AUTOMATIC CONTINUITY AND REPRESENTATION OF
GROUP HOMOMORPHISMS DEFINED BETWEEN GROUPS OF
CONTINUOUS FUNCTIONS

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Abstract. Let $C(X, G)$ be the group of continuous functions from a topological space $X$ into a topological group $G$ with the pointwise multiplication as the composition law. We investigate to what extent the group structure of $C(X, G)$ determines the topology of $X$. More generally, when the existence of a group homomorphism $H$ between the groups $C(X, G)$ and $C(Y, G)$ implies that there is a continuous map $h$ of $Y$ into $X$ such that $H$ is canonically represented by $h$. Along this line, we prove that, for any topological group $G$ and compact spaces $X$ and $Y$, every non-vanishing $C$-isomorphism (defined below) $H$ of $C(X, G)$ into $C(Y, G)$ is automatically continuous and can be canonically represented by a continuous map $h$ of $Y$ into $X$. Some applications to specific groups and examples are given in the paper.

1. Introduction

For a topological space $X$ and a topological group $G$, let $C(X, G)$ be the group of continuous functions from $X$ into $G$ under pointwise multiplication. In this paper we are concerned with the following question: given a topological group $G$ and topological spaces $X$ and $Y$, which homomorphisms defined between the groups $C(X, G)$ and $C(Y, G)$ are represented by continuous maps defined between the spaces $Y$ and $X$? The literature along this line of research is vast and the problem above has been considered for several specific groups. For example, there are classical results when $G$ is the field of real numbers or complex numbers (in [7]), and it is also well-known in case $G$ is a Banach space. Moreover, this question has been studied when $G$ is a non-Archimedean field (see [1]) or the group of integers (in [4],[8], [10]). Nevertheless, little is known if $G$ is assumed to be a topological group. As far as we know, this question has been proposed and treated initially by Yang in [13] and [14], but there is an initial mistake in his approach that invalids the main result obtained in that paper ([15]). In a subsequent paper [16], Yang imposed much stronger and restrictive conditions in order to obtain the representation of a group homomorphism defined between groups of continuous functions. However, the question of which less restrictive homomorphisms of $C(X, G)$ into $C(Y, G)$ are represented by continuous maps of $Y$ into $X$ remains unanswered.

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In a different context, groups of group-valued continuous functions have been considered by many authors, using different approaches. For instance, Varopoulos [12] has investigated the group $C(X, T)$, of all continuous functions on a compact space $X$ into the group of complex numbers with modulus one, in connection to certain questions of interpolation of functions in locally compact Abelian groups. More recently, groups of functions taking values in the ring of integers or a non-Archimedean fields have also been used to respond to open questions in Abelian groups (see [4],[8],[10]). In a different direction, Carey and Grundling [5] have proved the amenability of some groups of continuous functions that take value in the unitary group $U(n)$. On the other hand, groups of scalar-valued (resp. vector valued) continuous or measurable functions are common place in functional analysis. Thus, it seems clear the interest of obtaining a better understanding of the properties of the groups of continuous functions. Here, we investigate certain group homomorphisms $H : C(X, G) \to C(Y, G)$ defined between groups of continuous functions. We prove that under fairly general conditions the homomorphism $H$ is automatically continuous and can be represented by means of a continuous function $h$ of $Y$ into $X$. Some applications to specific groups and examples are also given in the paper.

2. Preliminaries and basic results

All spaces are assumed to be Hausdorff. We consider a compact topological space $X$, a topological group $G$ and some notions which were introduced in Yang's articles ([13] and [14]).

From now on, if $x \in X$, we will denote by $\delta_x$ the evaluation map, that is,

$$\delta_x : C(X, G) \to G$$

$$f \mapsto f(x),$$

which will appear throughout this work. Its kernel will be denoted by $M_x$, that is,

$$M_x := \{f \in C(X, G) : f(x) = 1_G\}.$$

For any $f \in C(X, G)$, we denote $N(f) = \{x \in X : f(x) = 1_G\}$. If $g \in G$, the symbol $\overline{g}$ designates the constant function which maps every point to the constant $g$.

**Definition 2.1.** Let $(X, G)$ be a pair of a topological space $X$ and a topological group $G$. We say that $(X, G)$ is $G$-regular, if whenever $C$ is a closed subset of $X$ and $x \notin C$, then given $\alpha \in G$, $\alpha \neq 1_G$, there is $f \in C(X, G)$ such that $C \subseteq N(f)$ and $f(x) = \alpha$.

In the sequel, all groups are assumed to be Hausdorff and with cardinality greater or equal than 2, and all pairs of the form $(X, G)$ are assumed to be $G$-regular.

**Remark 2.2.** If $X$ is a completely regular space and $G$ is path connected ($G = \mathbb{T}$, for instance), then $(X, G)$ is automatically $G$-regular. Moreover, when $X$ is zero dimensional, the pair $(X, G)$ is trivially $G$-regular for every topological group $G$. In particular, this implies that for any $\{p, q\} \subset X$ with $p \neq q$, we have $M_p \neq M_q$. 
Definition 2.3. (1) A normal subgroup $M$ of $C(X,G)$ is called $G$-filter, if \( \{N(f) : f \in M\} \) has the finite intersection property.

(2) A normal subgroup $M$ of $C(X,G)$ is called $G$-ultrafilter if it is maximal with respect to the inclusion of $G$-filters.

Remark 2.4. If $X$ is compact, for every $G$-filter $M$ of $C(X,G)$ the family $\mathcal{U} := \{N(f) : f \in M\} \subseteq X$ has the finite intersection property. Therefore, $\bigcap \{N(f) : f \in M\} \neq \emptyset$.

Now we state some preliminary results that will be needed in the up-coming sections:

Proposition 2.5. Let $M$ be a $G$-filter. Then $M$ is a $G$-ultrafilter if and only if for all $f \in C(X,G)$ such that $N(f) \cap N(f_1) \cap \ldots \cap N(f_n) \neq \emptyset$ for all $\{f_1, \ldots, f_n\} \subseteq M$ it holds $f \in M$.

Proof. ($\Leftarrow$) This implication is clear.

($\Rightarrow$) Let $\mathcal{U}$ denote the normal subgroup generated by $\{f\} \cup M$. We have that an arbitrary element $u$ of $\mathcal{U}$ can be expressed in the form

$$u = h(h_1 f_1 h_1^{-1} f_1 \ldots h_n f_n h_n^{-1} f_n) h^{-1}$$

with $\{f_1, \ldots, f_n\} \subseteq M$, $\{h\} \cup \{h_1, \ldots, h_n\} \subseteq C(X,G)$ and $\{\epsilon_1, \ldots, \epsilon_n\} \subseteq \mathbb{Z}$. From this fact, it is readily verified that $\mathcal{U}$ is a $G$-filter. Thus, we obtain $\mathcal{U} = M$, which yields $f \in M$.

Corollary 2.6. The subgroup $M_p$ is a $G$-ultrafilter for all singleton $p$ in $X$.

Proof. Let $f$ be in $C(X,G)$ such that $N(f) \cap N(f_1) \cap \ldots \cap N(f_n) \neq \emptyset$ for all $\{f_1, \ldots, f_n\} \subseteq M_p$. Assuming that $f(p) = \alpha \neq 1_G$, we set $g = \alpha^{-1} f$. Clearly, $g$ is in $M_p$ and $N(f) \cap N(g) = \emptyset$. This is a contradiction, which completes the proof.

Proposition 2.7. Let $X$ be a compact topological Hausdorff space and $G$ a Hausdorff topological group. Then for every $G$-ultrafilter $\mathcal{U} \subseteq C(X,G)$ there exists $p \in X$ such that $\mathcal{U} = M_p$.

3. Main Results

In this section, we assume that $X$ and $Y$ are compact topological Hausdorff spaces and $G$ a Hausdorff topological group. From here on, we will suppose that there is a homomorphism

$$H : C(X,G) \to C(Y,G),$$

where, when $H$ is further assumed to be continuous, the groups $C(X,G)$ and $C(Y,G)$ are endowed with the topology of the uniform convergence. Assuming $Y$ to be a singleton, we will identify the group $G$ to a group of the form $C(Y,G)$. For every $\alpha \in G$, $\alpha$ will denote the constant map from a topological space $X$ (or $Y$) into $G$, and $G \subseteq C(X,G)$ (or $C(Y,G)$) will be the set of all these constant functions. We now introduce some definitions that will be needed in the rest of the paper.
**Definition 3.1.** We say that \( H \) commutes with continuous endomorphisms of \( G \) if, for all \( \theta \in \text{End}(G) \), \( f \in C(X,G) \), we have \( H(\theta \circ f) = \theta \circ (Hf) \).

**Definition 3.2.** We say that \( H \) is a \( C \)-homomorphism if there is a continuous cross section \( \psi \), which is also a group homomorphism, of \( G \subseteq C(Y,G) \) into \( C(X,G) \).

We recall that \( \psi : \overline{G} \rightarrow C(Y,G) \) is a cross section when the mapping \( H \circ \psi \) is the identity on \( \overline{G} \). Observe that when \( H \) preserves the constant functions (i.e., \( H(\alpha) = \alpha \) for all \( \alpha \in G \)) or \( H^{-1} \) is a continuous homomorphism, we have that \( H \) is a \( C \)-homomorphism. On the other hand, it follows from the definition that the range of a \( C \)-homomorphism \( H \) always contains \( \overline{G} \).

**Definition 3.3.** \( H \) is non-vanishing if it preserves finite families of functions with disjoint neutral sets. That is to say, for all \( \{f_1, \ldots, f_n\} \subseteq C(X,G) \) with \( N(f_1) \cap \ldots \cap N(f_n) = \emptyset \), we have \( N(H(f_1)) \cap \ldots \cap N(H(f_n)) = \emptyset \).

The following result is essential in order to define the continuous mappings that are associated to homomorphisms commuting with endomorphisms of \( G \).

**Proposition 3.4.** Let \( X \) be a compact Hausdorff space and \( G \) an arbitrary topological group such that the pair \((X,G)\) is \( G \)-regular. If a \( G \)-filter \( M \) of \( C(X,G) \) is the kernel of a \( C \)-homomorphism \( \phi : C(X,G) \rightarrow G \) and \( \phi \) commutes with the continuous endomorphisms of \( G \), then there exists a unique \( p \in X \) such that \( M \subseteq M_p \).

**Proof.** Since the \( G \)-filter \( M \) is contained in some \( G \)-ultrafilter, the existence of a point \( p \in X \) such that \( M \subseteq M_p \) is clear by Proposition 2.7. In order to prove the uniqueness, we assume that there are \( p, q \in X \), \( p \neq q \), such that \( M \subseteq M_p \cap M_q \), with \( M \not\subseteq M_p \) and \( M \not\subseteq M_q \). We notice that the evaluation map

\[
\delta_q : C(X,G) \rightarrow G \\
f \mapsto f(q)
\]

is onto even if we restrict \( \delta_q \) to \( M_p \). Indeed, Let \( g \in G \), \( g \neq 1_G \). As \( p \) and \( q \) are different points of \( X \), we can find some open neighborhoods \( U_p \) of \( p \) and \( U_q \) of \( q \) such that \( U_p \cap U_q = \emptyset \). Due to the property of the pair \((X,G)\) of being \( G \)-regular, we can find a function \( f \in C(X,G) \) such that \( f(X \setminus U_q) = \{1_G\} \) and \( f(q) = g \).

This map is continuous and verifies that \( f \in M_p \) and \( f(q) = \delta_q(f) = g \). Therefore, due to the first isomorphism theorem, we have that

\[
\frac{M_p}{M_p \cap M_q} \cong G.
\]

By hypothesis, there is a \( C \)-homomorphism \( \phi : C(X,G) \rightarrow G \), which, as a consequence, is also surjective. Hence, by the first Isomorphism Theorem,

\[
\frac{C(X,G)}{M} \underbrace{\phi \circ}_{\cong} G.
\]
At this point, we want to introduce some useful notation. By \( \overline{\delta}_q \), we denote the quotient mapping
\[
\overline{\delta}_q : \frac{C(X,G)}{M_q} \to G
\]
\[
fM_q \mapsto f(q),
\]
which is an isomorphism. Hence, we have
\[
G \cong \frac{C(X,G)}{M_q} \cong \frac{C(X,G)}{M_q} \cong \frac{C(X,G)}{\phi_M(M_q)}
\]
In conclusion, the information we get from this chain of isomorphisms is that
\[
G \cong \frac{G}{\phi_M(M_q)} \text{ and } \phi_M(M_q) \neq 1_G.
\]
Moreover, it follows that \( \phi_M(M_q) \neq G \). We have the following commutative diagram:
\[
\begin{array}{ccc}
C(X,G) & \xrightarrow{\delta_q} & G \\
\pi_M \downarrow & & \uparrow \overline{\delta}_q \\
\frac{C(X,G)}{M} & \xrightarrow{\pi_{M_q}} & \frac{C(X,G)}{M_q}
\end{array}
\]
where \( \pi_M \) and \( \pi_{M_q} \) are canonical quotient mappings. Now, we restrict the last diagram to \( M_p \). So we have
\[
M_p \xrightarrow{\delta_q} G
\]
\[
\pi_M \downarrow \quad \uparrow \overline{\delta}_q \\
M_p \cong M_q \quad \cong M_p \cap M_q
\]
Finally, in order to complete the proof, we add one more branch to the last diagram:
\[
\begin{array}{ccc}
M_p & \xrightarrow{\pi_M} & M_q \\
\phi_M = 1 & \xrightarrow{\phi_M} & \frac{M_p}{M_p \cap M_q} \\
\phi_M(M_q) & \xrightarrow{\phi_M^{-1}} & \frac{M_p}{M_p \cap M_q} \\
& & \xrightarrow{\overline{\delta}_q} G
\end{array}
\]
Consider the following chain of mappings:
\[
\phi_M(M_q) \xrightarrow{\phi_M^{-1}} M_q \xrightarrow{\pi_{M_q}} \frac{M_p}{M_p \cap M_q} \xrightarrow{\overline{\delta}_q} G,
\]
Replacing \( q \) by \( p \) in (1) and (2), we have \( \phi_M(M_q) \subsetneq G \). Hence, there is \( \alpha \in G \setminus \phi(M_p) \). Nevertheless, due to the surjectivity of the mappings in (3) there is \( \beta \in \phi_M(M_q) \) such that \( (\overline{\delta}_q \circ \pi_{M_q} \circ \phi_M^{-1})(\beta) = \alpha \). Let \( \psi : \overline{G} \to C(X,G) \)
be a continuous cross section for $\phi$. Then $\psi(\beta) \in \phi^{-1}(\beta)M$ and, consequently, 
$(\pi_M \circ \psi)(\beta) = \phi_M^{-1}(\beta)$. Hence, $(\delta_q \circ \pi_M \circ \pi_M \circ \psi)(\beta) = \alpha$.

So we have a continuous homomorphism $\Psi : G \to G$, defined by $\Psi = \delta_q \circ \pi_M \circ \pi_M \circ \psi$, such that $\Psi(\beta) = \alpha$. On the other hand, there is $f \in M_p$ with $\beta = \phi_M(fM) = \phi(f)$. This yields,

$$\alpha = \Psi(\beta) = \Psi(\phi(f)).$$

Since $\phi$ commutes with the continuous endomorphisms of $G$, we have

$$\alpha = \phi(\Psi \circ f).$$

Now, $\Psi \circ f \in M_p$ whereas $\alpha$ was supposed to verify that $\alpha \in G \setminus \phi_M(M_p)$. Thus, we have reached a contradiction, which completes the proof. $\square$

**Corollary 3.5.** Let $X$ be a compact Hausdorff space and $G$ a topological group. If a $G$-filter $M$ of $C(X,G)$ is the kernel of a homomorphism $\phi : C(X,G) \to G$ such that $\phi(\overline{c}) = c$, for all $c \in G$, then there exists a unique $p \in X$ such that $M \subset M_p$.

**Proof.** Since, the $G$-filter $M$ is contained in some $G$-ultrafilter, the existence of a point $p \in X$ such that $M \subset M_p$ is clear. On the other hand, the identity mapping defined from $\overline{G}$ into $C(X,G)$ yields a continuous cross section for $\phi$. Thus, in order to apply Proposition 3.4, we must verify that $\phi$ commutes with endomorphisms of $G$. Moreover, it will suffice to show that $\theta \circ f \in M$, whenever $f \in M$, for all $\theta \in End(G)$. Now, suppose that $\theta \circ f \notin M$. Then, we have $\phi(\theta \circ f) = \beta \neq 1_G$. Therefore, $(\theta \circ f)\beta^{-1} \in M$ and, as a consequence, $(\theta \circ f)\beta^{-1} \in M_p$. This fact implies that $(\theta \circ f) \notin M_p \supseteq M$, which completes the proof. $\square$

We notice that Proposition 3.4 and Corollary 3.5 fix the proof of [13, Remark 5], which was incomplete (see [15]). Hence, the statement of Yang’s [13, Theorem 6] is in the end correct. In our principal result, Theorem 3.6, we prove the continuity of certain $C$-homomorphisms and we also obtain a canonical representation for them under certain circumstances. Firstly, we recall some basic definitions which will used along the rest of the paper. A topological group $G$ is called Čech complete

Next, it follows the main result of this paper, where it is proved the continuity and representation of certain group homomorphisms, defined between spaces of continuous functions.

**Theorem 3.6.** Let $X$ and $Y$ be compact Hausdorff spaces and $G$ a Hausdorff topological group. Let $H$ be a non-vanishing $C$-homomorphism of $C(X,G)$ into $C(Y,G)$ that commutes with the continuous endomorphisms of $G$.

(a) Then there exist mappings $h : Y \to X$ and $w : Y \to End(G)$ such that

$$w[y](Hf(y)) = f(h(y))$$

for every $f \in C(X,G)$ and $y \in Y$. 
(b) Furthermore, if the cross section associated to $H$ is also non-vanishing, the homomorphism $H$, restricted to the constant functions over $X$, is continuous, and $G$ is also a Čech-complete group, then there exist continuous mappings $h : Y \to X$ and $w : Y \to \text{Aut}(G)$ such that

$$(Hf)(y) = w[y]^{-1}(f(h(y))),$$

and $H$ is also continuous.

Proof. (a) Let us fix some point $y \in Y$; if we compose $H$ with the evaluation map $\delta_y$, we obtain:

$$\delta_y \circ H : C(X, G) \to G$$

such that, if $f \in C(X, G)$, then $(\delta_y \circ H)(f) = (Hf)(y)$. We will denote by $M^y$ the kernel of $\delta_y \circ H$, that is, $M^y := \{f \in C(X, G) : (Hf)(y) = 1_G\}$. Our first goal is to find a unique point $p \in X$ such that $M^y \subseteq M_p$. By Proposition 2.7, in order to prove the existence of such a $p \in X$, it suffices to verify that $M^y$ is a $G$-filter of $C(X, G)$. That is, whether for every finite subset $F \subseteq M^y$ it holds that $\{N(f) : f \in F\}$ has non empty intersection. Proceeding by contradiction, we assume WLOG that there are two functions $f, g \in M^y$ such that $N(f) \cap N(g) = \emptyset$. As $H$ is non-vanishing, we obtain that $N(Hf) \cap N(Hg) = \emptyset$. On the other hand, we know that $y \in N(Hf) \cap N(Hg)$, which is a contradiction. Hence $M^y$ is a $G$-filter and, as a consequence, it is contained in a $G$-ultrafilter $\mathcal{U}$ of $C(X, G)$. Applying Proposition 2.7 to $\mathcal{U}$, there is $x \in X$ such that $\mathcal{U} = M_x$. So, we have $M^y \subseteq M_x$. On the other hand, it is readily verified that $\delta_y \circ H$ is a $C$-homomorphism. Thus, Proposition 3.4 yields the uniqueness of the point $x$ with the property above. Therefore, we can define the function $h : Y \to X$ so that $h(y)$ is the unique point in $X$, satisfying $M^y \subseteq M_{h(y)}$. Now, take $f \in C(X, G)$ and $y \in Y$. For the sake of simplicity, let us denote by $\phi$ the map $\delta_y \circ H$ and let $\alpha$ denote the singleton $\phi(f) \in G$. If $\Psi$ is the continuous cross section associated to $H$ and we take $g \in C(X, G)$ to be $\Psi(\alpha)$, then we have $\phi(f \cdot g^{-1}) = 1_G$ and, since $M^y \subset M_{h(y)}$, this yields $f(h(y)) = g(h(y))$. Thus, $f(h(y)) = \Psi(\alpha)(h(y))$ or, equivalently, $f(h(y)) = (\delta_{h(y)} \circ \Psi)(\phi(f))$. Now, with some notational abuse, define $w[y] = \delta_{h(y)} \circ \Psi$. Using that $M^y \subset M_{h(y)}$, it is readily seen that $w[y]$ is well defined and belongs to $\text{End}(G)$. This completes the proof.

(b) By part (a), we know that there are mappings $h : Y \to X$ and $w : Y \to \text{End}(G)$ such that

$$(4) \quad w[y](Hf(y)) = f(h(y))$$

for every $f \in C(X, G)$ and $y \in Y$. We recall that $w$ has the following structure: $w[y] = \delta_{h(y)} \circ \Psi$. Let us see that for every $y \in Y$, $w[y]$ is an automorphism of $G$. Choose some $y \in Y$.

- $w[y]$ is surjective:
  Let $g \in G$, so we define $g' := (H\varnothing)(y) \in G$. If we use the representation (4) of $H$, we obtain that

$$w[y](g') = w[y](H\varnothing(y)) = \varnothing(h(y)) = g$$
• $w[y]$ is injective:  
  Let $c \in G$ be such that $w[y](c) = 1_G$. As $w[y]$ has the structure $\delta_h(y) \circ \Psi$, we have that 
  $$\Psi(\mathfrak{r})(h(y)) = 1_G,$$
  that is, $N(\Psi(\mathfrak{r})) \neq \emptyset$. We are supposing that $\Psi$ is non-vanishing, therefore $N(\mathfrak{r}) \neq \emptyset$ and $\mathfrak{r} = 1_G$, so $w[y]$ is injective.

  Hence, for every $y \in Y$, $w[y] \in \text{Aut}(G)$, and as a consequence, we are able to construct its inverse homomorphism which is also an automorphism. Then, from (4), it can be deduced that
  \begin{equation}
  (Hf)(y) = w[y]^{-1}(f(h(y))),
  \end{equation}
  for every $y \in Y$. What follows is the proof of the continuity of $w[y]^{-1}$ for every $y \in Y$. Indeed, we notice that for every $g \in G$,
  \begin{equation}
  (H\mathfrak{g})(y) = w[y]^{-1}(g) \quad \text{(in (5))}.
  \end{equation}

  So, as $H\mathfrak{g}$ is continuous by hypothesis, one concludes that $w[y]^{-1}$ is also continuous.

  With the help of the last assertion, we will prove that $H$ is continuous with respect to the topology of the pointwise convergence. Let $(f_n)_n \subseteq C(X,G)$ be a convergent sequence to some $f \in C(X,G)$. We know that for every $y \in Y$, the automorphisms $w[y]^{-1}$ is continuous, so we can choose some open neighborhoods $U$, $V$ of $1_G$ in $G$ such that
  $$w[y]^{-1}(U) \subseteq V.$$

  On the other hand, for $U \in \mathcal{N}(1_G)$ there exists $n_0$ such that
  $$f(h(y))f_n(h(y))^{-1} \in U \quad \forall n \geq n_0.$$

  At this manner, $w[y]^{-1}(f(h(y))f_n(h(y))^{-1}) \in V$ for every $n \geq n_0$, an this implies that
  $$\lim_{n \to \infty} (Hf)(y)(Hf_n)(y)^{-1} \in V \quad \forall n \geq n_0.$$

  Therefore, $(Hf_n)_n$ converges pointwisely to $Hf$. Let us consider now the map
  $$w^{-1} : Y \longrightarrow \text{Aut}_p(G),$$
  which takes every $y \in Y$ to the element $w[y]^{-1}$. Besides, we know that $w[y]^{-1}(g) = (H\mathfrak{g})(y)$. Let $(y_i)_i \subseteq Y$ be a convergent net to some $y_0 \in Y$. Then, as $H\mathfrak{g} \in C(Y,G)$, we have that
  $$(H\mathfrak{g})(y_i) \rightarrow (H\mathfrak{g})(y_0),$$

  and therefore, $w[y_i]^{-1}(g) \rightarrow w[y_0]^{-1}(g)$. So, $w^{-1}$ is continuous with respect to the topology of the pointwise convergence. Hence, $w[y]^{-1}(Y)$ is a compact subset of $\text{Hom}_p(G,G)$. By the Corson-Glicksberg’s Theorem (in [6]), which asserts that, whenever every closed subgroup of a group $G$ is a Baire space, then $A \subseteq \text{Hom}_c(G,K)$ is compact if, and only if is compact in $\text{Hom}(G_D,K)$, where $K$ is a topological group and $G_D$ is $G$ endowed with the discrete topology, we obtain that $w^{-1}(Y)$ is compact in $\text{Hom}_c(G,G)$, due to the property of $G$ of being a Čech-complete group. Hence, $w^{-1}(Y)$ is equicontinuous over $G$ and besides, $H$ is continuous with respect to the uniform convergence topology. Indeed, given an
open neighborhood $V \in \mathcal{N}(1_G)$ we can choose another open neighborhood of $1_G$
such that

$$w[y]^{-1}(U) \subseteq V \forall y \in Y.$$ 

Now, let $(f_n) \subseteq C(X, G)$ be a sequence which converges uniformly to some $f \in
C(X, G)$, then there will be $n_1$ such that $f(h(y))f_n(h(y))^{-1} \in U$ for very $n \geq n_1$
uniformly. So, it can be deduced that $w[y]^{-1}(f(h(y))f_n(h(y))^{-1}) \in V$ for every $n \geq
n_1$ (uniformly). Hence, $(Hf_n)$ converges uniformly to $Hf$ and $H$ is automatically
continuous.

On the other hand, let us see that the map

$$w : Y \rightarrow Aut_p(G),$$
is also continuous. In order to prove that, let us consider the following map

$$W^{-1} : Aut_p(G) \rightarrow Aut_p(G),$$
where $j(\phi) = \phi^{-1}$. The mappings $j$ and $w^{-1}$ are continuous and $Y$ is a compact
space, then, if we reason out as in the proof of the continuity of $w^{-1}$, we obtain
that $w[Y]$ is an equicontinuous subset of $Aut(G)$. From this fact, it follows that
the map

$$Y \rightarrow G$$
y $\mapsto w[y](Hf(y))$
is continuous for every $f \in C(X, G)$. Since the space $X$ is $G$-regular, it follows
that the map $h$ is continuous, which completes the proof.

**Corollary 3.7.** Let $X$ and $Y$ be compact Hausdorff spaces, $G$ a Čech-complete
topological group and $H : C(X, G) \rightarrow C(Y, G)$ a homomorphism. If $H$ and $H^{-1}$
are also non-vanishing mappings and continuous, if restricted to $G$, then there exist
a homeomorphism $h : Y \rightarrow X$ and a continuous mapping $w : Y \rightarrow Aut(G)$ such
that

$$(Hf)(y) = w[y]^{-1}(f(h(y))),$$
and $H$ is also continuous.

**Proof.** If $H$ and $H^{-1}$ are non-vanishing mappings, then there exist automatically
two continuous cross sections

$$\Psi : \overline{G} \rightarrow C(X, G)$$
and

$$\Phi : \overline{G} \rightarrow C(Y, G)$$
such that $\Psi(\overline{G}) := H^{-1}(\overline{G})$ and $\Phi(\overline{G}) := H(\overline{G})^{-1}$. This implies that $H$ and $H^{-1}$
are $C$-homomorphisms and the cross sections are also non-vanishing. Now, we apply Theorem 3.6 (part (b)) to $H$ and $H^{-1}$, and obtain that there are continuous
 mappings $h : Y \rightarrow X$, $k : X \rightarrow Y$, $w : Y \rightarrow Aut(G)$ and $v : X \rightarrow Aut(G)$ such that

$$(Hf)(y) = w[y]^{-1}(f(h(y))).$$
for every \( f \in C(X,G) \), and
\[
(H^{-1}t)(x) = v[y]^{-1}(t(k(x)))
\]
for every \( t \in C(Y,G) \). Moreover, both mappings \( H \) and \( H^{-1} \) are also continuous.
It remains to prove that \( h \) and \( k \) are inverse mappings one of the other. Let, therefore, be \( y \in Y \), then, if \( g \in C(Y,G) \), we have that
\[
g(y) = (H \circ H^{-1})(g)(y) = (H(H^{-1}(g))(y) = w[y]^{-1}((H^{-1}g)(h(y)))
\]
\[
= w[y]^{-1}(v[h(y)]^{-1}(g(k(h(y))))),
\]
so \( w[y](g(y)) = v[h(y)]^{-1}(g(k(h(y)))) \) for every \( y \in Y \) and every \( g \in C(Y,G) \). As \( w[y] \) and \( k(h(y)) \) are automorphisms of \( G \), then \( g(y) = g(k(h(y))) \). If there was \( y_0 \in Y \) such that \( y_0 \neq k(h(y_0)) \), then there would be open neighborhoods \( U \) and \( V \) of \( y_0 \) and \( k(h(y_0)) \), respectively, such that \( U \cap V = \emptyset \). Hence, given \( \alpha \neq 1_G \), we could find a map \( F \in C(Y,G) \) such that
\[
F(y_0) = \alpha \quad \text{and} \quad F(Y \setminus U) = \{1_G\},
\]
and this contradicts with the fact that \( g(y) = g(k(h(y))) \) for every \( g \in C(Y,G) \). As a consequence, we obtain that for every \( y \in Y \), \( k(h(y)) = y \). Analogously, one can prove that \( h(k(x)) = x \) for every \( x \in X \). Thus, \( h \) is also an open mapping, that is, a homeomorphism.

\[\square\]

**Remark 3.8.** For non-vanishing mappings and for compact spaces, the next Corollary improves the main result of [14], where the map \( H \) is required to be continuous with respect to the compact open topology.

**Corollary 3.9.** Given \( X \) and \( Y \) compact Hausdorff spaces and \( G \) a Hausdorff topological group, let \( H \) be a non-vanishing homomorphism of \( C(X,G) \) into \( C(Y,G) \) such that \( H \) coincides with the identity mapping on the constant functions. Then there exist a continuous mapping \( h : Y \to X \) such that
\[
(Hf)(y) = f(h(y))
\]
whenever \( f \in C(X,G) \) and \( y \in Y \), and \( H \) is continuous.

**Proof.** The identity mapping on \( G \) defines a continuous cross section for \( H \). Thus, from Corollary 3.5 and Theorem 3.6, we obtain that for every \( y \in Y \) there is a unique \( h(y) \in X \) such that \( w[y][(Hf)(y)] = f(h(y)) \). Moreover, it is readily seen that \( w(y) \) is the identity on \( G \) for all \( y \in Y \). Hence, \( (Hf)(y) = f(h(y)) \) and \( H \) is a Banach-Stone map. Now, it can be deduced that \( H \) is continuous with respect to the uniform convergence topology. In fact, let \( (f_n)_n \subseteq C(X,G) \) be a sequence which converges uniformly to some \( f \in C(X,G) \). Let \( U \in \mathcal{N}(1_G) \). Then, there is \( n_0 \in \mathbb{N} \) such that for every \( n \geq n_0 \)
\[
f(x)f_n(x)^{-1} \in U, \forall x \in X.
\]
On the other hand, we have that \((Hf)(y)(Hf_n)(y)^{-1} = f(h(y))f_n(h(y))^{-1}\), and, as \(H\) is a Banach-Stone map, then we obtain that for every \(n \geq n_0\),
\[
(Hf)(y)(Hf_n)(y)^{-1} = f(h(y))f_n(h(y))^{-1} \in U,
\]
for every \(y \in Y\). Hence, \(H\) is continuous with respect to the uniform convergence topology. After that, it can be readily seen that \(h : Y \to X\) is also continuous. This completes the proof. \(\square\)

We notice that it is not necessary for \(H\) to be the identity on the constant functions to obtain also a representation of \(H\). In fact, the following result shows that it is enough to assume that \(H\) is a topological isomorphism onto \(G\).

**Corollary 3.10.** Given \(X\) and \(Y\) compact Hausdorff spaces and \(G\) a topological group, let \(H\) be a non-vanishing homomorphism of \(C(X,G)\) into \(C(Y,G)\) such that \(H\) is a topological isomorphism onto \(G\). Then there exist a continuous mapping \(h : Y \to X\) such that \((Hf)(y) = Hf(\overline{f(h(y)})\)) whenever \(f \in C(X,G)\) and \(y \in Y\), and \(H\) is continuous.

**Proof.** It suffices to apply Corollary 3.9 to the map \(f \mapsto H^{-1}_{|G} \circ (Hf)\), which coincides with the identity on the constant functions. \(\square\)

### 4. Consequences for the scalar-valued functions

Let us see some direct consequences of the results above when we replace \(G\) by \(\mathbb{K}\), the field of real (or complex) numbers. Let \(C(X)\) designate the group \(C(X,\mathbb{K})\). We say that a homomorphism \(H : C(X) \to C(Y)\) preserves non-vanishing functions when, if we have \(Hf(y) = 0\) for some \(y \in Y\), then \(f(x) = 0\) for some point \(x \in X\). The symbol \(f \geq 0\) (resp. \(f > 0\), etc.) means that \(f(x) \geq 0\) (resp. \(f(x) > 0\), etc.) for all \(x \in X\). Given an arbitrary function \(f \in C(X)\), we define \(f^+ = (f + |f|)/2\) and \(f^- = (f - |f|)/2\), as usual. Next, we present an application of the methods developed in the sections above to obtain a new proof of a known result that was communicated to us by Araujo [2] (in fact, a proof of the non-Archimedean case appears in [1]). We wish to thank him for letting us know about this result. A variant for discrete spaces has been given by Arazy and Fisher in [3].

**Theorem 4.1.** Let \(X\) and \(Y\) be compact spaces and let \(H : C(X) \to C(Y)\) be a linear bijection such that \(H\) and \(H^{-1}\) preserve non-vanishing functions. Then there exists an homeomorphism \(h : Y \to X\) such that
\[
(Hf)(y) = (HT)(y) \cdot f(h(y)),
\]
whenever \(f \in C(X)\) and \(y \in Y\).

**Proof.** We may assume WLOG that \(H(\mathbb{T}) = \mathbb{T}\). Otherwise, we replace \(H\) by the map \(T = \frac{1}{H(0)} \cdot H\). As a consequence, it is easily verified that the subsets \(f(X)\) and \((Hf)(Y)\) coincide in \(\mathbb{K}\) for all \(f \in C(X)\), which implies \(Hf > 0\) if and only
if $f > 0$. Now, in order to apply Corollary 3.9, we must show that $H$ is non-vanishing. Let $f$ and $g$ be in $C(X)$ with $N(f) \cap N(g) = \emptyset$ and assume there is $y_0 \in Y$ with $Hf(y_0) = Hg(y_0) = 0$. Then $(Hf)^+(y_0) = (Hf)^-(y_0) = (Hg)^+(y_0) = (Hg)^-(y_0) = 0$. On the other hand, since $H$ sends the positive cone of $C(X)$ into the positive cone of $C(Y)$, there are positive functions, say $h_+, h_-, k_+, k_-$, such that $Hf^+ = (Hf)^+ + h_+, Hf^- = (Hf)^- + h_-, Hg^+ = (Hf)^+ + k_+$ and $Hg^- = (Hg)^- + k_-$. Thus, $0 = (Hf)^+(y_0) - (Hf)^-(y_0) = Hf(y_0) = Hf^+(y_0) - Hf^-(y_0)$, which yields $h_+(y_0) = h_-(y_0)$. In like manner, $k_+(y_0) = k_-(y_0)$. We also have $0 = (Hf)^+(y_0) + (Hf)^-(y_0) + (Hg)^+(y_0) + (Hg)^-(y_0)$; that is, if $\phi = (Hf)^+ + (Hf)^- + (Hg)^+ + (Hg)^-$, we obtain $\phi(y_0) = 0$. Hence, there is $x_0 \in X$ such that $H^{-1}\phi(x_0) = 0$. Equivalently,

$$[(f^+ - H^{-1}(h_+)) + (f^- - H^{-1}(h_-)) + (g^+ - H^{-1}(k_+)) + (g^- - H^{-1}(k_-))](x_0) = 0.$$ 

Since $f^+ - H^{-1}(h_+) \geq 0$, $f^- - H^{-1}(h_-) \geq 0$, $g^+ - H^{-1}(k_+) \geq 0$ and $g^- - H^{-1}(k_-) \geq 0$, we obtain

$$f(x_0) =
\begin{align*}
&H^{-1}[(Hf^+ - h_+) - (hf^- h_-)](x_0) =
&(f^+ - H^{-1}(h_+))(x_0) - (f^- - H^{-1}(h_-))(x_0) = 0
\end{align*}$$

In similar way, we would obtain that $g(x_0) = 0$. Thus, $x_0 \in N(f) \cap N(g)$, which is impossible. This contradiction proves that $H$ is non-vanishing. Therefore, it is enough to apply Corollary 3.9 and the proof is done.

In order to settle better the results presented in the previous sections, we next provide an example of two non-homeomorphic compact spaces $X$ and $Y$ such that there is a topological isomorphism $H$ of $C(X)$ onto $C(Y)$, which coincides with the identity mapping on the constant functions. As a matter of fact, there are many examples of this sort (see [4, Corollary 6.17] or [16, page 499]). Nevertheless, we have decided to include an example here for the reader’s sake. It shows that the hypotheses imposed on Theorem 3.6 and Corollaries 3.9 and 3.10 are essential in the statements of such results. In the sequel, let $c$ denote the Banach space of convergent sequences. Observe that that $c \times c$ is topologically isomorphic to $c$ via the map

$$H : c \times c \rightarrow c$$

$$\{(a_n, b_n)\} \mapsto \{d_n\},$$

where

$$d_1 := \lim a_n,$$

$$d_{2n} := a_n - \lim a_n + \lim b_n$$

and

$$d_{2n+1} := b_n.$$ 

We can also identify $c$ with the real Banach space of continuous functions $C(\mathbb{N}^*)$, where $\mathbb{N}^*$ is the Alexandroff compactification of $\mathbb{N}$. Since $c \times c \cong c$, we obtain
\( C(\mathbb{N}^* \cup \mathbb{N}^*) \cong C(\mathbb{N}^*) \) (here, \( \mathbb{N}^* \cup \mathbb{N}^* \) denotes the topological sum of both spaces). This mapping is a topological isomorphism and takes the constant functions over \( \mathbb{N}^* \times \mathbb{N}^* \) into the corresponding ones over \( \mathbb{N}^* \). However, the map \( H \) is not related to any homeomorphism of \( \mathbb{N}^* \cup \mathbb{N}^* \) onto \( \mathbb{N}^* \).

**References**


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