Existence of periodic solutions, global attractivity and oscillation of impulsive delay population model

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Abstract

In this paper we consider the nonlinear impulsive delay population model. The main objective is to systematically study the qualitative behavior of the model including existence of periodic solutions, global attractivity and oscillation. The main oscillation results are the results of the prevalence of the mature cells about the periodic solutions and the global attractivity results are the conditions for nonexistence of dynamical diseases on the population.

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1. Introduction

Impulsive delay differential equations have attracted the interests of many researchers in the last two decades [1,4,9,10,14] since they provide a natural description of the motion of several real-world processes which, on one hand, depends on the processes prehistory that often turns out to be the cause of phenomena substantially affecting the motion and, on other hand, is subject to short-time perturbations whose duration is almost negligible. Such processes are often investigated in various fields of science and technology, such as physics, population dynamics, ecology, biological systems, optimal control, etc., (for more details we refer the reader to the papers [11,15–18,21] and reference therein).

The nonlinear delay differential equation

\[ x'(t) + px(t) - \frac{q x(t)}{r + x^n(t-\tau)} = 0, \quad t \geq 0, \]

where \( p, q, r, \tau \in (0, \infty), \ n \in \mathbb{N} = \{1, 2, \ldots\} \) and \( q/p > r \) has been first proposed by Nazarenko [13] for studying the control of a single population of cells. The qualitative behavior of (1) has been studied by Kubiacyk and Saker [7].
Indeed, the authors established conditions for oscillation of all positive solutions about the unique positive fixed point $\bar{x} = [(q/p) - r]^{1/n}$ and also proved that every nonoscillatory solution tends to $\bar{x}$.

Many dynamical systems that model biological or ecological phenomena contain several parameters. Biologists are tasked to determine the exact parameter values in order to use the model for prediction purposes. Unfortunately, in the real world the parameters are not fixed constants and the parameters are estimated using statistical methods and at each stage in time the estimate will be improved. On the other hand, the variation of the environment plays an important role in the behavior of the biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as selective forces in systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). Thus, it is realistic to assume that the parameters in the models are periodic functions of period $\omega$.

One more effective factor is that the existence of the external forces. Indeed, it is of biological significance to consider the effect of interacting species by adding an additional forced term. In natural world, many evolution processes are characterized by the fact that at certain moments of time, they undergo an abrupt change of state. These phenomena are best described by the so-called impulsive differential equations. In particular, in this paper we consider equation of the form

$$x'(t) + p(t)x(t) - \frac{q(t)x(t)}{r + x^n(t - m\omega)} = \lambda(t), \quad t \neq t_k,$$

$$x(t_k^+) = \frac{1}{(1 + b_k)}x(t_k), \quad k \in \mathbb{N}, \quad (2)$$

where $x(t_k^+) = \lim_{t \to t_k^+} x(t)$ and $m \geq 0$ is an integer. Our first observation is that the invariant transformation $x(t) = 1/y(t)$ reduces (2) to the following impulsive delay differential equation:

$$y'(t) = y(t) \left[p(t) - \frac{Q(t)y^n(t - m\omega)}{R + y^n(t - m\omega)} - \lambda(t)y(t)\right], \quad t \neq t_k,$$

$$y(t_k^+) = (1 + b_k)y(t_k), \quad i \in \mathbb{N}, \quad (3)$$

where

$$Q(t) = \frac{q(t)}{r} \quad \text{and} \quad R = \frac{1}{r}.$$ 

In this paper, we investigate the asymptotic properties (existence, global attractivity and oscillation) of solutions of Eq. (3). Our results imply that under appropriate impulsive conditions, the impulsive delay differential equation (3) preserves the original periodicity and global attractivity of the nonimpulsive delay differential equation.

2. Preliminaries

For system (3), we assume the following conditions:

(A1) $0 < t_1 < t_2 < \cdots$ are fixed impulsive points such that $\lim_{k \to \infty} t_k = \infty$;

(A2) $p(t), Q(t), \lambda(t) \in (0, \infty)$ are locally summable functions;

(A3) $\{b_k\}$ is a real sequence such that $b_k > -1, k \in \mathbb{N}$;

(A4) $p(t), Q(t), \lambda(t)$ and $\prod_{0 < t_k < t} (1 + b_k)$ are positive periodic functions of period $\omega$.

Throughout the rest of the paper, we always assume that a product equals unity if the number of factors is zero. Due to certain biological aspects, we shall consider Eq. (3) accompanied with the initial condition

$$y(t) = \phi(t), \quad t \in [-m\omega, 0], \quad \phi(t) \in C([-m\omega, 0], [0, +\infty)); \quad \phi(0) > 0. \quad (4)$$

**Definition 1.** A function $y \in ([-m\omega, +\infty), (0, +\infty))$ is said to be a solution of (3) on $[-m\omega, +\infty)$ if:

(i) $y(t)$ is absolutely continuous on $(0, t_1]$ and on each interval $(t_k, t_{k+1}], k = 1, 2, \ldots$;
By Lemma 4, it suffices to prove that Eqs. (5) and (6) are defined on
From (5), one can easily see that
\[ y(t) = \phi(t), \quad t \in [-m\omega, 0], \]
where
\[ R(t) = \prod_{0 < t_k < t - m\omega} (1 + b_k)^n \quad \text{and} \quad \gamma(t) = \lambda(t) \prod_{0 < t_k < t} (1 + b_k). \]

By a solution of (5) and (6) we mean an absolutely continuous function \( z(t) \) defined on \([-m\omega, +\infty]\) satisfying (5) almost everywhere for \( t \geq 0 \) and \( z(t) = \phi(t) \) on \([-m\omega, 0]\).

**Definition 2.** Suppose that \( y(t) \) and \( y^*(t) \) are two positive solutions of (3) on \([t - m\omega, \infty)\). The solution \( y^*(t) \) is said to be asymptotically attractive to \( y(t) \) provided that \( \lim_{t \to \infty} [y(t) - y^*(t)] = 0 \). Further, \( y^*(t) \) is called globally attractive if \( y^*(t) \) is asymptotically attractive to all positive solutions of (3).

**Definition 3.** A function \( y(t) \) of (3) is said to oscillate about \( y^*(t) \) if \( (y(t) - y^*(t)) \) has arbitrarily large zeros. Otherwise, \( y(t) \) is called nonoscillatory. When \( y^*(t) = 0 \), we say that \( y(t) \) oscillates about zero or simply oscillates.

The following lemma proves very helpful. The proof is similar to that of Theorem 1 in [12] and hence is omitted.

**Lemma 4.** Assume that (A1)–(A4) hold. Then

(i) if \( z(t) \) is a solution of (5) on \([-m\omega, \infty)\), then \( y(t) = \prod_{0 < t_k < t} (1 + b_k)z(t) \) is a solution of (3) on \([-m\omega, \infty)\);

(ii) if \( y(t) \) is a solution of (3) on \([-m\omega, \infty)\), then \( z(t) = \prod_{0 < t_k < t} (1 + b_k)^{-1} y(t) \) is a solution of (5) on \([-m\omega, \infty)\).

It is clear that the transformation
\[ y(t) = \prod_{0 < t_k < t} (1 + b_k)z(t) \]
preserves the asymptotic properties of Eqs. (3) and (5). Thus, it suffices to consider Eq. (5).

**Lemma 5.** Assume that (A1)–(A4) hold. Then the solutions of (3) are defined on \([-m\omega, \infty)\) and are positive on \([0, \infty)\).

**Proof.** By Lemma 4, it suffices to prove that Eqs. (5) and (6) are defined on \([-m\omega, \infty)\) and are positive on \([0, \infty)\). From (5), one can easily see that \( z(t) \) for any \( \phi(t) \in C([-\omega, 0], [0, +\infty)) \) is defined on \([-m\omega, \infty)\) and is positive on \([0, \infty)\). \( \square \)

### 3. The main results

In this section, we prove that there exists a unique positive \( \omega \)-periodic solution \( y^*(t) \) of Eq. (3). Then, we provide some sufficient conditions for the global attractivity of \( y^*(t) \). Moreover, the oscillatory behavior of all positive solutions of (3) about \( y^*(t) \) are studied.
Denote
\[ f_m = \min_{t \in [0, \omega]} f(t) \quad \text{and} \quad f_M = \max_{t \in [0, \omega]} f(t), \]
where \( f \) is a positive periodic function of period \( \omega \).

### 3.1. Existence

Consider Eqs. (3) and (5) without delay. That is,
\[ y'(t) = y(t) \left[ p(t) - \frac{Q(t)y^n(t)}{R + y^n(t)} - \gamma(t)y(t) \right], \quad t \neq t_k, \]
\[ y(t_k^+) = (1 + b_k)y(t_k), \quad i \in \mathbb{N}, \]

and
\[ z'(t) = z(t) \left[ p(t) - \frac{Q(t)z^n(t)}{R(t) + z^n(t)} - \gamma(t)z(t) \right]. \quad (9) \]

We shall prove that there exists a unique \( \omega \)-periodic positive periodic solution \( z^*(t) \) of Eq. (8).

**Theorem 6.** Assume that (A1)–(A4) hold, and
\[ (A5) \quad Q_M > p_m \quad \text{and} \quad Q_m > p_M. \]

Then there exists a unique \( \omega \)-periodic positive solution \( z^*(t) \) of (8).

**Proof.** To prove the theorem, it suffices to prove that Eq. (9) has a unique \( \omega \)-periodic positive solution \( z^*(t) \). Consider the function
\[ f(z) = p - \frac{Qz^n}{R + z^n} - \gamma z, \quad z \in (0, \infty), \]
where \( p, Q, R \) and \( \gamma \) are positive constants. Clearly, if \( Q > p \), then the equation \( f(z) = 0 \) has a unique positive root. This is justified by the fact that \( f(z) = (\gamma z^{n+1} + (Q - p)z^n + \gamma R z - p R)/(R + z^n) = 0 \) has a unique positive root \( z_0 \) and this follows since \( f(0) = -p R < 0 \) and \( f(\infty) = \infty \). Furthermore, \( f(z) < 0 \) for \( z \in [0, z_0) \) and \( f(z) > 0 \) for \( z \in (z_0, \infty) \).

We define the functions
\[ f_1(z) = p_m - \frac{Q M z^n}{R M + z^n} - \gamma M z \quad \text{and} \quad f_2(z) = p_M - \frac{Q m z^n}{R M + z^n} - \gamma M z. \]

Then from the above discussion, it is clear that \( f_1(z) \) and \( f_2(z) \) have positive zeros \( z_1 \) and \( z_2 \), respectively, that is, \( f_1(z_1) = 0 \) and \( f_2(z_2) = 0 \). In addition, the inequality
\[ p_M - \frac{Q M z^n}{R M + z^n} - \gamma M z_2 < p_M - \frac{Q m z^n}{R M + z^n} - \gamma M z_2 = 0 \]
implies that \( z_2 > z_1 \). Suppose \( z(t) = z(t, 0, z_0) \) with \( z_0 > 0 \) is the unique solution of (9) through \( (0, z_0) \). We claim that if \( z_0 \in [z_1, z_2] \) then \( z(t) \in [z_1, z_2] \) for all \( t \geq 0 \). Otherwise, let \( t^* = \inf \{ t > 0 : z(t) > z_2 \} \). Then, there exists \( \tilde{t} \geq t^* \) such that \( z(\tilde{t}) > z_2 \) and \( z'(\tilde{t}) \geq 0 \). However, from (9) and the fact that \( z(\tilde{t}) > z_2 \), we conclude that
\[ z'(\tilde{t}) = z(\tilde{t}) \left[ p(\tilde{t}) - \frac{Q(\tilde{t})z^n(\tilde{t})}{R(\tilde{t}) + z^n(\tilde{t})} - \gamma(\tilde{t})z(\tilde{t}) \right] < z(\tilde{t}) \left[ p_M - \frac{Q m z^n}{R M + z^n} - \gamma M z_2 \right] = 0, \]
which is a contradiction. Therefore, \( z(t) \leq z_2 \). By a similar argument, we can show that \( z(t) > z_1 \) for all \( t \geq 0 \). Hence, in particular, \( z_0 = z(\omega, 0, z_0) \in [z_1, z_2] \).
We define a mapping \( F : [z_1, z_2] \rightarrow [z_1, z_2] \) as follows: for each \( z_0 \in [z_1, z_2] \), \( F(z_0) = z_0 \). Since the solution \( z(t, 0, z_0) \) depends continuously on the initial value \( z_0 \), it follows that \( F \) is continuous and invariant, that is, it maps the interval \([z_1, z_2]\) into itself. By the Brouwer’s Fixed Point Theorem, \( F \) has a fixed point \( z^*_0 \). In view of the periodicity of \( p(t) \), \( Q(t) \), \( R(t) \) and \( y(t) \), it follows that the unique solution \( z^*(t) = z(t, 0, z^*_0) \) of (9) through the initial point \((0, z^*_0)\) is a positive periodic solution of period \( \omega \). Let \( y^*(t) = \prod_{0 < k < t} (1 + b_k)z^*(t) \). Then by Lemma 4 and (A4), \( y^*(t) \) is \( \omega \)-periodic solution of (8). The proof is complete. \( \square \)

**Theorem 7.** Assume that (A1)–(A5) hold. Then, there exists a unique \( \omega \)-periodic positive solution \( y^*(t) \) of (3).

**Proof.** In view of Theorem 6, Eq. (9) has a unique \( \omega \)-periodic positive solution \( z^*(t) \). However, \( z^*(t) \) is also an \( \omega \)-periodic positive solution of (5). Thus by Lemma 4 and condition (A3), \( y^*(t) = \prod_{0 < k < t} (1 + b_k)z^*(t) \) is \( \omega \)-periodic positive solution of (3). On the other hand, if \( y^*(t) \) is a periodic positive solution of (3), it is also a positive periodic solution of (8). By Theorem 6, the positive periodic solution of (3) is unique. \( \square \)

### 3.2. Attractivity

In this subsection, in the nondelay case we shall prove that the unique positive periodic solution \( y^* \) of Eq. (8) is globally attractive. Moreover, it is shown that every positive solution of (3) which does not oscillate about \( y^*(t) \) converges to \( y^*(t) \). Finally, we shall establish sufficient conditions for \( y^*(t) \) to be a global attractor of all other positive solutions of (3).

**Theorem 8.** Assume that (A1)–(A5) hold. Let \( y(t) \) be a positive solution of (8). Then the limit
\[
\lim_{t \to \infty} \| y(t) - y^*(t) \| = 0.
\]

**Proof.** Let \( y^*(t) \) be a periodic positive solution of (8). Thus, by Lemma 4, \( z^*(t) = \prod_{0 < k < t} (1 + b_k)^{-1}y^*(t) \) is the periodic solution of (9). Assume that \( y(t) > y^*(t) \) for \( t \) sufficiently large then \( \lim_{t \to \infty} y(t) = z^*(t) \) (the case \( y(t) < y^*(t) \) is similar and hence is omitted).

Set \( z(t) = z^*(t)e^{v(t)} \).

Then, \( v(t) > 0 \) for \( t \) sufficiently large and
\[
v'(t) = \gamma(t)z^*(t)(1 - e^{v(t)}) - \frac{Q(t)R(t)[z^*(t)]^n}{(R(t) + [z^*(t)]^n)((R(t) + [z^*(t)]^n)e^{v(t)})}((e^{u(t)} - 1))<0.
\]

However, since \( e^{v(t)} - 1 > 0 \) for \( t \) sufficiently large, it follows that
\[
v'(t) < -\frac{Q(t)R(t)[z^*(t)]^n}{(R(t) + [z^*(t)]^n)((R(t) + [z^*(t)]^n)e^{v(t)})}((e^{u(t)} - 1)) < 0.
\]

Thus, \( v(t) \) is decreasing and therefore \( \lim_{t \to \infty} v(t) = \alpha \in [0, \infty) \). Now, we shall prove that \( \alpha = 0 \). Suppose that \( \alpha > 0 \), then there exists \( \varepsilon > 0 \) and \( T_\alpha > 0 \) such that for \( t \geq T_\alpha \), \( 0 < \alpha - \varepsilon < v(t) < \alpha + \varepsilon \). However, from (13), we have
\[
v'(t) \leq -\frac{Q_mR_m[z^*(t)]^n}{(R_M + [z^*(t)]^n)((R_M + [z^*(t)]^n)e^{\alpha})}((e^{u(t)} - 1)) < 0,
\]

Integrating (14) from \( T_\varepsilon \) to \( \infty \), we immediately obtain a contradiction. Hence, \( \alpha = 0 \) and therefore \( v(t) \) tends to zero as \( t \to \infty \). Thus, we have
\[
\lim_{t \to \infty} \| y(t) - y^*(t) \| = \lim_{t \to \infty} z^*(t) \prod_{0 < k < t} (1 + b_k)[e^{v(t)} - 1] = 0.
\]

The proof is completed. \( \square \)
**Theorem 9.** Assume that (A1)–(A5) hold. Let \( y(t) \) be a positive solution of (3) which does not oscillate about \( y^*(t) \). Then \( \lim_{t \to \infty} |y(t) - y^*(t)| = 0 \).

**Proof.** Proceeding exactly as in the proof of Theorem 8, we obtain the desired result. \( \square \)

To show that \( y^*(t) \) is a global attractor of (3), we need to prove that \( z^*(t) \) is a global attractor of (5). To do this, the following lemma is needed.

**Lemma 10.** Assume that (A1)–(A5) hold, and let \( z(t) \) be a positive solution of (5) which oscillates about \( z^*(t) \). Then, there exists a \( T \) such that for all \( t \geq T \),

\[
\begin{align*}
&\text{(a) } z(t) \leq z_2 \exp\{mwp_M\} = Z_2; \\
&\text{(b) } z(t) \geq z_1 \exp\{m\omega(p_m - (Q_MZ^n_2)/(\bar{R}_m + z^n_1) - \gamma_MZ_2)\} = Z_1.
\end{align*}
\]

**Proof.** First we shall show (a). Let \( m\omega \leq t_1 < t_2 < \cdots < t_i < \cdots \) be a sequence of zeros of \( z(t) - z^*(t) \) with \( \lim_{t_i \to \infty} t_i = \infty \). Our strategy is to show that (a) holds in each interval \((t_i, t_{i+1})\). For this, let \( \xi_i \in (t_i, t_{i+1}) \) be a point where \( z(t) \) attains its maximum in \((t_i, t_{i+1})\). Then, it suffices to show that

\[
z'_{i+1} \leq z_2 \exp\{mwp_M\} = Z_2.
\]

We can assume that there exists a \( \xi_i \) where \( z(\xi_i) > z_2 \), (see Theorem 1) otherwise there is nothing to prove. Since \( z'(\xi_i) = 0 \), it follows that:

\[
0 = z'(\xi_i) < z(\xi_i) \left( p_M - \frac{Q_mz^n_2(\xi_i - m\omega)}{\bar{R}_m + z^n_2(\xi_i - m\omega)} - \gamma_Mz(\xi_i) \right)
\]

and hence

\[
p_M - \frac{Q_mz^n_2(\xi_i - m\omega)}{\bar{R}_m + z^n_2(\xi_i - m\omega)} - \gamma_Mz_2 > 0.
\]

Thus, if \( z(t) \) attains its maximum at \( \xi_i \), then it follows that \( z(\xi_i - m\omega) < z_2 \). Since \( z(\xi_i) > z_2 \) and \( z(\xi_i - m\omega) < z_2 \), then there exists \( \tilde{\xi}_i \) in \((\xi_i - m\omega, \xi_i)\), such that \( z(\tilde{\xi}_i) = z_2 \). Integrating (5) from \( \tilde{\xi}_i \) to \( \xi_i \), we get

\[
\ln \left( \frac{z(\xi_i)}{z(\tilde{\xi}_i)} \right) = \int_{\tilde{\xi}_i}^{\xi_i} \left( p(t) - \frac{Q(t)z^n(t - m\omega)}{\bar{R}(t) + z^n(t - m\omega)} - \gamma(t)z(t) \right) dt < \int_{\tilde{\xi}_i}^{\xi_i} p(t) dt < \int_{\tilde{\xi}_i}^{\xi_i} m\omega p_M dt = m\omega p_M,
\]

which immediately gives (15).

Now, we shall show that (b) is valid for \( t \geq T_1 + m\omega \). For this, following as above let \( \mu_i \in (t_i, t_{i+1}) \) be a point where \( z(t) \) attains its minimum in \((t_i, t_{i+1})\). Then, it suffices to show that

\[
z(\mu_i) \geq z_1 \exp \left\{ m\omega \left( p_m - \frac{Q_MZ^n_2}{\bar{R}_m + z^n_1} - \gamma_MZ_2 \right) \right\} = Z_1.
\]

Since \( Z_2 > z_2 > z_1 \). It follows that:

\[
p_m - \frac{Q_MZ^n_2}{\bar{R}_m + z^n_1} - \gamma_MZ_2 < 0.
\]

Thus, \( Z_1 < Z_2 \). Now, assume that there exists a \( \mu_i \geq T_1 + m\omega \) where \( z(\mu_i) < z_1 \), otherwise there is nothing to prove. Since \( z'(\mu_i) = 0 \), we have

\[
0 = z'(\mu_i) > z(\mu_i) \left( p_m - \frac{Q_Mz^n(\mu_i - m\omega)}{\bar{R}_m + z^n(\mu_i - m\omega)} - \gamma_Mz(\mu_i) \right)
\]
and hence

\[ p_m = \frac{Q_M z^n (\mu_i - m\omega)}{R_m + z^n (\mu_i - m\omega)} - \gamma_M z(\mu) < 0. \]

Thus, it is necessary that \( z(\mu_i - m\omega) > z_1 \). Hence, there exists a \( \bar{\mu}_i \in (\mu_i - m\omega, \mu_i) \) where \( z(\bar{\mu}_i) = z_1 \). Integrating (5) from \( \bar{\mu}_i \) to \( \mu_i \) and using \( z(t) \leq Z_2 \) and (19), we get

\[
\ln \left( \frac{z(\mu_i)}{z(\bar{\mu}_i)} \right) = \int_{\bar{\mu}_i}^{\mu_i} \left( p(t) - \frac{Q(t)z^n(t - m\omega)}{R(t) + z^n(t - m\omega)} - \gamma(t)z(t) \right) \, dt
\]

\[
> \int_{\bar{\mu}_i}^{\mu_i} \left( p_m - \frac{Q_M Z_2^n}{R_m + z_1^n} - \gamma_M Z_2 \right) \, dt
\]

\[
= m\omega \left( p_m - \frac{Q_M Z_2^n}{R_m + z_1^n} - \gamma_M Z_2 \right),
\]

which immediately leads to (18). \( \square \)

**Lemma 11.** Assume that (A1)–(A5) hold, and let \( y(t) \) be a positive solution of (3) which oscillates about \( y^*(t) \). Then, there exists a \( T \) such that for all \( t \geq T \),

(a) \( y(t) \leq \prod_{0 \leq t_k < t} (1 + b_k) \exp\{mwp_M\} \);

(b) \( y(t) \geq \prod_{0 \leq t_k < t} (1 + b_k) \exp\{m\omega(p_m - (Q_M Z_2^n)/(R_m + z_1^n) - \gamma_M Z_2)\}, \)

where \( Z_2 \) is defined as in Lemma 10.

The following result provides sufficient conditions for the global attractivity of \( y^*(t) \).

**Theorem 12.** Assume that (A1)–(A5) hold, and

\[
\lim_{t \to \infty} \int_{t - m\omega}^{t} \left( \frac{nQ(s)[Z_2]^n}{R(s)} \exp(Z_2m\omega\gamma_M) \right) \, ds < \frac{\pi}{2}.
\]

(20)

Then, (10) is valid for any positive solution \( y(t) \) of (3).

**Proof.** Clearly, it suffices to prove that

\[
\lim_{t \to \infty} [z(t) - z^*(t)] = 0,
\]

(21)

where \( z^*(t) = \prod_{0 \leq t_k < t} (1 + b_k)^{-1} y^*(t) \) is the unique \( \omega \)-periodic positive solution of (5) and \( z(t) = \prod_{0 \leq t_k < t} (1 + b_k)^{-1} z(t) \) is any other positive solution of (5). In the nondelay case we have established (21) in Theorem 8, and for the positive solutions of (5) which are nonoscillatory about \( z^*(t) \) we have shown (21) in Theorem 9. Thus, it remains to prove (21) for the positive solutions of (5) which oscillate about \( z^*(t) \). If \( z(t) \) is an arbitrary positive solution of (5), then from the transformation (11), (5) reduces to

\[
v'(t) + \gamma(t)z^*(t)f_1(v(t)) + \frac{Q(t)R(t)[z^*(t)]^n}{(R(t) + [z^*(t)]^n)} f_2(v(t - m\omega)) = 0,
\]

(22)

where

\[
f_1(u) = e^u - 1, \quad f_2(u) = \frac{e^{nu} - 1}{R(t) + [z^*(t)]^n e^{nu}}.
\]
Let
\[ G_1(t, u) = \gamma(t)z^*(t)f_1(u), \quad G_2(t, u) = \frac{Q(t)\overline{R}(t)[z^*(t)]^p}{(\overline{R}(t) + [z^*(t)]^p)}f_2(u). \]

Then we have
\[ \frac{\partial G_1(t, u)}{\partial u} = \gamma(t)z^*(t)e^u, \quad \frac{\partial G_2(t, u)}{\partial u} = \frac{nQ(t)\overline{R}(t)[z^*(t)]^p e^{nu}}{(\overline{R}(t) + [z^*(t)]^p e^{nu})^2}. \]

Eq. (22) becomes
\[ v'(t) + G_1(t, v(t)) - G_1(t, 0) + G_2(t, v(t - m\omega)) - G_2(t, 0) = 0. \tag{23} \]

By the mean value theorem, (23) can be written as
\[ v'(t) + F_1(t)v(t) + F_2(t)v(t - m\omega) = 0, \tag{24} \]

where
\[ F_1(t) = \left. \frac{\partial G_1(t, u)}{\partial u} \right|_{u=z^*_1(t)} = \gamma(t)z^*(t)e^{z^*_1(t)} = \gamma(t)\eta_1(t) \]

and
\[ F_2(t) = \left. \frac{\partial G_2(t, u)}{\partial u} \right|_{u=z^*_2(t)} = \frac{nQ(t)\overline{R}(t)[z^*(t)]^p e^{n(z^*_2(t))}}{(\overline{R}(t) + [z^*(t)]^p e^{n(z^*_2(t))})^2} = \frac{nQ(t)\overline{R}(t)\eta_2^2(t)}{(\overline{R}(t) + \eta_2^2(t))^2}, \]

where \( \eta_1(t) \) lies between \( z^*(t) \) and \( z(t) \) and \( \eta_2(t) \) lies between \( z^*(t) \) and \( z(t - m\omega) \).

Let
\[ v(t) = X(t) \exp\left( -\int_0^t F_1(s) \, ds \right). \]

Then (24) can be written as
\[ X'(t) + F_2(t) \exp\left( \int_{t-m\omega}^t F_1(s) \, ds \right) X(t - m\omega) = 0. \tag{25} \]

Then by Lemma 11, we obtain
\[ F_2(t) \exp\left( \int_{t-m\omega}^t F_1(s) \, ds \right) \leq \frac{nQ(t)[Z_2]^p}{\overline{R}(t)} \exp\left( Z_2 \int_{t-m\omega}^t \gamma(s) \, ds \right). \]

Thus, in view of (22), we find
\[
\begin{align*}
\lim_{t \to \infty} \int_{t-m\omega}^t F_2(s) \exp\left( \int_{s-m\omega}^s F_1(u) \, du \right) \, ds \\
\leq \lim_{t \to \infty} \int_{t-m\omega}^t \left( \frac{nQ(s)[Z_2]^p}{\overline{R}(s)} \exp\left( Z_2 \int_{s-m\omega}^s \gamma(r) \, dr \right) \right) \, ds \\
\leq \lim_{t \to \infty} \int_{t-m\omega}^t \left( \frac{nQ(s)[Z_2]^p}{\overline{R}(s)} \exp(Z_2m\omega\gamma_M) \right) \, ds < \frac{\pi}{2}.
\end{align*}
\]

Using the well known result in [8], we conclude that every solution of (25) satisfies \( \lim_{t \to \infty} X(t) = 0 \) which implies that \( \lim_{t \to \infty} v(t) = 0 \) and hence \( \lim_{t \to \infty} [z(t) - z^*(t)] = 0. \) \( \square \)
3.3. Oscillation

In this subsection, we prove that every solution of Eq. (3) oscillates about $y^*(t)$.

We first give the following fundamental lemma which states that the oscillation of all solutions of (3) is equivalent to the oscillation of those of (5).

**Lemma 13.** Assume that (A1)–(A5) hold. Then every solution of (3) oscillates if and only if every solution of (5) oscillates.

**Proof.** Suppose that $z(t)$ is a solution of (5) on $[0, \infty)$. Let $y(t) = \prod_{0<\tau<t} (1+b_\tau) z(t)$. Then, by Lemma 4, $y(t)$ is a solution of (3) on $[0, \infty)$. Since that $\prod_{0<\tau<t} (1+b_\tau) > 0$, $y(t)$ is oscillatory if and only if $z(t)$ is oscillatory. □

**Theorem 14.** Assume that (A1)–(A5) hold and every solution of the delay differential equation

$$W'(t) + \exp\left((1-\varepsilon)\int_{t-m_0}^t \gamma(s)z^*(s)\,ds\right)\frac{(1-\varepsilon)nQ(t)R(t)[z^*(t)]^n}{(R(t) + [z^*(t)]^n)^2} W(t - m\omega) = 0 \quad (26)$$

oscillates. Then, every solution of (3) oscillates about $y^*(t)$.

**Proof.** Assume on the contrary that (5) has a solution $z(t)$ which does not oscillate about $z^*(t)$. Without loss of generality, we assume that $z(t) > z^*(t)$ so that $v(t) > 0$ (the case $z(t) < z^*(t)$, i.e., when $z(t) < 0$ is similar and hence is omitted). Set

$$z(t) = z^*(t) e^{v(t)}. \quad (27)$$

It is clear that a solution $z(t)$ of (5) oscillates about $z^*(t)$ if and only if $z(t)$ oscillates about zero. The transformation (27) transforms (5) to the equation

$$v'(t) + \gamma(t)z^*(t)f_1(z(t)) + \frac{nR(t)Q(t)[z^*(t)]^n}{(R(t) + [z^*(t)]^n)^2} f_2(v(t - m\omega)) = 0, \quad (28)$$

where

$$f_1(u) = e^u - 1 \quad \text{and} \quad f_2(u) = \frac{R + [z^*(t)]^n}{n} \frac{(e^{nu} - 1)}{R + [z^*(t)]^n} e^{nu}. \quad (29)$$

Note that

$$uf_1(u) > 0 \quad \text{for} \quad u \neq 0 \quad \text{and} \quad \lim_{u \to 0} \frac{f_1(u)}{u} = 1, \quad (29)$$

and

$$uf_2(u) > 0 \quad \text{for} \quad u \neq 0 \quad \text{and} \quad \lim_{u \to 0} \frac{f_2(u)}{u} = 1. \quad (30)$$

From (29) and (30) it follows that for any given arbitrarily small $\varepsilon$ there exists a $\delta > 0$ such that for all $0 < u < \delta$, $f_1(u) \geq (1-\varepsilon) u$ and $f_2(u) \geq (1-\varepsilon) u$ (for all $-\delta < u < 0$, $f_1(u) \leq (1-\varepsilon) u$ and $f_2(u) \leq (1-\varepsilon) u$). In view of Theorem 9, $v(t) \to 0$ as $t \to \infty$ for sufficiently large $t$, we can use these estimates in (28) to conclude that eventually $v(t)$ is a positive solution of the differential inequality

$$v'(t) + \gamma(t)z^*(t)(1-\varepsilon) v(t) + \frac{(1-\varepsilon)nR(t)Q(t)[z^*(t)]^n}{(R(t) + [z^*(t)]^n)^2} v(t - m\omega) \leq 0. \quad (31)$$

The transformation

$$v(t) = \exp\left(-(1-\varepsilon)\int_{0}^{t} \gamma(s)z^*(s)\,ds\right) V(t)$$
in (31) implies that $V(t)$ is also an eventually positive solution of the differential inequality

$$V'(t) + \exp\left((1 - \varepsilon) \int_{t - m(t)}^{t} \gamma(s) z^*(s) \, ds\right) \left(1 - \varepsilon\right) nR(t) Q(t) [z^*(t)]^n \left(\frac{1}{(R(t) + [z^*(t)]^n)}\right)^2 V(t - m(t)) \leq 0.$$  

By Corollary 3.2.2 in [5], there exists an eventually positive solution of the delay differential equation (26) which satisfies that $W(t) \geq V(t)$. This contradicts our assumption that every solution of (26) is oscillatory. Hence, every positive solution of (5) oscillates about $z^*(t)$. Then, it follows from Lemma 13 that every solution of (3) oscillates about $y^*(t)$ if every solution of (26) oscillates. \(\square\)

For the oscillation of the delay differential equation (26) several known criteria can be employed. For example, the results given in [5] when applied to (26) lead to the following corollary

**Corollary 15.** Assume that (A1)–(A5) hold. Then

$$\Theta_e = (1 - \varepsilon) \liminf_{t \to \infty} \int_{t - m(t)}^{t} \Psi(s) \exp\left((1 - \varepsilon) \int_{s - m(t)}^{s} \gamma(r) z^*(r) \, dr\right) \, ds > \frac{1}{e},$$  

or

$$\Gamma_e = (1 - \varepsilon) \limsup_{t \to \infty} \int_{t - m(t)}^{t} \Psi(s) \exp\left((1 - \varepsilon) \int_{s - m(t)}^{s} \gamma(r) z^*(r) \, dr\right) \, ds > 1,$$  

where

$$\Psi(s) = \frac{nR(s) Q(s) [z^*(s)]^n}{(R(s) + [z^*(s)]^n)^2},$$

implies that every solution of (26) is oscillatory.

If the strict inequalities in (32) and (33) hold for $\varepsilon = 0$, then the same must be true for all sufficiently small $\varepsilon > 0$ also. Thus, we can restate Corollary 15 as follows:

**Corollary 16.** Assume that (A1)–(A5) hold. Then

$$\Theta_0 = \liminf_{t \to \infty} \int_{t - m(t)}^{t} \Psi(s) \exp\left(\int_{s - m(t)}^{s} \gamma(r) z^*(r) \, dr\right) \, ds > \frac{1}{e},$$  

or

$$\Gamma_0 = \limsup_{t \to \infty} \int_{t - m(t)}^{t} \Psi(s) \exp\left(\int_{s - m(t)}^{s} \gamma(r) z^*(r) \, dr\right) \, ds > 1,$$  

implies that every solution of (26) is oscillatory.

From Theorem 14 and Corollary 16 the following oscillation criterion for (3) is an immediate.

**Theorem 17.** Assume that (A1)–(A5) hold. Then, (34) or (35) implies that every positive solution of (3) oscillates about $y^*(t)$.

For the oscillation of (26) it is clear that between the conditions (32) and (33) there is a gap when the limit

$$(1 - \varepsilon) \lim_{t \to \infty} \int_{t - m(t)}^{t} \Psi(s) \exp\left((1 - \varepsilon) \int_{s - m(t)}^{s} \gamma(r) z^*(r) \, dr\right),$$

does not exist. To fill this gap partially we can employ some known results from the literature. For example, the criterion of Erbe and Zhang [3] when applied to first order delay differential equation (26) guarantees that every solution of (3) oscillates about $y^*(t)$ provided that

$$0 < \Theta_e \leq \frac{1}{e}.$$
and
\[ I_0 > 1 - \frac{\theta_0^2}{4} \]  
(36)

We also note that, on following the work of Chao [2], condition (36) can be improved as
\[ I_0 > 1 - \frac{\theta_0^2}{2(1 - \theta_0^2)}, \]  
(37)

whereas a result of Yu et al. [19,20] gives an improvement of (37) to become
\[ I_0 > 1 - \frac{(1 - \theta_0) - \sqrt{1 - 2\theta_0 - \theta_0^2}}{2}. \]  
(38)

Finally, we remark that condition (38) can be improved further by employing the recent results presented in [6], however,
due to the lack of space the details are left to the interested reader.

References