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Difference Scheme for Semilinear Reaction-Diffusion Problems on a Mesh of Bakhvalov Type

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The paper examines a semilinear singular reaction-diffusion problem. Using the collocation method with naturally chosen splines of exponential type, a new difference scheme on a mesh of Bakhvalov type is constructed. A difference scheme generates the system of nonlinear equations, and the theorem of existence and this system's solution uniqueness are also provided. At the end, a numerical example is given as well, which points to the convergence of the numerical solution to the exact one.

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1 Introduction

We consider the semilinear problem

$$\begin{aligned} \epsilon^2 y'' = f(x,y), \quad x \in I = [0,1]\,, \\ y(0)=0,\; y(1)=0. \end{aligned} \tag{1.1}$$

Here ϵ is a positive parameter, $f(x,y) \in C^2(I \times \mathbf{R})$ has bounded partial derivates and

$$\gamma \geq f_y(x,y) \geq m > 0, \tag{1.2}$$

for all $(x,y) \in I \times \mathbf{R}$. Problem (1.1) under condition (1.2), has a unique solution.

Problems like (1.1) are differential equations that depend on a small positive parameter ϵ, and whose solution (or their derivates) approach a discontinuous limit as ϵ approaches zero. Such problems are said to be singularly perturbed, where we regard ϵ as the perturbation parameter.

The solutions of singular perturbation problems typically contain layers. If any discretization technique is applied to a parameter-dependent problem, then the behavior of the discretization depends on the parameter. For singularly perturbed problems, conventional techniques often lead to discretization that are worthless if the singular perturbation parameter is close to some critical values. Our paper is devoted to the construction of approximations using the collocation method with the natural choice exponential splines on Bakhvalov mesh.

Many authors have considered the problem (1.1), under various hypotheses on $f(x,y)$, for example Vulanović [5], Uzelac and Surla [4]. Bakhvalov [1] was the first to use the special mesh to solve singularly perturbed problems.

Numerical examples show the convergence of the numerical solution to the exact one and they also offer better results when compared to previous difference schemes (i.e. [4]).

2 Construction of the nonlinear difference scheme on a Bakhvalov mesh

We apply the mesh of Bakhvalov type $0 = x_0 < x_1 < \ldots < x_N < 1$ on the interval $[0,1]$. The interval $[0,1]$ will be divided to three subintervals, namely $[0,h_\epsilon]$, $[h_\epsilon, 1-h_\epsilon]$ and $[1-h_\epsilon, 1]$ where $h_\epsilon = \frac{2\epsilon}{m_1}\ln|\epsilon|$ and $m_1 = \frac{\sqrt{m}}{2}$.
The subinterval $[0,h_\epsilon]$ is divided by the points $x_i = -\frac{2\epsilon}{m_1}\ln\left[1-(1-\epsilon)i\delta\right]$, $(i = 1,\ldots,j)$, and the subinterval $[h_\epsilon, 1-h_\epsilon]$ is divided by $x_{j+1} = x_j + ih$, $(i = 1,\ldots,k)$, while the last interval $[1-h_\epsilon, 1]$ is divided by
$x_{j+k+i} = 1 + \frac{2\epsilon}{m_1}\ln\left[1-(1-\epsilon)(j-i)\delta\right]$, $(i = 1,\ldots,j)$, where $\delta = \frac{1}{j}$, $j = \frac{N}{4}$ and $k = \frac{N}{2} - 1$ represent the chosen integer so that $N = 2j + k + 1$. It can be noticed that the choice of j, k and N, as well as δ, does not depend on the parameter ϵ. In order to construct a difference scheme, which will later help us to get a numerical solution of the problem (1.1), the following function is introduced $\psi(x,y) = f(x,y) - \gamma y$. Now, the problem (1.1) becomes

$$L_\epsilon y(x) := \epsilon^2 y''(x) - \gamma y(x) = \psi(x, y(x)) \text{ on } [0,1]. \tag{2.1}$$

The following mesh expresses further problems:

$$\begin{aligned} &L_\epsilon u_i(x) := 0 \text{ on } (x_i, x_{i+1}),\, u_i(x_i) = 1,\ u_i(x_{i+1}) = 0,\ (i = 0,1,\ldots,N-1),\\ &L_\epsilon u_i(x) := 0 \text{ on } (x_i, x_{i+1}),\ u_i(x_i) = 0,\ u_i(x_{i+1}) = 1, (i = 0,1,\ldots,N-1) \end{aligned} \tag{2.2}$$

and

$$L_\epsilon y_i(x) = \psi(x, y_i) \text{ on } (x_i, x_{i+1}),\ y_i(x_i) = y(x_i),\ y_i(x_{i+1}) = y(x_{i+1}), \quad (i = 1, 2, ..., N-1). \tag{2.3}$$

We denote the solutions of problems (2.2) by $u_i^I(x)$, $u_i^{II}(x)$, $(i = 1, 2, ..., N-1)$, respectively. Using the solution of the above mentioned problems (2.2) and the Green's function for the operator L_ϵ, the solution of the problem (2.3) is

$$y_i(x) = y_i u_i^I(x) + y_{i+1} u_i^{II}(x) + \int_{x_i}^{x_{i+1}} G_i(x, s)\psi(s, y(s))ds, \tag{2.4}$$

where $G_i(x, s)$ is the Green's function. While $y_i(x) \equiv y(x)$ on $[x_i, x_{i+1}]$ and $y_i'(x)\big|_{x=x_i} = y_{i-1}'(x)\big|_{x=x_i}$, $(i = 1, 2, ..., N-1)$, after the differentiation of the solution $y_{i-1}(x)$ and $y_i(x)$, it can be observed that

$$\begin{gathered} a_i y_{i-1} - c_i y_i + b_i y_{i+1} = \\ \frac{1}{\epsilon^2}\left[\int\limits_{x_{i-1}}^{x_i} u_{x_{i-1}}^{II}(s)\psi(s, y(s))ds + \int\limits_{x_i}^{x_{i+1}} u_{x_i}^{I}(s)\psi(s, y(s))ds\right], \\ y_0 = 0,\ \ y_N = 0,\ \ (i = 1, 2, ..., N-1). \end{gathered} \tag{2.5}$$

Clearly, we cannot generally explicitly compute the integrals in (2.5).
We approximate the function $\psi(x, y(x))$ on the interval $[x_{i-1}, x_i]$ by

$$\overline{\psi}_{i-1} = \overline{\psi}(x, y(x)) = \psi\left(\tfrac{x_{i-1}+x_i}{2}, \tfrac{\overline{y}_{i-1}+\overline{y}_i}{2}\right) \text{ on } [x_{i-1}, x_i], \quad (i = 1, 2, ..., N-1), \tag{2.6}$$

where $\overline{y}_i$, $(i = 1, 2, ..., N-1)$, are approximation values of the solution $y(x)$ of the problem (1.1) – (1.2) in the points x_i, $(i = 1, 2, ..., N-1)$. Finally, from (2.5) and (2.6) we get the difference scheme

$$a_i\overline{y}_{i-1} - (d_i + d_{i+1})\overline{y}_i + b_i\overline{y}_{i+1} = \frac{1}{\gamma}\overline{\psi}_{i-1}(d_i - a_i) + \frac{1}{\gamma}\overline{\psi}_i(d_{i+1} - a_{i+1}), \quad (i = 1, 2, ..., N-1), \tag{2.7}$$

where $a_i = \frac{\beta}{\sinh(\beta h_{i-1})}$, $b_i = \frac{\beta}{\sinh(\beta h_i)}$, $d_i = \frac{\beta}{\tanh(\beta h_{i-1})}$ and $\beta = \frac{\sqrt{\gamma}}{\epsilon}$, (see [2]).

Theorem 2.1. *The difference scheme* (2.7), *has a unique solution* $\overline{y}$, *where* $\overline{y} = (\overline{y}_0, \overline{y}_1, \overline{y}_2, ..., \overline{y}_{N-1}, \overline{y}_N)$.

Proof. We use a technique from [5]. From (2.7), since $\psi(x,y) = f(x,y) - \gamma y$, we get

$$
\begin{aligned}
a_i\overline{y}_{i-1} - c_i\overline{y}_i + b_i\overline{y}_{i+1} - \ & \frac{1}{\gamma}\left\{\left(\frac{\beta}{\tanh(\beta h_{i-1})} - a_i\right)\left[\overline{f}_{i-1} - \gamma\frac{\overline{y}_{i-1}+\overline{y}_i}{2}\right]\right. \\
& \left. + \left(\frac{\beta}{\tanh(\beta h_i)} - b_i\right)\left[\overline{f}_i - \gamma\frac{\overline{y}_i+\overline{y}_{i+1}}{2}\right]\right\} = 0,
\end{aligned} \tag{2.8}
$$

$$\overline{y}_0 = \overline{y}_N = 0, \ (i = 1, 2, ..., N-1).$$

Let us denote the left-hand side (2.8) with $G\overline{y}$, then (2.8) becomes

$$G\overline{y} = 0.$$

The Fréchet derivate $A := G'(\overline{y})$ is a tridiagonal matrix, and the non-zero elements of this tridiagonal matrix are

$$
\begin{aligned}
a_{i,i} = -c_i - \frac{1}{\gamma} & \left\{\left(\frac{\beta}{\tanh(\beta h_{i-1})} - a_i\right)\left[\frac{1}{2}f_{y_{i-1}}\left(\frac{x_{i-1}+x_i}{2}, \frac{\overline{y}_{i-1}+\overline{y}_i}{2}\right) - \frac{\gamma}{2}\right]\right. \\
& \left. + \left(\frac{\beta}{\tanh(\beta h_i)} - b_i\right)\left[\frac{1}{2}f_{y_i}\left(\frac{x_i+x_{i+1}}{2}, \frac{\overline{y}_i+\overline{y}_{i+1}}{2}\right) - \frac{\gamma}{2}\right]\right\},
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
a_{i,i-1} &= a_i - \frac{1}{\gamma}\left(\frac{\beta}{\tanh(\beta h_{i-1})} - a_i\right)\left[\frac{1}{2}f_{y_{i-1}}\left(\frac{x_{i-1}+x_i}{2}, \frac{\overline{y}_{i-1}+\overline{y}_i}{2}\right) - \frac{\gamma}{2}\right], \\
a_{i,i+1} &= b_i - \frac{1}{\gamma}\left(\frac{\beta}{\tanh(\beta h_i)} - b_i\right)\left[\frac{1}{2}f_{y_i}\left(\frac{x_i+x_{i+1}}{2}, \frac{\overline{y}_i+\overline{y}_{i+1}}{2}\right) - \frac{\gamma}{2}\right].
\end{aligned}
$$

It can be shown that

$$a_{i,i-1} > 0 \ a_{i,i+1} > 0 \text{ and } a_{i,i} < 0.$$

Hence, A is an L matrix. Let us show that A is an M matrix. Now,

$$a_{i,i} - a_{i,i-1} - a_{i-1,1} = -c_i - a_i - b_i < 0. \tag{2.10}$$

Based on (2.10), we have proved that A, is $M-$ matrix. Since A is an $M-$ matrix, $Ae^h \geq me^h$ holds. Now, we obtain that $\|A^{-1}\| \leq \frac{1}{m}$. Now, by the Hadamard Theorem (5.3.11 from [3]), the statement of our theorem follows. ■

$\epsilon \backslash n$	64	128	256	512	1024	
2^{-4}	$1.05e-4$	$2.60e-5$	$6.48e-6$	$1.62e-6$	$4.05e-7$	E_N
	2.01	1.99	2.00	2.00		*Ord*
2^{-6}	$1.17e-4$	$2.77e-5$	$6.69e-6$	$1.65e-6$	$4.06e-7$	E_N
	2.08	2.05	2.02	2.02		*Ord*
2^{-8}	$1.26e-4$	$3.01e-5$	$7.34e-6$	$1.72e-6$	$4.22e-7$	E_N
	2.03	2.07	2.09	2.02		*Ord*
2^{-10}	$1.24e-4$	$3.03e-5$	$7.37e-6$	$1.79e-6$	$4.30e-7$	E_N
	2.04	2.04	2.04	2.04		*Ord*
2^{-12}	$1.24e-4$	$3.03e-5$	$7.39e-6$	$1.80e-6$	$4.49e-7$	E_N
	2.04	1.99	2.04	2.04		*Ord*
2^{-15}	$1.24e-4$	$3.03e-5$	$7.39e-6$	$1.80e-6$	$4.49e-7$	E_N
	2.04	1.99	2.04	2.04		*Ord*

Table 1: Error E_N and convergence rates Ord for approximate solution

3 The numerical example

Example 3.1. Consider the following problem

$$\epsilon^2 y'' = (1+y)(1+(1+y)^2) \quad \text{on} \quad (0,1)\,, \tag{3.1}$$

$$y(0) = y(1) = 0. \tag{3.2}$$

The exact solution of the problem (3.1)−(3.2) is unknown. The nonlinear system of equations is solved by Newton's method with initial guess $y_0 = -1$. Because the exact solution is unknown, we define E_N in the usual way

$$E_N = \max_{0\le i\le N} \left|\overline{y}^{2N}(x_i) - \overline{y}^{N}(x_i)\right|, \tag{3.3}$$

where $\overline{y}^N(x_i)$ and $\overline{y}^{2N}(x_i)$ are the numerical solutions on a mesh with N and $2N$ subintervals, respectively. Also, we define in the usual way the order of convergence Ord

$$\text{Ord} = \frac{\ln E_N - \ln E_{2N}}{\ln 2}. \tag{3.4}$$

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