathbb{Z}$, where

\[
x_1 = \exp \left( \frac{1}{q} \ln \left( \frac{\tilde{a}}{c} \right) - 2\omega \tilde{a} \right),
\]

\[
x_2 = \exp \left( \frac{1}{q} \ln \left[ \tilde{a} + \tilde{b} \exp \left( \frac{p}{q} \ln \left( \frac{\tilde{a}}{c} \right) - 2p\omega \tilde{a} \right) \right] + \omega \tilde{a} \right).
\]

**Proof.** Define

\[
l_1 = \{ y = y(n) : y(n) \in \mathbb{R}, n \in \mathbb{Z} \}.
\]

Let $l^\omega \subseteq l_1$ denotes the subspace of all $\omega$-periodic sequences equipped with the usual norm $||\cdot||$, i.e.,

\[
||y|| = \max_{n \in \mathbb{I}_\omega} y(n) \text{ for } y = \{y(n) : n \in \mathbb{Z} \} \in l^\omega.
\]

It is easy to see that $l^\omega$ is a finite-dimensional Banach space. Let the linear operator $S : l^\omega \to \mathbb{R}$ be defined by

\[
S(y) = \frac{1}{\omega} \sum_{n=0}^{\omega-1} y(n), \quad y = \{y(n)\} \in l^\omega.
\]

Then we obtain two subspaces $l^\omega_0$ and $l^\omega_c$ defined by

\[
l^\omega_0 = \left\{ y = y(n) \in l^\omega : S(y) = \frac{1}{\omega} \sum_{n=0}^{\omega-1} y(n) = 0 \right\}
\]

and

\[
l^\omega_c = \{ y = y(n) \in l^\omega : y(n) = \beta, \text{ for some } \beta \in \mathbb{R} \text{ and for all } n \in \mathbb{Z} \},
\]

respectively. Then from Lemma 2.1 in [22], we find that $l^\omega_0$ and $l^\omega_c$ are closed linear subspaces of $l^\omega$ and

\[
l^\omega = l^\omega_0 \oplus l^\omega_c, \text{ dim } l^\omega_c = 1.
\]

To prove our main results, we let $x_n = \exp \{y(n)\}$, so that (1.10) can be written as

\[
y(n + 1) - y(n) = a_n + b(n)e^{\gamma y(n-\omega)} - c(n)e^{\gamma y(n-\omega)}.
\]

(2.3)

In order to embed our problem into the framework of continuation theorem, we define $\mathbb{X} = \mathbb{Y} = l^\omega$, $L(y(n)) = y(n + 1) - y(n)$, and

\[
N(y(n)) = a_n + b(n)e^{\gamma y(n-\omega)} - c(n)e^{\gamma y(n-\omega)}, \text{ for } y \in \mathbb{X} \text{ and } n \in \mathbb{Z}.
\]

Again, from Lemma 2.1 in [22] we see that $L$ is a bounded linear operator with
Ker$L = l^0_c$, $\text{Im } L = l^0_0$

and

$$\dim \text{Ker } L = 1 = \text{codim } \text{Im } L,$$

therefore, $L$ is a Fredholm mapping of index zero. Define

$$Py = \frac{1}{\omega} \sum_{n=0}^{\infty} y(n), \quad y \in \mathbb{X}, \quad Qz = \frac{1}{\omega} \sum_{n=0}^{\infty} z_n, \quad z \in \mathbb{N}.$$  

It is not difficult to show that $P$ and $Q$ are continuous projectors such that

$$\text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L = \text{Im } (I - Q).$$  

Furthermore, the generalized inverse (of $L$) $K_P : \text{Im } L \to \text{Ker } P \cap \text{Dom } L$ exists and is given by

$$K_P(z) = \sum_{s=0}^{n-1} z_s - \frac{1}{\omega} \sum_{s=0}^{\infty} (\omega - s)z_s.$$  

Then $QN : \mathbb{X} \to \mathbb{Y}$ read

$$QN y = \frac{1}{\omega} \sum_{s=0}^{\infty} [a(s) + b(s)e^{\eta y(s-\omega)} - c(s)e^{\eta y(s-\omega)}].$$  

Obviously, $QN : \mathbb{X} \to \mathbb{Y}$ and $K_P(I - Q)N : \mathbb{X} \to \mathbb{X}$ are continuous with respect to $s$ and they are mapping bounded continuous functions to bounded continuous functions. Since $\mathbb{X}$ is a finite dimensional Banach space, using the Ascoli–Arzela theorem, we see that $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, $N$ is $L$-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.  

Now we reach the position to search for an appropriate open, bounded subset $\Omega$ for the application of Lemma 2.1. Corresponding to the operator equation $L y = \lambda Ny$, $\lambda \in (0, 1)$, we have

$$y(n+1) - y(n) = \lambda [a(n) + b(n)e^{\eta y(n-\omega)} - c(n)e^{\eta y(n-\omega)}]. \quad (2.4)$$  

Suppose that $y = y(n) \in \mathbb{X}$ is a an arbitrary solution of (2.4) for a certain $\lambda \in (0, 1)$. Summing on both sides of (2.4) from 0 to $\omega - 1$, we have

$$0 = \sum_{n=0}^{\omega-1} (y(n+1) - y(n)) = \sum_{n=0}^{\omega-1} \lambda [a(n) + b(n)e^{\eta y(n-\omega)} - c(n)e^{\eta y(n-\omega)}],$$

that is,

$$\sum_{n=0}^{\omega-1} [c(n)e^{\eta y(n-\omega)} - b(n)e^{\eta y(n-\omega)}] = \sum_{n=0}^{\omega-1} a(n) = \omega \bar{a}. \quad (2.5)$$
From (2.4) and (2.5), it follows that:

\[
\sum_{n=0}^{\alpha-1} |y(n + 1) - y(n)| = \lambda \sum_{n=0}^{\alpha-1} |a(n) + b(n)e^{\eta(n-\omega)} - c(n)e^{\eta(n-\omega)}| \\
= \lambda \sum_{n=0}^{\alpha-1} |a(n) - (c(n)e^{\eta(n-\omega)} - b(n)e^{\eta(n-\omega)})| \\
\leq \sum_{n=0}^{\alpha-1} a(n) + \sum_{n=0}^{\alpha-1} (c(n)e^{\eta(n-\omega)} - b(n)e^{\eta(n-\omega)}) \\
= 2\omega \alpha,
\]

that is,

\[
\sum_{n=0}^{\alpha-1} |y(n + 1) - y(n)| \leq 2\omega \alpha.
\]

(2.6)

Since \( y \in X \), there exist \( \zeta, \eta \in I_\omega \) such that

\[
y(\zeta) = \min_{n \in I_\omega} y(n), \quad y(\eta) = \max_{n \in I_\omega} y(n).
\]

(2.7)

From (2.5) and (2.7) we have

\[
\sum_{n=0}^{\alpha-1} c(n)e^{\eta y(n)} \geq \sum_{n=0}^{\alpha-1} [c(n)e^{\eta y(n-\omega)} - b(n)e^{\eta y(n-\omega)}] = \omega \alpha,
\]

so, we have

\[
y(\eta) \geq \frac{1}{q} \ln \left( \frac{a}{c} \right)
\]

and hence from Lemma 2.2, (2.1) and (2.6), it follows that:

\[
y(n) \geq y(\eta) - \sum_{n=0}^{\alpha-1} |y(n + 1) - y(n)| \geq \frac{1}{q} \ln \left( \frac{a}{c} \right) - 2\omega \alpha := M_1 > 0.
\]

(2.8)

On the other hand, from (2.5), (2.7) and (2.8), we also have

\[
\sum_{n=0}^{\alpha-1} [c(n)e^{\eta y(\zeta)} - b(n)e^{\theta M_1}] \leq \sum_{n=0}^{\alpha-1} [c(n)e^{\eta y(n-\omega)} - b(n)e^{\eta y(n-\omega)}] = \omega \alpha,
\]

then

\[
y(\zeta) \leq \frac{1}{q} \ln \left[ \frac{a + b e^{\theta M_1}}{c} \right] > 0,
\]

and hence from Lemma 2.2, and (2.6) it follows that by (2.1):
\[ y(n) \leq y(\zeta) + \sum_{n=0}^{\omega-1} |y(n+1) - y(n)| \leq \frac{1}{q} \ln \left[ \frac{a + b e^{\omega M_1}}{c} \right] + \omega \alpha := M_2 \]

\[ > 0, \quad (2.9) \]

which together with (2.8), leads to
\[ |y(n)| \leq \max\{|M_1|, |M_2|\} := B_1. \]

Clearly, \( B_1 \) is independent on the choice of \( \lambda \). From Lemma 1.1 it is clear that there exists a positive unique solution of the equation
\[ a^* + b^* e^{p \tau} - c e^{q \tau} = 0 \]

satisfying \( |\ln(\mu^*)| < B_2 \), where \( B_2 \) is chosen sufficiently large. Take \( B = B_1 + B_2 \), then \( |y| < B \). Let
\[ \Omega := \{y \in \mathbb{X} : |y| < B\}. \]

It is clear that \( \Omega \) verifies requirement (a) of Lemma 2.1. When \( y \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap \mathbb{R} \), \( y \) is a constant with \( |y| = B \). Then we have
\[ Q Ny = \frac{1}{\omega} \sum_{s=0}^{\omega-1} \left[ a_s + b_s e^{p \tau s - a} - c_s e^{q \tau s - a} \right] \neq 0. \]

Furthermore, it is easy to see that
\[ \text{deg}(J Q Ny, \Omega \cap \text{Ker} L, 0) = \text{deg}(J Q Ny, \Omega \cap \mathbb{R}, 0) = \text{sign}(pbe^{p \tau} - qce^{q \tau}) \]

\[ \neq 0, \]

where \( \text{deg}(.) \) the Brouwer degree, \( J \) can be the identity mapping, since \( \text{Im} P = \text{Ker} L \). By now we have proved that \( \Omega \) verifies all the requirements of Lemma 2.1. Hence (2.3) has at least one \( \omega \)-periodic solution \( y_n^* \) in \( \Omega \). Set \( x_n^* = \exp(y_n^*) \), then \( x_n^* \) is a positive \( \omega \)-periodic solution of (1.10). The boundedness of \( \{y_n^*\} \) implies that the existence of positive constants \( z_1, z_2 \) such that
\[ x_1 \leq x_n^* \leq x_2. \]

References