Evaluating Wireless Sensor Node Longevity through Markovian Techniques

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Abstract

Wireless sensor networks are constituted of a large number of tiny sensor nodes randomly distributed over a geographical region. In order to reduce power consumption, nodes undergo active-sleep periods that, on the other hand, limit their ability to send/receive data. The aim of this paper is to analyze the longevity of a battery-powered sensor node. A battery discharge model able to capture both linear and non linear discharge processes is presented. Then, two different models are proposed to investigate the longevity, in terms of reliability, of sensor nodes with active-sleep cycles. The first model, well known in the literature, is based on the Markov reward theory and on the evaluation of the accumulated reward distribution. The second model, based on continuous phase type distributions and Kronecker algebra, represents the main contribution of the present work, since it allows to relax some assumptions of the Markov reward model, thus increasing its applicability to more concrete use cases. In the final part of the paper, the results obtained by applying the two techniques to a case study are compared in order to validate and highlight the benefits of our approach and demonstrate

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the utility of the proposed model in a quite complex and real scenario.

Keywords: Wireless sensor networks, Longevity, Reliability, Energy consumption, Battery discharge process, Markov reward models, Continuous phase type distributions, Kronecker algebra.

1. Introduction

Wireless sensor networks (WSN) are networks composed of tiny sensors equipped with radio interfaces and distributed over a geographical region. The task of each sensor is to perform measurements and to send data to a node collector or sink. The WSN application areas are numerous and different, ranging from disaster recovery to field monitoring. Interesting applications have also been found in industrial scenarios where hostile environments can preclude the human intervention or the deployment of networking infrastructures.

In the last years, researches on WSN have mainly focused on networking aspects [1, 2] as well as on data management [3]. However, many specific applications introduce strict dependability requirements [4, 5] that are primarily related to data security and reliability [6] and to the longevity of the single node and the WSN as whole [7, 8, 9, 10, 11]. In fact, in this latter case, cheap sensors do not guarantee their functioning over the time and they are normally equipped with low voltage batteries that limit their lifetime or longevity. From such perspective, a WSN is a power constrained system, since nodes run on limited power batteries.

In order to reduce and optimize the energy consumption, a common practice is to switch WSN nodes into a lower powered state, usually identified as
sleep mode, by deactivating the radio equipments when no data have to be transmitted [7]. Since quite often the radio is the highest power consumer subsystem in a WSN node, this allows to save battery and therefore to improve the node lifetime. In this way, by associating the battery discharge to the WSN node aging process as in [7, 8, 9], the node longevity can be expressed, in terms of reliability, as a function of the battery state of charge.

Several authors deal with WSN and power management topics, at different layers and achieving different goals. In the specific context of reliability, an interesting investigation is performed in [9], where a simulation technique for evaluating blind flooding over WSN is proposed. The authors base the evaluation on a realistic and accurate battery model, also considering capacity and recovery effects as well as switching energy, instead of starting from the linear discharge model usually assumed in literature.

The choice of the battery model is of primary importance in evaluating the WSN longevity as also highlighted in [10]. Mathematical models [12] apply empirical equations to address the battery charge/discharge behavior; electrochemical models [13] provide a more detailed representation of the process but, on the other hand, they usually require greater efforts in their evaluation; electrical models [14] are instead mainly used for circuit analysis, simulation and optimization.

Mathematical battery models are usually used in the node/WSN longevity and power management modeling and evaluation. For example, in [11] a new sensor node deployment scheme is proposed to increase the sensor network longevity assuming a non-linear battery discharge model. More recently, [10] proposes a flexible and extensible simulation framework to estimate power
consumption of sensor network applications for arbitrary hardware platforms in which energy and timing parameters are partially obtained through direct measurements.

The problem of WSN longevity and power management has also been investigated through the use of analytical models. For example, in [8] Markov models are used in order to evaluate WSN reliability. In such case the context is slightly different: WSN composed of several nodes are considered and the final model reflects such choice, representing the WSN as a whole and introducing some higher level approximations.

The aim of this work is to analyze the longevity of a single WSN node undergoing cycles of active-sleep periods. Both linear and non linear battery models are taken into consideration, thus proposing and comparing two different analytical techniques. The former is based on a Markov reward model able to capture the linear battery depletion process of a WSN node and to derive the reliability parameters under specific assumptions. The latter is based on continuous phase type distributions (CPHs) and Kronecker algebra and is able to relax some of the assumptions related to the Markov reward solution technique. In particular, non linear battery discharge model and different active-sleep modes sojourn time distributions can be taken into account. Moreover, some numerical problems that could affect the implementation of the first technique can be overcome by our approach.

From an operational/practical viewpoint, the proposed technique can be exploited to perform parametric evaluations on the WSN also taking into account the application requirements, as for example: finding out the appropriate duty cycle to be used for achieving the expected lifetime required
by the application, deriving the expected node’s lifetime, depending on its assigned duty cycle, etc.

The paper is organized as follows: Section 2 introduces and discusses the battery discharge phenomena, also proposing a possible model and its validation. Section 3 describes WSN nodes characterizing the problem under evaluation. In Section 4 a Markov reward model of a WSN node is described, while in Section 5 a technique based on CPHs and Kronecker algebra is detailed. From the evaluation of such models specific results are obtained, compared and discussed in Section 6, while Section 7 concludes the paper with final remarks and future work.

2. Battery discharge model

As stated above, WSN can be considered as power-constrained systems in which the power management problem has to be adequately addressed. It is therefore required to use adequate techniques and models in order to satisfy and optimize the power management, also taking into account the application requirements and specifications. With regards to batteries, that usually provide the WSN power supply system, it is necessary to define adequate models to describe their discharge process.

The battery lifetime or time-to-failure is the time to discharge. Once the battery is exhausted the powered system, in the specific the WSN node, shuts down; therefore, maximizing the battery lifetime is an important goal to be addressed.

Starting from the existing literature [9, 15], one of the most adopted analytical models to represent the discharge process of a battery subjected
to a constant load is based on the Peukert’s law [16] that expresses the capacity of a battery in terms of the rate at which it is discharged: if the rate increases, the available charge capacity decreases. Generally such effect can be quantified by the formula:

\[ T = H \cdot \left( \frac{C}{H \cdot I} \right)^\eta \]  

(1)

that analytically formalizes the relationships among the time of discharge \( T \), expressed in hours, the rated capacity of the battery \( C \) at the specified hour rating \( H \) (hours), expressed in Ampere-hours and the (constant) discharge current \( I \), expressed in Ampere. In (1) \( \eta \) is the Peukert’s exponent or constant, ranging in the interval [1, 2], depending on the chemical process characterizing the battery. The Peukert constant increases with age for any of the battery types, but generally ranges from [1.05, 1.15] for lead-acid batteries, [1.1, 1.25] for gel, [1.2, 1.6] for flooded batteries, [1.06, 1.13] for lithium ion batteries while [1.2, 1.4] characterizes alkaline batteries. A value of \( \eta \) equal to 1 identifies the ideal case, in which the discharge process is linear, i.e., the battery capacity \( C \) linearly decreases with the current \( I \). In such case the actual capacity would be independent of the current.

The Peukert’s law expresses the battery lifetime given an initial capacity. In the proposed modeling technique, we need to know the trend of the battery discharge process with respect to the time. To this end, by inverting eq. (1) we can obtain the relationship for the capacity \( C \) as follows:

\[ C = I \cdot H \cdot \left( \frac{T}{H} \right)^{(1/\eta)} . \]  

(2)
Then, considering $C$ as a function of the time variable $t$, we can express its evolution in time as $c(t)$ from eq. (2):

$$c(t) = c_0 - I \cdot H \cdot \left(\frac{t}{H}\right)^{(1/\eta)}$$ (3)

where $c_0$ is the initial capacity of the battery.

By eq. (3) we can argue that, when a constant load is applied, a time-varying discharge rate commonly characterizes the batteries discharge process. A typical trend of the discharge process in time $c(t)$ is shown in Figure 1.

Since such trend is not linear, it is necessary to recur to an adequate model to represent it. Let us identify the battery useful charge range as $[c_0, c_{min}]$, that we split into $n$ contiguous intervals $[c_i, c_{i+1}]$ of equal size $\frac{c_0 - c_{min}}{n}$, with $i = 0, \ldots, n - 1$. In this way, we discretize the battery capacity into $n + 1$ charge levels with generic value $c_i$ ($i = 0, \ldots, n$), where $c_n = c_{min}$. Since $c(t)$ is usually strictly decreasing, the time instants $t_i$ such that $c(t_i) = c_i$, with $i = 0, \ldots, n$, can be univocally identified. Let $\tau_i = t_{i+1} - t_i$, with
\( i = 0, \ldots, n - 1 \), be the duration of the \( i \)-th time interval, in which the charge assumes values ranging into \([c_i, c_{i+1}]\). By discretizing the value assumed by the charge in \([t_i, t_{i+1}]\) with \(c_i\), we can represent the discharge phenomenon through the \(n + 1\)-state continuous time Markov chain (CTMC) shown in Figure 2 and defined by the stochastic process

\[
B = \{ B(t), t \geq 0 \}
\]

where the state \(b_i\) encodes the \(i\)-th charge interval. Since \(\forall t \in [t_i, t_{i+1}] \Rightarrow c(t) \in [c_i, c_{i+1}]\), \(\tau_i\) can be considered as the sojourn time into the state \(b_i\) and, as a consequence, the transition rate between states \(b_i\) and \(b_{i+1}\) has to be set to \(\lambda_i = \frac{1}{\tau_i}\).

\[
\begin{array}{cccccccccccc}
& & b_0 & \lambda_1 & b_1 & \lambda_2 & b_2 & \lambda_3 & b_3 & \lambda_4 & b_4 & \lambda_5 & \cdots & b_{n} \\
& \lambda_0 & & & & & & & & & & & \\
\end{array}
\]

Figure 2: CTMC representing the battery discharge process of Figure 1.

The introduced model is a CTMC with an absorbing state whose absorption time is a random variable that represents the time-to-discharge of the battery. As a consequence, the distribution of the time to absorption of the CTMC represents the distribution of the time-to-discharge. The CTMC thus obtained describes a single discharge behavior corresponding to a constant load; if different loads have to be considered, as a number of CTMCs equal to the number of different loads have to be specified accordingly. Anyway, all the CTMCs modeling the battery discharge in the different load conditions are characterized by the same number of states if the discretization levels are the same; considering the same battery this is not a restrictive assumption,
since the initial charge level $c_0$ and its minimum $c_{\text{min}}$ are fixed for a given battery. The sojourn time into each state of the chain of Figure 2 has been considered as exponentially distributed because our aim is to characterize the function describing the discharge process of the sensor node battery through a CPH distribution. Such a characterization allows us to approximate a non-Markovian stochastic process with an expanded CTMC that captures the discharge process of the sensor node.

2.1. Validating the battery discharge model

In order to validate the CPH battery discharge model proposed we solved the CTMC depicted in Figure 2 and compared the results against the battery discharge function $c(t)$ obtained by the Peukert’s law expressed in eq. (3). More formally, we obtained an estimation $\overline{c}(t)$ of the $c(t)$ function as the expected value of the stochastic process $B$ of eq. (4) at time $t$:

$$\overline{c}(t) = \sum_{i=0}^{n} c_i \times Pr\{B(t) = b_i\}.$$  \hspace{1cm} (5)

We also evaluated an estimation of the total battery depletion distribution $\overline{F}(t)$ as the probability the CTMC is in the absorbing state, i.e., the probability for the battery to be depleted at time $t$. Such function is compared with the $F(t)$ that is defined as:

$$F(t) = \begin{cases} 1 & \text{if } t \leq T \\ 0 & \text{if } t > T \end{cases}$$  \hspace{1cm} (6)

where $T$ is the battery lifetime expressed in eq. (1).
Figure 3: Battery discharge function (a) and corresponding total depletion time distribution (b) varying the number of states.

Figure 3 shows the obtained results. We referred to an alkaline battery (AA/R6) with Peukert constant $\eta = 1.3$, initial capacity $c_0 = 2500mAh$, constant load $I = 100mA$ and, consequently, $H$ is $25h$. As expected, by increasing the number $n$ of states of the CTMC, we obtained a better approximation. In particular, it can be observed that with a 1000 states-CTMC we obtain a very good approximation of the battery discharge process.

3. Longevity of WSN nodes with active-sleep cycles

In typical WSN applications, wireless sensor nodes are randomly scattered in large geographical regions and it is not always possible to perform node maintenance after the WSN deployment by intervening on the single node. For this reason, sensor nodes have to adapt their behavior to the environmental changes. In particular, each node has to reduce its power consumption in order to increase the WSN lifetime. Sensor nodes are usually powered by low voltage batteries and they are composed of a processing unit, a wireless communication unit, and a sensing unit. Wireless communications
represent the most expensive task a node has to accomplish [7], but even if no data are actually transmitted, the energy consumed by the wireless communication unit is significant. Then, in order to reduce energy consumption, a good strategy is to deactivate the radio equipments when no data have to be transmitted. According to this strategy, nodes periodically go through two different functioning states: active and sleep. When the node is in the active state, it is able to send/receive data while, when sleeping, it is only able to perform off-line tasks such as sensing and data processing.

In order to analyze the battery discharge process of a node with active-sleep cycles, it is possible to associate to each operating mode the corresponding battery discharge function \( c(t) \). Being the discharge process strictly related to the battery type, functions \( c(t) \) associated to different operating modes have the same value of \( \eta \) and \( H \) of eq. (3) and only differ in the discharge current \( I \).

Assuming that the only failure node condition is due to battery depletion, we can study the node longevity by associating the node reliability with the battery charge. Let us assume that, when the battery charge goes below a certain threshold \( c_{\text{min}} \), the node cannot fulfill its tasks and it can be considered failed. Then, it is possible to define the reliability of a sensor node at time \( t \), \( R(t) \) as:

**Definition 1.** The reliability of a sensor node at time \( t \), \( R(t) \), is the probability that its battery is not depleted in the time interval \([0, t]\).

\[
R(t) = Pr\{c(t) \geq c_{\text{min}}\}. \quad (7)
\]
In the following, starting from the battery discharge model presented in Section 2 and knowing the energy consumption parameters associated to the two operating modes and the period of the active-sleep cycles, we show how the reliability function $R(t)$, and then the node longevity, can be evaluated.

This paper has been focused on the reliability distribution of a single sensor node. Once such distribution is obtained it is possible to extend the investigation to the WSN as a whole. Two possible cases can be identified. If the failure events (in this case the battery discharge) are statistically independent the reliability of the whole WSN can be obtained using combinatorial techniques such as fault trees. We focused on this aspect on [17, 18]. If the failure events are statistically dependent the reliability of WSN can be studied using dynamic reliability tools such DRBD [19] or modeling techniques able to capture the interactions among sensors, such as Markovian Agents [20]. We plan to investigate such a second aspect in future works.

4. Markov reward model

Let us assume that the WSN node sojourn times in the active and sleep states are exponentially distributed. Under such assumption, we can model the node activity with the CTMC depicted in Figure 4, defined by the stochastic process

$$X = \{X(t), t \geq 0\}$$

where states $S_a$ and $S_s$ are associated to the active and sleep conditions, respectively, and $\lambda$ and $\mu$ represent the active-sleep and the sleep-active transition rates.

To derive the expression of $R(t)$ from the analysis of the CTMC of Figure
4 we can recur to the Markov reward theory [21]. Let us associate to states $S_a$ and $S_s$ indices $i = 0$ and $i = 1$, respectively, and let us indicate with $S = \{0, 1\}$ the state space associated to the CTMC of the WSN node. Then, we can define a reward function $r : S \rightarrow \mathbb{R}$ in which, for each state $i \in S$, $r(i) = r_i$ represents the reward obtained per unit time spent by process $X$ in that state.

We can associate the reward to the battery consumption by defining $r$ as:

$$r_i = \begin{cases} q_a & i = 0 \\ q_s & i = 1 \end{cases}$$

where $q_a$ and $q_s$ represent the charge absorbed by the sensor per unit time in the active and sleep states, respectively. If the charge absorbed by the node is not constant over the time [22], it is possible to consider reward functions $r(i, t)$ that are also time-dependent on time [21].

Let $Z(t) = r_{X(t)}$ be the system reward rate (i.e., the node charge absorption rate) at time $t$. We can define the accumulated reward $Y(t)$ in the interval $[0, t)$ as:

$$Y(t) = \int_0^t Z(\tau)d\tau.$$  

The value of $Y(t)$ can be interpreted as the charge consumed by the node.
from the activation instant till time $t$. By and indicating with $c_0$ the initial battery charge of the WSN node, from (10) we can express the battery charge level $c(t)$ at time $t$ as:

$$c(t) = c_0 - Y(t).$$

(11)

An interesting measure to compute is the distribution of the accumulated reward that can be expressed as:

$$F(t, y) = Pr\{Y(t) \leq y\}.$$  

(12)

In fact, from (7), (11), and (12) we can derive the expression of $R(t)$ as:

$$R(t) = Pr\{c(t) \geq c_{min}\}$$

$$= Pr\{c_0 - Y(t) \geq c_{min}\}$$

$$= Pr\{Y(t) \leq c_0 - c_{min}\}$$

$$= F(t, c_0 - c_{min}).$$

(13)

Then, to perform a quantitative analysis of $R(t)$, we need to evaluate the distribution of the accumulated reward $F(t, y)$. Such distribution can be computed starting from the analysis of the CTMC described in Figure 4.

4.1. Closed-form solution

A method for evaluating $F(t, y)$ during a finite mission time and assuming the reward rates to be time independent is provided in [23]. If $A = [a_{ij}]$ is the infinitesimal generator of the process $X$, it is possible to indicate with $P = [p_{ij}] = A/\gamma + I$ the stochastic matrix of the associated Poisson process,
where $\gamma \geq \max \{|a_{ij}|\}$ and $I$ is the identity matrix. Then, $F(t, y)$ can be then expressed as:

$$F(t, y) = \sum_{k=0}^{\infty} \left[ \alpha_{0}^{(k)}(y - r_{0}t) + \sum_{h=1}^{k} \sum_{w=0}^{1} \left( \frac{k}{h - 1} \right) \cdot \beta_{0}^{(k)}(w, h) \left( \frac{y - r_{w}t}{t} \right)^{k-h+1} \cdot u(y - r_{w}t) \right] \cdot \frac{(\gamma t)^{k}e^{-\gamma t}}{k!} \tag{14}$$

where

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \tag{15}$$

and coefficients $\alpha_{i}^{(k)}$ and $\beta_{i}^{(k)}(w, h)$ are recursively defined by:

$$\alpha_{i}^{(k)} = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{j=0}^{1} p_{ij} \alpha_{j}^{(k-1)} & \text{if } k > 0 \end{cases} \tag{16}$$

$$\beta_{i}^{(k)}(w, h) = \begin{cases} -\sum_{j=0}^{1} p_{ij} \alpha_{j}^{(k-1)} \cdot \frac{1}{r_{w} - r_{i}} & \text{if } w \neq i, h = k \\ -\sum_{h=1}^{k} \beta_{i}^{(k)}(w, h) \cdot \frac{p_{ij} \beta_{j}^{(k-1)}(w, h)}{r_{w} - r_{i}} & \text{if } w \neq i, h < k \\ -\sum_{w=0}^{1} \beta_{i}^{(k)}(w, 1) & \text{if } w = i, h = 1 \\ \sum_{j=0}^{1} p_{ij} \beta_{j}^{(k-1)}(i, h - 1) & \text{if } w = i, h > 1 \end{cases} \tag{17}$$
To numerically solve eq.(14), the infinite sum has to be truncated to a value $k^*$ that depends on the required error tolerance. An estimation of the error truncation is given by:

$$\varepsilon(k^*) \leq 1 - e^{-\gamma t} \sum_{k=0}^{k^*} \frac{(\gamma t)^k}{k!}$$

(18)

5. The CPH distributions and Kronecker algebra model

Eq. (14) cannot be used to compute the WSN node reliability in case of time dependent reward rates. However, even in the simple case of constant reward rates, the algorithm that can be used to evaluate eq. (14) and to compute the recursive coefficients $\alpha_i^{(k)}$ and $\beta_i^{(k)}(w, h)$ could be subject to numerical problems (overflow, underflow, stiffness). Such issues could prevent eq. (14) to be used for the computation of the node reliability $R(t)$ for large values of $t$.

Moreover, the described technique cannot be used when the relationship between the charge of the battery and the reliability of the node is more complex than that of eq. (7): for example if we assume that there is a non-null probability for the node to fail, even if the battery charge is greater than $c_{\min}$. Finally, the stochastic process $X$ could be more complex than a simple CTMC, being the sojourn times of the node in active and sleep states non-exponentially distributed. As an example, the sojourn times could depend on the load of the node in terms of number of packets it is processing at a specific time instant.

For such reasons we propose an alternative analytical technique based on CPH distributions and Kronecker algebra. It can be used for evaluating the
WSN node reliability relaxing some of the assumptions specified in Section 4.

The state space model depicted in Figure 5 represents the behavior of a sensor node also considering its battery depletion. As in Figure 4, states $S_a$ and $S_s$ are related to the active and the sleep modes of the WSN node under exam, respectively, while the absorbing state $S_f$ represents the node failure due to the battery depletion. Events $e_a$ and $e_s$ are related to the transitions between active and sleep modes, while event $e_f$ represents the failure of the WSN node.

Let us assume that the cumulative distribution functions (CDFs) $F_a(t)$ and $F_s(t)$ are associated to events $e_a$ and $e_s$, while two different time-to-failure distributions $F_{af}(t)$ and $F_{sf}(t)$ are associated to event $e_f$ in states $S_a$ and $S_s$, respectively. $F_{af}(t)$ and $F_{sf}(t)$ are the node lifetime distributions in the two functioning states in isolation, as specified in Section 3. They are obtained from the battery discharge functions of states $S_a$ and $S_s$ as detailed in Section 2, so both of them are represented by $n + 1$ states CTMCs.

Since the WSN node cyclically changes its functioning state triggered by
events $e_u$ and $e_s$, the distribution associated to event $e_f$ changes from $F^a_f(t)$ to $F^s_f(t)$ and vice-versa. Moreover, at the generic change point $\bar{t}$ the battery charge has to be preserved. For example, the changing from $F^a_f(t)$ to $F^s_f(t)$ at $\bar{t}$ means that the battery discharge process of the node continues according to $F^s_f(t)$, thus preserving the battery charge level reached so far, i.e., $F^s_f(\bar{t})$. In terms of the battery model introduced in Section 2, the battery charge level is preserved by the CTMC state reached when $F^a_f(t)$ is disabled at $\bar{t}$, since each state corresponds to a specific charge level.

Based on these considerations, in the following we describe a technique based on CPH distributions and Kronecker algebra for evaluating such model. In this way, the unreliability and the sojourn time distribution of the WSN node in the two functioning modes are represented by CPHs, moving the problem towards the solution of an expanded CTMC. Kronecker algebra is therefore used in order to deal with the well-known state space explosion problem, since this allows the infinitesimal generator matrix of the expanded CTMC is never generated nor stored as a whole, but it can be algorithmically evaluated on-the-fly in a block-wise fashion during the model solution. In this way, only few information about the stochastic process is permanently stored, with consequent memory saving.

5.1. Representing events through CPHs

The use of phase type (PH) distributions dates back to the pioneering work of Erlang on congestion in telephone systems at the beginning of the last century [24] and was popularized in 1981 by Neuts [25] who formally defined a PH distribution as the distribution of the time until absorption in a finite state Markov chain with a single absorbing state. In particular, the
class of CPH distributions is defined over CTMCs.

More specifically, let us consider a CTMC $\chi$ with $\ell$ transient states and a single absorbing state (labeled $\ell + 1$) whose infinitesimal generator matrix $\hat{G} \in \mathbb{R}^{\ell+1} \times \mathbb{R}^{\ell+1}$ is in the form:

$$\hat{G} = \begin{bmatrix} G & U \\ 0 & 0 \end{bmatrix}$$

where $G \in \mathbb{R}^{\ell} \times \mathbb{R}^{\ell}$ describes the transient behavior of the CTMC and $U \in \mathbb{R}^{\ell}$ is a vector grouping the transition rates to state $\ell + 1$. Moreover, let us suppose that the chain is started with an initial probability vector $\pi(0) = [\alpha, \alpha_{\ell+1}]$ such that $\alpha_{\ell+1} = 1 - \sum_{i=1}^{\ell} \alpha_i$. We say that a random variable $T$ is distributed according to the CPH distribution with representation $(\alpha, G)$ and order $\ell$ if its CDF $F_T(t)$ is the probability to reach the absorbing state of $\chi$; $F_T(t)$ can be expressed as:

$$F_T(t) = 1 - \alpha \cdot e^{Gt \cdot 1}, \quad t \geq 0 \quad (19)$$

The theory of PH distributions found several applications in reliability (see Neuts [26], Pérez-Ocón et al. [27, 28, 29] and references therein) and, in general, in the analysis of stochastic models where non-exponentially distributed events are considered. Other interesting and powerful techniques exist for the analysis of such class of non-Markovian models such as supplementary variables, semi-Markov processes, and Markov regenerative models [30], but they usually present severe restrictions on the structure of the manageable systems. On the contrary, the use of PH distributions for the
representation of non-exponentially distributed events \((\text{state space expansion})\) allows to manage a more general class of models.

The state space expansion approach consists in the representation of a non-Markovian stochastic process by mean of a Markov chain defined over an augmented state space \([31]\). Each state in the non-Markovian process is expanded into a set of states within the Markov chain, called \textit{macro-state}, with the purpose to capture the evolution of the non-exponentially distributed events within it. In particular, when CPH distributions are used, the non-Markovian process is represented via an expanded CTMC.

By applying the state space expansion approach to the WSN node under evaluation, we have to represent \(F_a(t), F_s(t), F_{af}(t), \) and \(F_{sf}(t)\) through CPH distributions. In such model, special care has to be devoted to the representation of the node unreliability in the two operating modes, since the charge level has to be preserved at change points.

5.2. \textit{From CPHs to Kronecker algebra}

Let us consider a discrete-state discrete-event non-Markovian model and let \(S\) be the system state space and \(\mathcal{E}\) the set of system events. Following the state space expansion approach, the events are represented by CPH distributions, and the expanded CTMC thus resulting is composed of \(|S|\) macro-states, characterized by a \(|S| \times |S|\) block infinitesimal generator matrix \(Q\) in which:

- the generic diagonal block \(Q_{ii}\) \((1 < i < |S|)\) is a square matrix that describes the evolution of the CTMC inside the macro-state related to state \(i\), and it depends on the possible events occurring in that state;
• the generic off-diagonal block $Q_{ij}$ ($1 < i < ||S||$, $1 < j < ||S||$) describes
  the transition from the macro-state related to state $i$ to that related to
  state $j$, and it depends on the event occurred in state $i$ and the possible
  events able to still occur in state $j$.

The main drawback of the state space expansion approach is the explosion
of the state space. This limits the applicability of the approach thus reducing
its benefits and potentialities. A possible solution to overcome the problem
is to recur to Kronecker algebra [25, 32, 28], a well known and effective
technique able to provide a compact representation of CTMCs. Kronecker
algebra is based on two main operators [33]: Kronecker product ($\otimes$) and
Kronecker sum ($\oplus$). Given two rectangular matrices $A$ and $B$ of dimensions
$m_1 \times m_2$ and $n_1 \times n_2$ respectively, their Kronecker product $A \otimes B$ is a matrix
of dimensions $m_1 n_1 \times m_2 n_2$. More specifically, it is a $m_1 \times m_2$ block matrix
in which the generic block $i, j$ has dimension $n_1 \times n_2$ and is given by $a_{i,j} \cdot B$.
On the other hand, if $A$ and $B$ are square matrices of dimensions $m \times m$ and
$n \times n$ respectively, their Kronecker sum $A \oplus B$ is the matrix of dimensions
$mn \times mn$ written as $A \oplus B = A \otimes I_n + I_m \otimes B$ where $I_n$ and $I_m$ are the
identity matrices of order $n$ and $m$, respectively.

By exploiting Kronecker algebra, matrix $Q$ does not need to be generated
and stored as a whole but can be symbolically represented through Kronecker
expressions and algorithmically evaluated on-the-fly when needed [34, 35]. As
detailed below, the matrix $Q$ blocks have the following form:

$$Q_{ii} = \bigoplus_{1 < e < ||e||} Q_e$$  \hspace{1cm} (20)
Diagonal blocks are computed as Kronecker sums (off-diagonal blocks are computed as Kronecker products) of a series of matrices $Q_e$ each of which is associated to one of the system events and depends on the behavior of such event in the involved states. In this context, operators $\oplus$ and $\otimes$ assume particular meanings. Let us consider two CTMCs $\chi_1$ and $\chi_2$ with infinitesimal generator matrices $Q_1$ and $Q_2$. The matrix resulting by the Kronecker sum of these matrices ($Q_1 \oplus Q_2$) is usually interpreted as the infinitesimal generator matrix of the CTMC that models the concurrent evolution of $\chi_1$ and $\chi_2$. It is important to remark that the resulting CTMC is composed of a set of states that derives from the combination of all the states of the two initial CTMCs.

On the other hand, let us consider a CTMC $\chi_1$ with a single absorbing state and let $U$ be the column vector containing the transition rates to such state. Moreover, let us consider a CTMC $\chi_2$ and let $P$ be a matrix that contains the probabilities to enter into the $\chi_2$ states. The matrix obtained by the Kronecker product of $U$ and $P$ ($U \otimes P$) is usually interpreted as the matrix containing the transition rates to the CTMC resulting from the combination of the absorbing state of $\chi_1$ with all the states of $\chi_2$. For a complete treatment of all the possible forms of matrices $Q_e$, in the case of discrete-time models, see [35]. Here, we apply such technique to the evaluation of a WSN node reliability in order to derive the corresponding $Q_e$ matrices.
In order to represent the WSN node with active-sleep cycles, let us consider the three states model of Figure 5. In terms of Kronecker algebra terms, the expanded CTMC, obtained by introducing the sojourn time distributions in states $S_a$ and $S_s$ and the battery discharge models, is represented by the infinitesimal generator matrix $Q$, that is the $3 \times 3$ block matrix:

$$
Q = \begin{bmatrix}
Q_{00} & Q_{01} & Q_{02} \\
Q_{10} & Q_{11} & Q_{12} \\
Q_{20} & Q_{21} & Q_{22}
\end{bmatrix}
$$

where states $S_a$, $S_s$ and $S_f$ are identified as 0, 1 and 2, respectively.

In the following, $(\alpha_a, G_a)$ and $(\alpha_s, G_s)$ are the CPH representations of the node sojourn time CDFs $F_a(t)$ and $F_s(t)$ in the two functioning states, and $(\alpha_a^f, G_a^f)$ and $(\alpha_s^f, G_s^f)$ are the CPH representations of the node time-to-failure CDFs $F_a^f(t)$ and $F_s^f(t)$.

Let us start with the description of the diagonal blocks of matrix $Q$. In state $S_a$, events $e_s$ and $e_f$ are enabled and can evolve concurrently according with their CPH distributions while event $e_a$ is disabled. For such a reason, matrix $Q_{00}$ is:

$$
Q_{00} = G_s \oplus G_a^f \oplus [0]
$$

where matrix $[0]$ (i.e., a matrix with a single element equal to 0) associated to event $e_a$ is the neutral element for the Kronecker sum operator. Similarly, in state $S_s$ events $e_a$ and $e_f$ are enabled and can evolve concurrently according to their CPH distributions while event $e_s$ is disabled, thus:

$$
Q_{11} = [0] \oplus G_s^f \oplus G_a.
$$
On the other hand, in state \( S_f \) no event is enabled, therefore \( Q_{22} \) is a \( 1 \times 1 \) null matrix.

With regards to off-diagonal blocks of matrix \( Q \), let us start with matrix \( Q_{01} \) associated to the transition from \( S_a \) to \( S_s \). Event \( e_s \) is enabled in \( S_a \) and causes the state transition, becoming disabled in state \( S_s \). Thus, the stochastic process leaves the phases of the CPH associated to event \( e_s \) with the rates specified in \( U_s \). On the other hand, \( e_a \) is disabled in \( S_a \) and enabled in \( S_s \). As a consequence, the stochastic process enters the phases of the CPH associated to \( e_a \) with the probabilities given by vector \( \alpha_a \). Finally, \( e_f \) is enabled in both \( S_a \) and \( S_s \) but changes its distribution from \( F_f^a(t) \) to \( F_f^s(t) \).

Since, as stated above, the CPHs representing \( F_f^a(t) \) and \( F_f^s(t) \) have been specified by coding the battery charge levels in their states, and the charge level has to be preserved changing from \( S_a \) to \( S_s \), the \( (j) \)th state reached by the CPH corresponding to \( F_f^a(t) \) has to be saved by restarting from the same state \( (j) \) of the CPH corresponding to \( F_f^s(t) \). Such behavior can be implemented by using an \( n \times n \) identity matrix \( I_{a \rightarrow s} \):

\[
Q_{01} = U_s \otimes I_{a \rightarrow s} \otimes \alpha_a. \tag{24}
\]

Similarly \( Q_{10} \), representing the transition from \( S_s \) to \( S_a \), has the form:

\[
Q_{10} = \alpha_s \otimes I_{s \rightarrow a} \otimes U_a. \tag{25}
\]

The matrix representing the transition from \( S_a \) to \( S_f \) triggered by the firing of \( e_f \) is:

\[
Q_{02} = e_s \otimes U_f^a \otimes [1]. \tag{26}
\]
since \( e_s \) is enabled in \( S_a \) but it is disabled in \( S_f \) without firing and it is not required to keep memory of the state reached (the corresponding matrix is the column vector \( e_s \) whose elements are equal to 1). Such vector represents the fact that, regardless of the probabilities to be in a stage of the CPH associated to the event \( e_s \), in \( S_f \) the failure event disables all the other events.

Matrix \([1]\) (i.e., a matrix with a single element equal to 1) is the neutral element for the Kronecker product operator and it is associated to \( e_s \) since it does not contribute to the state transition, being disabled in both the involved states and having no memory in the reached state. Event \( e_f \) is associated to \( U_f^a \) since its firing causes the state transition. Similarly, \( Q_{12} \) is given by:

\[
Q_{12} = [1] \otimes U_f^a \otimes e_a.
\] 

Finally, since \( S_f \) is an absorbing state and no event is enabled in it, \( Q_{20} \) and \( Q_{21} \) are null matrices.

In this way, by applying the CPH-Kronecker algebra technique, the stochastic process underlying the WSN node lifecycle considering non-Markovian sojourn times and time-to-failure distributions is reduced to a CTMC that can be easily solved by applying a classical transient solution method (e.g., the uniformization method) for the computation of the WSN node reliability \( R(t) \).

6. Numerical results

In this section, we present some numerical results obtained by using the described techniques. First of all, we provide a comparison between the Markov reward model and the CPH-Kronecker model in order to demonstrate
the equivalence of the two approaches. Then, by evaluating the WSN node reliability function $R(t)$, we show how to apply the CPH-Kronecker technique in order to derive important parameters that can be used during the setup phase of a WSN.

6.1. Techniques comparison

![Figure 6: Node reliability $R(t)$ computed with the Markov reward approach and the CPH-Kronecker approach.](image)

In order to demonstrate the equivalence between the modeling techniques described in the previous sections, we need to take into consideration a scenario that can be analyzed by mean of the Markov reward model. As we have already discussed, only a linear battery model can be evaluated through the accumulated reward distribution closed-form equation (14), meaning that constant reward rates have to be associated to the WSN node active and sleep modes. Moreover, due to some numerical issues encountered in the implementation of the Markov reward model technique and specifically of eq. (14), it is not possible to evaluate the WSN node reliability for large values
of \( t \) and therefore it is not possible to validate the CPH-Kronecher algebra
technique by the previously analyzed scenario.

We thus consider a WSN node characterized by a linear battery discharge
of 10 s in the active mode of and 20 s in the sleep one. The WSN node
lifecycle is regulated by the CTMC depicted in Figure 4 with \( \lambda = 1.0 \ \text{s}^{-1} \)
and \( \mu = 0.2 \ \text{s}^{-1} \). In order to apply the CPH-Kronecher technique the WSN
node battery linear discharge process has been approximated by a 1000-phase
CPH.

Figure 6 reports the comparison between the WSN node reliability func-
tion \( R(t) \) computed with the Markov reward model and with the CPH-
Kronecker technique varying the number of the CPH phases \( n \). It can be
observed that, in the case of \( n = 1000 \), the two curves are totally overlapped
and small differences are present only in the two knees of the curves, thus
demonstrating the effectiveness of the CPH-Kronecker technique and also
validating it.

6.2. Reliability analysis

Once the reliability model has been validated, it is possible to use it in
the evaluation of the WSN sensor node longevity as specified in Section 3
and more specifically by eq. (7). We apply the CPH-Kronecker technique
described in Section 5. The WSN node is therefore characterized by the ac-
tive and sleep states, and it periodically switches between the two operating
modes. The proposed technique is not strictly related to a particular appli-
cation or communication protocol, and it is general enough to be adopted at
large. For this reason, we do not focus on a specific duty cycle period and
the values of parameters \( \lambda \) and \( \mu \) are specified in order to provide a full nu-
merical example. In particular, to derive the expression of $R(t)$, we assume that the stochastic processes associated to events $e_a$ and $e_s$ of Figure 5 are exponentially distributed with rates $\lambda = 3600 \, h^{-1}$ and $\mu = 60 \, h^{-1}$, meaning that the node switches from active to sleep with a mean time of 1s, while from sleep to active with a mean time of 60 s.

The battery discharge process has been characterized by using the same battery parameters listed in Section 2.1. To differentiate the discharge processes in the active and sleep modes we have to evaluate the corresponding energy consumptions in terms of the discharge current $I$. Starting from empirical data obtained by experiments on MeshNetics MeshBean2 boards, we observed that such WSN nodes in the active mode absorb a current $I_a = 100 \, mA$, while in the sleep mode a current $I_s = 20 \, mA$.

With regard to the CPH-Kronecker model, we can consider the representation of the battery discharge process shown in Figure 2 as a specific CPH. As discussed in Section 2, the two CTMCs corresponding to the discharge from both active and sleep modes have the same number of states or phases identifying and discretizing the battery charge level reached. They differ only in the rates between the phases. Thus, to the same $i$–th phase of the two CPHs. In such a way the CPH-Kronecker technique described in Section 5 can be applied, keeping memory of the charge level reached by saving the corresponding CPH phase in the transitions between macrostates involving active-sleep and sleep-active mode switching.

The results obtained by evaluating the reliability function $R(t)$ through a 1000 phases CPH model are shown in Figure 7, in the specific the non linear discharge curve. The resulting function has been compared against
the one obtained by considering a linear discharge, using the same battery parameters of the non linear discharge case and applying the ideal Peukert constant $\eta = 1$. This latter function corresponds to the linear discharge curve plotted in Figure 7. The two curves show a similar trend is not really a step-shaped function, due to the stochastic nature of the model evaluated, corresponding to a Markov modulated active-sleep cycles process.

Comparing the curves, we can argue that the two models are really different. In numerical terms we can compute the mean time to failure (MTTF) that can be considered as an estimation of the expected lifetime of a sensor node. We obtain that the expected lifetime of the linear case is 117.291 h, while in the non linear case it is 181.435 h. This allows us to appreciate the differences of a linear discharge model approximation with respect to the more realistic Peukert model. Even though the two trends are similar, we can certainly state that in such specific case the linear approximation is not the right one. This is an important goal achieved by our technique that
encourages its use as a more accurate alternative to the evaluation of the accumulated reward distribution.

Figure 8: Expected lifetime varying the active-sleep transition rate $\lambda$.

In order to highlight how the knowledge of the reliability function $R(t)$ can help a designer during the setting phase of a WSN, we evaluate the trend of the expected lifetime by varying the active-sleep transition rate $\lambda$ in the range $[360, 9000]$ $h^{-1}$, using the non linear Peukert discharge model.

The results thus obtained, shown in Figure 8 considering $1/\lambda \in [0.4, 10] s = [1/9000, 1/360] h$, demonstrate that $1/\lambda$ (the mean time of the active-sleep CDF) has a significant impact on the WSN node power consumption. Moreover, a sub-linear trend can be observed, meaning that increasing $1/\lambda$ has not a proportional effect on the power consumption, and therefore has to be carefully evaluated when specifying a WSN power management strategy.
7. Conclusions and future work

In this paper, we have investigated the longevity of a battery-powered WSN node characterized by active-sleep cycles. In particular, the reliability function and the expected lifetime have been taken into consideration as evaluation parameters. First of all, an analytical model for the battery discharge process has been specified. Such model is able to describe both linear and non-linear discharge processes and has been validated using the well known Peukert’s law. Then, a Markov reward model has been described for the analytical evaluation of a WSN node reliability in the case of linear discharge. Finally, as the main contribution of the present work, a new technique based on the use of CPH distributions and Kronecker algebra has been introduced in order to relax some of the assumptions on which the Markov reward model relies. In particular, such technique allows to characterize a sensor node through non-linear battery discharge processes and non-exponentially distributed sojourn times in the active and sleep states.

The CPH-Kronecker algebra technique has been compared vs. the Markov reward approach in a simple case study in order to mutually validate both techniques. Therefore, by evaluating a more complex scenario considering both linear and non-linear battery discharges, we can argue that a linear approximation of the more realistic non-linear behavior provides wrong results. In order to further demonstrate such thesis, an in-depth analysis by varying the active-sleep rate has been performed, with the aim of evaluating the impact of such parameter on the expected lifetime of a node. By this, a sub-linear trend has been observed demonstrating how the precision in the computation of the expected lifetime offered by our technique is important.
during the WSN setting phase.

The obtained results strongly encourage future work. In particular, since the Kronecher algebra allows to overcome the state space expansion problem, the study can be extended to a WSN composed of several sensors with a complex topology and in presence of redundant nodes, specifically focusing on dependability aspects. A possible solution in such cases could be the adoption of hierarchical models: for example by considering combinatorial/high level models to represent the network on top of a model representing the single node, based on the proposed technique. Moreover, the impact of unreliable wireless links on the WSN longevity can be an interesting parameter to evaluate, as well as causes of failures not related to the battery discharge process.

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