(α, β)-Fuzzy Hyperideals in Semihyperrings

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Abstract: The concept of quasi-coincidence of fuzzy point with a fuzzy subset has considered. By using this idea, the notion of (α, β)-fuzzy hyperideal in a semihyperring introduced and consequently, a generalization of fuzzy hyperideals has defined. In this paper, we study the related properties of the (α, β)-fuzzy hyperideals and in particular, the (ε, vq)-fuzzy hyperideals in semihyperrings will be investigated. Moreover, we also consider the concept of implication-based fuzzy hyperideals in semihyperrings.

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INTRODUCTION

Hyperstructure theory was born in 1934 when Marty [18] defined hyper groups, began to analyze their properties and applied them to groups, rational algebraic functions. Now they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties.

The notion of a fuzzy set was introduced by Zadeh [20]. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, real analysis, measure theory etc.

Rosenfeld studied fuzzy subgroups of a group [21]. The study of fuzzy semigroups was studied by Kuroki in his classical papers [22]. Recently Jun and Kang [2] considered fuzzification of generalized Tarsaki filter in Tarsaki algebra. Recently, fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. Recently in [17], Davvaz and Leoreanu-Fotea studied the structure of fuzzy Γ-hyperideals in Γ-semihypergroups. Ameri and Noari [3] initiated fuzzy hyperalgebras and introduced some important results. Davvaz et al., proposed fuzzy Hm-ideals in Γ-Hm-rings [4] and fuzzy Γ-hypernearrings [5]. Davvaz [6], initiated fuzzy Krasner (m,n)-hyperrings. Sun et al deeply studied fuzzy hypergraphs on fuzzy relations in [7]. Using the notions "belong to" relation (ε) introduced by Pu and Liu [8]. In [9], Morali proposed the concept of a fuzzy point belonging to a fuzzy subset under natural equivalence on fuzzy subset. Bhakat and Das introduced the concepts of (α, β)-fuzzy subgroups by using the "belong to" relation (ε) and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup and defined an (ε, vq)-fuzzy subgroup of a group [10]. In [11], Tariq et al. introduced the concept of (α, β)-fuzzy hyperideals and (ε, vq)-fuzzy hyperideals in semihypergroups. In [12], M. Shabir and T. Mehmood studied (ε, vq)-fuzzy h-ideals of hemirings and characterized different classes of hemirings by the properties of (ε, vq)-fuzzy h-ideals. For further reading [13-15]. Recently, M. Aslam et al. [16] initiated the concept of (α, β)-fuzzy Γ-ideals of Γ-LA-semigroups and given some characterization of Γ-LA-semigroups by (α, β)-fuzzy Γ-ideals.

In this paper we concentrate on the concept of quasi-coincidence of fuzzy point with a fuzzy subset. By using this idea, the notion of (α, β)-fuzzy hyperideal in a semihyperring introduced and consequently, a generalization of fuzzy hyperideals is defined. In this paper, we study the related properties of the (α, β)-fuzzy hyperideals and in particular, an (ε, vq)-fuzzy hyperideals in semihyperrings will be investigated. Moreover, we also consider the concept of implication-

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based fuzzy hyperideals in a semihyperring and obtained some useful results.

**PRELIMINARIES**

By a semihyperring, we mean an algebraic hyper system \((R,+,\cdot)\) consisting of a non-empty set \(R\) together with binary operations on \(R\) called addition and multiplication, such that \((R,+)\) and \((R,\cdot)\) are semihypergroups and for all \(x,y,z\in R\), we have \(x(y+z) = xy+xz\), which is called distributivity. By a subsemihyperring of \(R\), we mean a non-empty subset \(S\) of \(R\) such that for all \(x,y\in S\) we have \(x+y\in S\) and \(x,y\in S\). By a left (right) hyperideal of \(R\), we mean a subsemihyperring \(I\) of \(R\) such that for all \(r\in R\) and \(x\in I\), we have \(rx\in I\) and \(xr\in I\).

By a hyperideal, we mean a subsemihyperring of \(R\) which is both a left and a right hyperideal of \(R\), we mean an element \(0\in R\) such that \(0+x = x+0 = x\) for all \(x\in R\) [4].

**Example 2.1:** Let \(X\) be a non-empty finite set and \(\tau\) is a topology on \(X\). We define the hyperoperation of the addition and the multiplication on \(\tau\) as:

For any \(A,B\in \tau\), \(A+B = \{A\cup B\}\) and \(A\cdot B = \{A\cap B\}\). Then \((\tau,+,\cdot)\) is a semihyperring with absorbing element and additive identity \(\Phi\) and multiplicative identity \(X\).

**Example 2.2:** Let us consider a set

\[
S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in W \right\}
\]

where \(W\) is a set of whole numbers. We define the hyperoperation of addition and multiplication as;

For \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) and \(B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) be taken from \(S\),

\[
A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+a & b+b \\ c+c & d+d \end{bmatrix} \subseteq S
\]

and

\[
A\cdot B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aa+bc & ab+bd \\ ca+dc & cb+dd \end{bmatrix} \subseteq S
\]

Then \((S,+,\cdot)\) is a semihyperring with additive identity as a null matrix and multiplicative identity as an identity matrix.

**Definition 2.3:** Let \(R\) be a semihyperring and \(\mu\) a fuzzy set in \(R\). Then, \(\mu\) is said to be a fuzzy hyperideal of \(R\) if for all \(r,x,y\in R\) the following axioms hold:

(i) \(\mu(z) \geq \mu(x) \land \mu(y)\), for all \(z\in x+y\).

(ii) \(\mu(rx) \geq \mu(x)\) and \(\mu(xr) \geq \mu(x)\).

**Theorem 2.4:** Let \(\mu\) be a fuzzy set in a semihyperring \(R\). Then, \(\mu\) is a fuzzy hyperideal of \(R\) if and only if for every \(t \in (0,1]\), the level subset \(\mu_t(\neq \Phi)\) is a hyperideal of \(R\), where \(\mu_t = \{x \in R | \mu(x) \geq t\}\).

**Proof:** The proof is straightforward by considering the definition.

**Lemma 2.5:** Let \(\mu\) be a fuzzy set in a semihyperring \(R\). If \(R\) has zero element, then \(\mu(0) \geq \mu(x)\) for all \(x\in R\).

**Proof:** The proof is clear. A fuzzy set \(\mu\) in a set \(R\) of the form

\[
\mu(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}
\]

is said to be a fuzzy point with support \(x\) and value \(t\) and is denoted by \(x_t\). A fuzzy point \(x_t\) is said to be belong to (resp. quasi-coincident with) a fuzzy set \(\mu\), written as \(x_t \in \mu\) (resp \(x_t \subseteq \mu\)) if \(\mu(x) \geq t\) (\(\mu(x) \subseteq t\)). If \(x_t \in \mu\) or \(x_t \subseteq \mu\), then we write \(x_t \in \mu\). If \(x_t \in \mu\) and \(x_t \subseteq \mu\), then we write \(x_t \in \mu\). The symbol \(\in \mu\) means neither \(\in\) nor \(\subseteq\) holds. The symbol \(\in \mu\) means \(\in\) or \(\subseteq\) does not hold.

**ON \((\alpha,\beta)\)-FUZZY HYPERIDEALS OF SEMIHYPERRINGS**

In what follows, \(R\) will denote a semihyperring and \(\alpha,\beta\) will denote one of \(\in, \subseteq, \cup, \cap\) unless otherwise specified. Also \(\bar{\alpha}\) means \(\alpha\) does not hold.

**Definition 3.1:** A fuzzy set \(\mu\) in \(R\) is called an \((\alpha,\beta)\)-fuzzy hyperideal of \(R\), where \(\alpha \neq \in \cup \cap\), if for all \(r,x,y\in R\) and \(t_1, t_2 \in (0,1]\) the following conditions hold:

(i) \(x_t \alpha \mu\) and \(y_t \alpha \mu\) imply \((z)_t \alpha \beta \mu\), for all \(z\in x+y\).

(ii) \(x_t \alpha \mu\) implies \((rx)_t \beta \mu\) and \((xr)_t \beta \mu\).
where \( t_1 \land t_2 = \min \{t_1, t_2\} \). Let \( \mu \) be a fuzzy set in \( R \) such that \( \mu(x) \leq 0.5 \) for all \( x \in R \). Suppose that \( x \in R \) and \( t \in (0,1] \), such that \( x_t \in \land \mu \). Then \( \mu(x) \leq t \land (\mu(x) + t) \). It follows that \( 1 < \mu(x) + t \leq (\mu(x) + t_2) = (2 \mu(x), so that \( \mu(x) > \frac{1}{2} \). This means that \( \{x_t \mid x_t \in \land \mu\} = \emptyset \). Therefore, the case \( \alpha \in \equiv \land \alpha \) in Definition 4 is omitted.

In the next theorem, by an \((\alpha, \beta)\)-fuzzy hyperideal of \( R \), we construct an ordinary hyperideal of \( R \).

**Theorem 3.2:** Let \( \mu \) be a non-zero \((\alpha, \beta)\)-fuzzy hyperideal of \( R \). Then, the set \( \text{supp} (\mu) = \{x \in R \mid \mu(x) > 0\} \) is a hyperideal of \( R \).

**Proof:** Suppose that \( x, y \in \text{supp} (\mu) \). Then \( \mu(x) > 0 \) and \( \mu(y) > 0 \). Assume that \( \mu(z) > 0 \) for all \( z \in x \land y \). If \( \alpha \in \{e, e \in \forall q\} \), then \( x_{\mu(z)}, \alpha \mu \) and \( y_{\mu(z)}, \alpha \mu \). But, for all \( z \in x \land y \), \( (z)i = (z), \beta \mu, \) for every \( \beta \in \{e, e \in \forall q, \in \land \land q\} \), which is a contradiction. Note that \( x_{\mu(q)} \) and \( y_{\mu(q)} \) but, for all \( z \in x \land y \), \( (z), \lambda \mu = (z), \beta \mu, \) for every \( \beta \in \{e, e \in \forall q, \in \land \land q\} \), which is a contradiction. Hence, \( \mu(z) > 0 \), for all \( z \in x \land y \), that is, for all \( z \in x \land y, z \in \text{supp} (\mu) \). Also, let there exists \( r \in R \) such that \( \mu(rz) = 0 \). If \( \alpha \in \{e, e \in \forall q\} \), then \( x_{\mu(z)}, \alpha \mu, \) \( (r), \beta \mu, \) for every \( \beta \in \{e, e \in \forall q, \in \land \land q\} \), which is a contradiction. We know that \( x_{\mu(q)} \).

But \( (r), \beta \mu, \) for every \( \beta \in \{e, e \in \forall q, \in \land \land q\} \), which is a contradiction. Hence, \( \mu(z) > 0 \), that is \( rz \in \text{supp} (\mu) \). Similarly, we can show that \( rz \in \text{supp} (\mu) \). Therefore, \( \text{supp} (\mu) \) is a hyperideal of \( R \).

In the next theorem, we see that a \((q, q, q)\)-fuzzy hyperideal is constant under suitable condition.

**Theorem 3.3:** Let \( R \) has zero element and \( \mu \) be a non-zero \((q, q, q)\)-fuzzy hyperideal of \( R \). Then, \( \mu \) is constant on \( \text{supp} (\mu) \).

**Proof:** By Lemma 2.5, we know that \( \mu(0) = v(\mu(x) \mid x \in R) \). Suppose that there exists \( x \in \text{supp} (\mu) \) such that \( t_x = (x) \neq t_y \), then \( x \neq t_y \). Choose \( t_1 \land t_2 \in (0,1] \) such that \( 1-t_0 = t_1 < 1-t_2 < 2 \). Then \( 0, q, \mu \) and \( x_t, \mu \) but \( (z)_i = x_q, \mu \) for all \( z \in 0+x \) and \( (z)_i = x_q, \mu \) for all \( z \in x+0 \), which is a contradiction. Thus, \( \mu(x) = \mu(0) \) for all \( x \in \text{supp} (\mu) \). Therefore, \( \mu \) is constant on \( \text{supp} (\mu) \). In the following theorem, we investigate some conditions that make a fuzzy set \( \mu \) in \( R \) as a \((q, e \in \forall q)\)-fuzzy hyperideal.

**Theorem 3.4:** Let \( I \) be a hyperideal of \( R \) and \( \mu \) a fuzzy set in \( R \) such that

(i) \( \forall x \in R \setminus I, \mu(x) = 0 \),

(ii) \( \forall x \in I, \mu(x) \geq 0.5 \).

Then, \( \mu \) is a \((q, e \in \forall q)\)-fuzzy hyperideal of \( R \).

**Proof:** Suppose that \( x, y \in R \) and \( t_1, t_2 \in (0,1] \) such that \( x_t, \mu \) and \( y_t, \mu \). Then \( x, y \in I \) and so \( z \in I \) for all \( z \in x+y \). We can consider the following cases:

1. \( t_1 \land t_2 \leq 0.5 \), then \( \mu(z) \geq 0.5 \geq t_1 \land t_2 \), for all \( z \in x+y \) and hence \( (z), \lambda \mu, \in \mu \) for all \( z \in x+y \).
2. \( t_1 \land t_2 > 0.5 \), then \( \mu(z) + t_1 \land t_2 > 0.5 + 0.5 = 1 \) and so \( (z), \lambda \mu \). Therefore, \( (z), \lambda \mu, \in \forall q, \in \forall q \) for all \( z \in x+y \).

Now, suppose that \( r \in R \) and \( t \in (0,1] \) such that \( x_t, \mu \). Then, \( x \in I \) and \( x \in I \). We can see two following cases:

1. \( t \leq 0.5 \), then \( \mu(x) > 0.5 \geq t \) and hence \( (x), \lambda \mu, \in \mu \).
2. \( t > 0.5 \), then \( \mu(x) + 1 > 0.5 + 0.5 = 1 \) and so \( (x), \lambda \mu, \in \mu \).

Similarly, \( (x), \lambda \mu, \in \mu \).

This completes the proof. Also, we have the converse of Theorem 3.4 as follows:

**Theorem 3.5:** Let \( R \) be a semihyperring with zero and \( \mu \) a \((q, e \in \forall q)\)-fuzzy hyperideal of \( R \), such that \( \mu \) is not constant on \( \text{supp} (\mu) \). Then, \( \mu(x) > 0.5 \) for all \( x \in \text{supp} (\mu) \).

**Proof:** By Lemma 2.5, we know that \( \mu(0) = v(\mu(x) \mid x \in R) \).

Assume that \( \mu(x) > 0.5 \) for all \( x \in R \). Since \( \mu \) is not constant on \( \text{supp} (\mu) \), there exists \( x \in \text{supp} (\mu) \) such that \( t_x = (x) \neq \mu \). Then \( t_x < t_y \). Choose \( t_1 > 0.5 \) such that \( t_x + t_1 < 1-t_0 + t_1 \). Then \( 0, q, \mu \) and \( x_t, \mu \). Since \( \mu(x) + t_1 = t_x + t_1 < 1 \), we have \( x_t, \mu \) and so \( (z), \lambda \mu, \in \forall q, \in \forall q \) for all \( z \in 0+x \) or \( z \in x+0 \). This contradicts \( \mu \) is a \((q, e \in \forall q)\)-fuzzy hyperideal of \( R \). Therefore, \( \mu \) is constant for some \( y \in R \). Also, since \( \mu(0) \geq \mu(y) \), then \( \mu(0) \geq 0.5 \). Finally, let \( t_x = (x) < 0.5 \) for some \( x \in \text{supp} (\mu) \). Take \( t_y > 0.5 \) such that \( t_y < t_x + t_1 < 0.5 \), then \( x_t, \mu \) and \( 0, q, \mu \). But \( \mu(x) + 0.5 + t_1 = t_x + 0.5 + t_1 < 0.5 + 0.5 = 1 \), which implies \( x_t, \mu \).

Thus, \( (z), \lambda \mu, \in \forall q, \in \forall q \) for all \( z \in 0+x \) or \( z \in x+0 \), which is a contradiction. Therefore, \( \mu(x) > 0.5 \) for all \( x \in \text{supp} (\mu) \).
A fuzzy set \( \mu \) in \( R \) is said to be proper if \( \text{Im}(\mu) \) has at least two elements. Two fuzzy sets are said to be equivalent if they have the same family of level subsets. Otherwise, they are said to be non-equivalent. Now, we can discuss on \((\varepsilon,\varepsilon)\)-fuzzy hyperideal of \( R \) which can be expressed as the union of two proper non-equivalent \((\varepsilon,\varepsilon)\)-fuzzy hyperideals.

**Theorem 3.6:** Let \( R \) has proper ideals. A proper \((\varepsilon,\varepsilon)\)-fuzzy hyperideal \( \mu \) of \( R \) such that \( 3 \leq \text{Im}(\mu) \). Then, \( X \in R \in \mathbb{I} \) can be expressed as the union of two proper non-equivalent \((\varepsilon,\varepsilon)\)-fuzzy hyperideals of \( R \).

**Proof:** Let \( \mu \) be a proper \((\varepsilon,\varepsilon)\)-fuzzy hyperideal of \( R \) with \( \text{Im}(\mu) = \{ t_0, t_1, t_2 \} \), where \( t_0 > t_1 > t_2 \). Then, \( \mu_0 \subseteq \mu_1 \subseteq \mu_2 = R \) is the chain of level hyperideals of \( R \). Define fuzzy sets \( \nu \) and \( \theta \) in \( R \) by

\[
\nu(x) = \begin{cases} 
  t_1 & \text{if } x \in \mu_0, \\
  t_2 & \text{if } x \in \mu_0 \setminus \mu_1, \\
  t_3 & \text{if } x \in \mu_0 \setminus \mu_2,
\end{cases}
\]

and

\[
\theta(x) = \begin{cases} 
  t_0 & \text{if } x \in \mu_0, \\
  t_1 & \text{if } x \in \mu_0 \setminus \mu_1, \\
  t_2 & \text{if } x \in \mu_0 \setminus \mu_2, \\
  t_3 & \text{if } x \in \mu_0 \setminus \mu_2,
\end{cases}
\]

where \( t_0 < t_1 < t_2 \) and \( z \subseteq x+y \). Then, \( \nu \) and \( \theta \) are \((\varepsilon,\varepsilon)\)-fuzzy hyperideal of \( R \),

\[
\mu_0 \subseteq \mu_1 \subseteq \mu_2 = R,
\]

and

\[
\mu_0 \subseteq \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_n = R,
\]

are the chains of level ideals respectively and \( \nu, \theta \subseteq \mu \). Therefore, \( \mu \) and \( \theta \) are non-equivalent and clearly, \( (z) \in \nu \cap \theta \).

**FUZZY HYPERIDEALS OF TYPE (\varepsilon,\varepsilon)\**

In this section, we investigate some results and properties of \((\alpha, \beta)\)-fuzzy hyperideals (specifically, \((\varepsilon,\varepsilon)\)-fuzzy hyperideals) of \( R \).

**Theorem 4.1:** Every \((\varepsilon,\varepsilon)\)-fuzzy hyperideal of \( R \) is an \((\varepsilon,\varepsilon)\)-fuzzy hyperideal of \( R \).

**Proof:** Suppose that \( x, y \in R \) and \( t, t_1 \in (0,1] \) such that \( x, y \in X \). Then, \( x, y \in \nu \cap \theta \). By the hypothesis, it follows that \( (z) \in \nu \cap \theta \) for all \( z \in x+y \).

**Proposition:** If \( I \) is a hyperideal of \( R \), then \( X \) (the characteristic function of \( I \) is an \((\varepsilon,\varepsilon)\)-fuzzy hyperideal of \( R \).

**Example 4.3:** On four element semihyperring \((R, +, \cdot)\) defined by the following two tables:

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<th>c</th>
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consider a fuzzy μ set as follows:

\[
\mu(x) = \begin{cases} 
0.9 & \text{if } x = 0 \\
0.7 & \text{if } x = a, b \\
0.3 & \text{if } x = c 
\end{cases}
\]

for all \(z \in x + y\).

condition for \((e, e \lor q)\)-fuzzy hyperideals.

**Theorem 4.4:** A fuzzy set μ in R is an \((e, e \lor q)\)-fuzzy hyperideal of R if and only if for all \(r, x, y \in R\) the following two conditions hold:

(i) \(\mu(z) \geq \mu(x) \land \mu(y) \lor 0.5\), for all \(z \in x + y\).

(ii) \(\mu(rx) \geq \mu(x) \lor 0.5\) and \(\mu(xr) \geq \mu(x) \lor 0.5\).

**Proof:** Let μ be an \((e, e \lor q)\)-fuzzy hyperideal of R and \(x, y \in R\). We can consider the following cases:

1. \(\mu(x) \land \mu(y) < 0.5\). In this case, if \(\mu(z) < \mu(x) \land \mu(y)\), for all \(z \in x + y\). We can choose \(t \in (0,0.5)\) such that for all \(z \in x + y\), \(\mu(z) < t < \mu(x) \land \mu(y)\). Then \(x \in \mu\) and \(y \in \mu\), but \((z)_t \in \lor \mu\), for all \(z \in x + y\), which is a contradiction. Thus, for all \(z \in x + y\), \(\mu(z) \geq \mu(x) \land \mu(y) = \mu(x) \land \mu(y) = 0.5\).

2. \(\mu(x) \land \mu(y) \geq 0.5\). In this case, we have \(\mu_{0.5} \in \mu\) and \(y_{0.5} \in \mu\). If for all \(z \in x + y\), \(\mu(z) < 0.5\), then, for all \(z \in x + y\), \((z)_{0.5} \in \lor \mu\) and \(\mu(z) + 0.5 < 1\) (or \((z)_{0.5} \in \lor \mu\), for all \(z \in x + y\)). Hence, \((z)_{0.5} \in \lor \mu\), which is a contradiction. Thus, \(\mu(z) \geq \mu(x) \land \mu(y) = 0.5\), for all \(z \in x + y\). Also, if \(r, x, y \in R\), we can consider two following cases:

1. \(\mu(x) \geq 0.5\). In this case, if \(\mu(rx) < \mu(x)\), we can choose \(t \in (0,0.5)\) such that \(\mu(rx) < t < \mu(x)\). Then \(x \in \mu\), but \((rx)_t \in \lor \mu\), which is a contradiction. Thus, \(\mu(rx) \geq \mu(x) = \mu(x) \land 0.5\). Similarly, \(\mu(xr) \geq \mu(x) \land 0.5\).

2. \(\mu(x) \geq 0.5\). In this case, we have \(\mu_{0.5} \in \mu\). If \(\mu(rx) < 0.5\), then \((rx)_{0.5} \in \lor \mu\) and \(\mu(rx) + 0.5 < 1\) (or \((rx)_{0.5} \in \lor \mu\)). Hence \((rx)_{0.5} \in \lor \mu\), which is a contradiction. Thus, \(\mu(rx) \geq \mu(x) = \mu(x) \land 0.5\). Similarly, \(\mu(xr) \geq \mu(x) \land 0.5\).

Conversely, suppose that μ satisfies condition (i) and (ii). Let \(x, y \in R\) and \(t_1, t_2 \in (0,1)\) such that \(x_t \in \mu\) and \(y_t \in \mu\). Then, \(\mu(x) \geq t_1\) and \(\mu(y) \geq t_2\). Suppose that \(\mu(z) < t_1 \land t_2\) for all \(z \in x + y\). If \(\mu(x) \land \mu(y) < 0.5\), then, for all \(z \in x + y\), \(\mu(z) \geq \mu(x) \land \mu(y) \land 0.5 = \mu(x) \land \mu(y) > t_1 \land t_2\), which is a contradiction. So \(\mu(x) \land \mu(y) \geq 0.5\). It follows that, for all \(z \in x + y\),

\[
\mu(z) + (t_1 \land t_2) > 2\mu(z) \geq 2(\mu(x) \land \mu(y) \land 0.5) = 1,
\]

Hence, \((z)_{t_1} \land t_2, q \mu\) for all \(z \in x + y\), which implies \((z)_{t_1} \land t_2, q \mu\) for all \(z \in x + y\). Also, let \(r, x, y \in R\) and \(t \in (0,1]\) such that \(x_t \in \mu\), then \(\mu(x) \geq 1\). Suppose that \(\mu(rx) < t\). If \(\mu(x) < 0.5\), then \(\mu(rx) \geq \mu(x) \land 0.5 = \mu(x) \geq 1\), which is a contradiction and so \(\mu(x) \geq 0.5\). It follows that \(\mu(rx) + t > 2\mu(rx) \geq 2(\mu(x) \land 0.5) = 1\). Hence, \((rx)_{t_1} q \mu\), which implies \((rx)_{t_1} q \mu\). Similarly, \((x)_{t_1} q \mu\). Therefore, μ is an \((e, e \lor q)\)-fuzzy hyperideal of R.

In the following theorem, we characterize \((e, e \lor q)\)-fuzzy hyperideals based on level subsets.

**Theorem 4.5:** Let μ be a fuzzy set in R. If μ is an \((e, e \lor q)\)-fuzzy hyperideal of R and for all \(0 < t \leq 0.5\), \(\mu_t = \Phi\) or \(\mu_t\) is a hyperideal of R.

Conversely, If \(\mu_t (\neq \Phi)\) is a hyperideal of R for all \(0 < t \leq 0.5\), then μ is an \((e, e \lor q)\)-fuzzy hyperideal of R.

**Proof:** Let μ be an \((e, e \lor q)\)-fuzzy hyperideal of R and \(0 < t \leq 0.5\). If \(x, y \in \mu_t\), then \(\mu(x) \geq t\) and \(\mu(y) \geq t\). Hence, for all \(z \in x + y\), \(\mu(z) \geq \mu(x) \land \mu(y) \land 0.5 \geq t \land 0.5 = t\), which implies \(\mu(z) \geq t\), for all \(z \in x + y\). That is \(z \in \mu_t\), for all \(z \in x + y\). Now, suppose that \(x \in \mu_t\) and \(\mu_t \in R\). Then, \(\mu(x) \geq t\) and hence \(\mu(rx) \geq \mu(x) \land 0.5 \geq t \land 0.5 = t\). It implies \(\mu(rx) \geq t\), that is \(rx \in \mu_t\). Similarly, \(xr \in \mu_t\). Therefore, μ is a hyperideal of R.

Conversely, Let μ be a fuzzy set in R such that \(\mu_t (\neq \Phi)\) is a hyperideal of R for all \(0 < t \leq 0.5\). If \(x, y \in R\) we have

\[
\mu(x) \geq \mu(x) \land \mu(y) \land 0.5 = t_0, \mu(y) \geq \mu(x) \land \mu(y) \land 0.5 = t_0.
\]

then \(x, y \in \mu_{t_0}\), and so \(z \in \mu_{t_0}\) for all \(z \in x + y\). Now, we have \(\mu(z) \geq t_0 = \mu(x) \land \mu(y) \land 0.5\) for all \(z \in x + y\). Hence, condition (i) of the Theorem 4.5 is verified. Now, if \(x \in \mu_t\), we have \(\mu(x) \geq \mu(x) \land 0.5 = t_0\). Then, \(x \in \mu_{t_0}\), so \(rx \in \mu_{t_0}\) for all \(r \in \mu_t\). Hence, \(\mu(x) \geq t_0 = \mu(a) \land 0.5\). Similarly, \(\mu(xr) \geq t_0 = \mu(a) \land 0.5\). This shows condition
(ii) of Theorem 4.5 holds. Therefore, μ is an \((\varepsilon, \varepsilon \wedge q)\)-fuzzy hyperideal of \(R\).

In Theorem 4.6, we discuss on level subsets in the interval \((0.0.5]\). In the next theorem, we see what happen to the subsets in interval \((0.5.1]\).

**Theorem 4.6:** Let \(\mu\) be a fuzzy set in \(R\). Then, \(\mu_t (\not= \emptyset)\) is a hyperideal of \(R\) for all \(t \in (0.5.1]\) if and only if for all \(x, y \in R\).

(i) \(\mu(x) \wedge \mu(x) \leq \mu(z) \vee 0.5\), for all \(z \in x+y\).

(ii) \(\mu(x) \leq \mu(\text{tr}) \vee 0.5\) and \(\mu(x) \leq \mu(\text{tx}) \vee 0.5\).

**Proof:** Let \(\mu\) be a hyperideal of \(R\) for all \(t \in (0.5.1]\). If there exists \(x, y \in R\) such that \(\mu(z) \wedge \mu(x) \wedge \mu(x) = t\), for all \(z \in x+y\), then \(t \in (0.5.1]\), \(\mu(z) \leq t\), \(x \in \mu_t\) and \(y \in \mu_t\) for all \(z \in x+y\). Hence, \(z \in \mu_t\), for all \(z \in x+y\) and so \(\mu(z) \leq t\), for all \(z \in x+y\), which is a contradiction. Therefore, for all \(x, y \in R\), we have \(\mu(z) \wedge 0.5 \leq \mu(x) \wedge \mu(y)\), for all \(z \in x+y\). Thus, (1) is proved. Also, if there exists \(x, y \in R\) such that \(\mu(\text{tx}) \wedge 0.5 \leq \mu(x) \wedge 0.5 \leq \mu(x) = t\), then \(t \in (0.5.1]\), \(\mu(\text{tx}) \leq t\) and \(x \in \mu_t\). Hence, \(\mu(\text{tx}) \geq t\), which is a contradiction. Therefore, for all \(x, y \in R\), we have \(\mu(\text{tx}) \wedge 0.5 \geq \mu(x) \wedge \mu(x) \wedge 0.5 \geq \mu(x)\). Thus, (2) is proved.

Conversely, let (1) and (2) hold. Assume that \(t \in (0.5.1]\) and \(x, y \in \mu_t\). Then, by (1) we have \(0.5 \leq t \leq \mu(x) \wedge \mu(y) \leq \mu(z) \vee 0.5\), for all \(z \in x+y\). It implies that \(0.5 \leq t \leq \mu(z) \vee 0.5\), for all \(z \in x+y\). Hence, \(\mu(z) \geq t\) for all \(z \in x+y\), which means \(z \in \mu_t\), for all \(z \in x+y\). Also, suppose that \(t \in (0.5.1]\), \(x \in \mu_t\) and \(r \in R\). Then, by (2) we have \(0.5 \leq t \leq \mu(r) \vee 0.5\). It implies \(0.5 \leq \mu(r) \wedge 0.5\). Hence, \(\mu(r) \geq t\), which means \(rx \in \mu_t\). Similarly, \(xr \in \mu_t\). Therefore, \(\mu_t\) is a hyperideal of \(R\). Let \(\mu\) be a fuzzy set in \(R\) and \(J\) be the set of \(t \in (0.1]\) such that \(\mu_t (\not= \emptyset)\) or \(\mu_t\) is a hyperideal of \(R\). If \(J = (0.1]\), then by Theorem 4.6, \(\mu\) is an \((\varepsilon, \varepsilon \wedge q)\)-fuzzy hyperideal of \(R\). Naturally, a corresponding result should be considered when \(J = (0.5.1]\).

**Definition 4.7:** A fuzzy set \(\mu\) in \(R\) is called \((\varepsilon, \varepsilon \wedge q)\)-fuzzy hyperideal of \(R\) if for all \(t_1, t_2 \in (0.1]\) and \(r, x, y \in R\), the following condition hold:

1. \((a)\) if \(\mu(rx) \geq \mu(x)\) and \(\mu(xr) \geq \mu(t)\), then \(x \in \mu_t\).

2. \((b)\) if \(\mu(x) \wedge \mu(y) \geq \mu(z) \vee 0.5\), then \(x \in \mu_t\).
(b) If \( \mu(rx) \geq \mu(x) \), then by (ii) we have \( 0.5 \geq \mu(x) \).

Hence, \( \mu(rx) \lor 0.5 \geq \mu(x) \). Now if \( x_1 \in \mu \), then \( t \leq \mu(x) \leq 0.5 \). It follows that \( x_1 \bar{q}_t \), which implies that \( x_1 \in \bar{q}_t \). Therefore, \( \mu \) is an \( (\bar{e}, \bar{q}) \)-fuzzy hyperideal of \( R \).

In the following theorem, we characterize \( (\bar{e}, \bar{q}) \)-fuzzy hyperideals based on level subsets.

**Theorem 4.9:** A fuzzy set \( \mu \) in \( R \) is an \( (\bar{e}, \bar{q}) \)-fuzzy hyperideal of \( R \) if and only if \( \mu_t(\neq \Phi) \) is a hyperideal of \( R \) for all \( t \in (0, 1] \).

**Proof:** It follows by Theorem 4.7 and 4.9. For any fuzzy set \( \mu \) in \( R \) and \( t \in (0, 1] \), we put \( \mu_t = \{ x \in R / x \in \mu_t \} \).

\[ \bar{\mu}_t = \{ x \in R / x \in \bar{\mu}_t \} \]

Clearly \( \bar{\mu}_t = \mu_t \cup \mu_t \). In fact, \( \mu_t \) and \( \bar{\mu}_t \) are generalized level subsets. Now, we can characterize \( (\bar{e}, \bar{q}) \)-fuzzy hyperideals based on generalized level subsets.

**Theorem 4.10:** A fuzzy set \( \mu \) in \( R \) is an \( (\bar{e}, \bar{q}) \)-fuzzy hyperideal of \( R \) if and only if \( \bar{\mu}_t \) is a hyperideal of \( R \) for all \( t \in (0, 1] \).

**Proof:** Let \( \mu \) be an \( (\bar{e}, \bar{q}) \)-fuzzy hyperideal of \( R \) and \( x, y \in \bar{\mu}_t \) for \( t \in (0, 1] \). Then, \( x_1 \in \bar{\mu}_t \) and \( y_1 \in \bar{\mu}_t \), which means \( \mu(x) \geq t \) or \( \mu(y) > t \) and \( \mu(y) > t \) or \( \mu(y) > t \).

On the other hand, by Theorem 4.5, we know \( \mu(z) \geq \mu(x) \land \mu(y) < t \) for all \( z \in x+y \), so if \( \mu(z) < t \land 0.5 \), for all \( z \in x+y \), then \( \mu(z) > t \land 0.5 \). Hence, \( \mu(x) \land t \) or \( \mu(y) < t \), that is, \( x_1 \in \bar{\mu}_t \) or \( y_1 \in \bar{\mu}_t \). Thus, \( x_1 \in \bar{\mu}_t \) or \( y_1 \in \bar{\mu}_t \) which is a contradiction. We know \( t \geq 0.5 \), then \( \mu(z) \geq t \land 0.5 \) and so \( z \in \mu_t \subseteq \bar{\mu}_t \) for all \( z \in x+y \). Also, let \( r \in R \) and \( x \in \bar{\mu}_t \) for \( t \in (0, 0.5] \).

Then, \( x_1 \in \bar{\mu}_t \) which means \( \mu(x) \geq t \) or \( \mu(x) < t \). On the other hand, by Theorem 4.5, we know that \( \mu(rx) \geq \mu(x) \land 0.5 \). Hence, \( \mu(x) \land t \) that is \( x_1 \in \bar{\mu}_t \), which is a contradiction. We know \( t \geq 0.5 \), then \( \mu(x) \geq t \land 0.5 \) and so \( rx \in \mu_t \subseteq \bar{\mu}_t \). Similarly, \( x \in \bar{\mu}_t \), therefore, \( \bar{\mu}_t \) is a hyperideal of \( R \).

Conversely, let \( \bar{\mu}_t \) be a hyperideal of \( R \) for \( t \in (0, 0.5] \). Suppose \( x, y \in R \) such that \( \mu(x) \land \mu(y) < t \). Then, there exists \( t \in (0, 0.5) \) such that \( \mu(z) < t \land 0.5 \), for all \( z \in x+y \). Hence, \( \mu(z) < t \land 0.5 \), which is a contradiction. Therefore, \( \mu(x) \land \mu(y) < t \) for all \( z \in x+y \). Also, suppose \( r, x \in R \) such that \( \mu(rx) < t \land 0.5 \), then there exists \( t \in (0, 0.5) \) such that \( \mu(rx) < t \land 0.5 \). It follows \( x \in \mu_t \subseteq \bar{\mu}_t \), which implies \( rx \in \bar{\mu}_t \). Hence, \( \mu(x) \geq t \) or \( \mu(x) > t \land t > 1 \), which is a contradiction. Thus, \( \mu(rx) \geq \mu(x) \land 0.5 \). Similarly, \( \mu(rx) \geq \mu(x) \land 0.5 \). Therefore, the proof is completed.

In the next theorem, we discuss on \( (\bar{e}, \bar{q}) \)-fuzzy hyperideal of \( R \) which can be expressed as the union of two proper non-equivalent \( (\bar{e}, \bar{q}) \)-fuzzy hyperideals.

**Theorem 4.11:** Let \( \mu \) be a proper \( (\bar{e}, \bar{q}) \)-fuzzy hyperideal of \( R \) such that \( 2 \leq |\mu(x)| |\mu(x) < 0.5| < \infty \).

Then, there exist two proper non-equivalent \( (\bar{e}, \bar{q}) \)-fuzzy hyperideal of \( R \) such that \( \mu \) can be expressed as the union of them.

**Proof:** Let \( \mu(x) |\mu(x) < 0.5| = [t_0, t_1, ..., t_n] \), where \( t_1 > t_2 > ... > t_n \) and \( r \geq 2 \). Then, the chain of \( (\bar{e}, \bar{q}) \)-level ideals of \( R \) is \( \bar{\mu}_{t_0} \subseteq \bar{\mu}_{t_1} \subseteq ... \subseteq \bar{\mu}_{t_n} = R \). Let \( v \) and \( \theta \) be fuzzy sets in \( R \) defined by

\[
v(x) = \begin{cases} 
    t_1 & \text{if } x \in \bar{\mu}_{t_1}, \\
    t_2 & \text{if } x \in \bar{\mu}_{t_2} \setminus \bar{\mu}_{t_1}, \\
    ... & \\
    t_r & \text{if } x \in \bar{\mu}_{t_r} \setminus \bar{\mu}_{t_{r-1}}, \\
\end{cases}
\]

and

\[
\theta(x) = \begin{cases} 
    \mu(x) & \text{if } x \in \bar{\mu}_{t_0}, \\
    k & \text{if } x \in \bar{\mu}_{t_0} \setminus \bar{\mu}_{t_1}, \\
    t_3 & \text{if } x \in \bar{\mu}_{t_3} \setminus \bar{\mu}_{t_2}, \\
    ... & \\
    t_4 & \text{if } x \in \bar{\mu}_{t_4} \setminus \bar{\mu}_{t_3}, \\
    ... & \\
    t_r & \text{if } x \in \bar{\mu}_{t_r} \setminus \bar{\mu}_{t_{r-1}}, \\
\end{cases}
\]
where \( t_1 < k < t_2 \). The, \( v \) and \( \theta \) are \((\epsilon, \epsilon \vee q)\)-fuzzy hyperideals of \( R \) and \( v, \theta \mu \). The chains of \((\epsilon \vee q)\)-level ideals of \( v \) and \( \theta \) are, respectively, given by \( |\mu|_{0.5} \subseteq |\mu|_{0.5} \subseteq |\mu|_{0.5} \subseteq \ldots \subseteq |\mu|_{0.5} \) and \( |\mu|_{0.5} \subseteq |\mu|_{0.5} \subseteq |\mu|_{0.5} \subseteq \ldots \subseteq |\mu|_{0.5} \). Thus, \( v \) and \( \theta \) are non-equivalent and clearly \( \mu = v \vee 0 \). Therefore, \( \mu \) be expressed as the union of two proper non-equivalent \((\epsilon, \epsilon \vee q)\)-fuzzy hyperideal of \( R \).

**t-IMPLICATION-BASED FUZZY HYPERIDEALS OF SEMIRINGS**

In this section, we generalize the notion of ordinary fuzzy hyperideals, \((\epsilon, \epsilon \vee q)\)-fuzzy hyperideals and \((\epsilon, \epsilon \vee q)\)-fuzzy hyperideals. Specially, we characterize fuzzy hyperideals, \((\epsilon, \epsilon \vee q)\)-fuzzy hyperideals and \((\epsilon, \epsilon \vee q)\)-fuzzy hyperideals based on implication operators.

**Definition 5.1:** Let \( m, n \in [0,1] \), \( m < n \) and \( \mu \) be a fuzzy set in \( R \). Then, \( \mu \) is said to be a fuzzy hyperideal with thresholds \((m,n)\) of \( R \), if and only if all \( r, x, y \in R \), the following conditions hold:

(i) \( \mu(x) \wedge \mu(y) \wedge n \leq \mu(z) \lor m \), for all \( z \in x+y \),

(ii) \( \mu(x) \wedge n \leq \mu(x) \lor m \) and \( \mu(x) \wedge n \leq \mu(x) \lor m \).

Clearly, every fuzzy hyperideal with thresholds \((m,n)\) of \( R \) is an ordinary fuzzy hyperideal when \( m = 0 \) and \( n = 1 \) (see definition 1). Also, it is an \((\epsilon, \epsilon \vee q)\)-fuzzy (resp. \((\epsilon, \epsilon \vee q)\)-fuzzy) hyperideals when \( m = 0 \) and \( n = 0.5 \) (resp. \( m = 0 \) and \( n = 0.5 \)) (see Theorem 4.7 and 4.9).

**Theorem 5.2:** A fuzzy set \( \mu \) in \( R \) is a fuzzy hyperideal with threshold \((m,n)\) of \( R \) if and only if \( \mu(x) \neq \emptyset \) is a hyperideal of \( R \) for all \( t \in [m,n] \).

**Proof:** Suppose that \( \mu \) is a fuzzy hyperideal with threshold \((m,n)\) of \( R \) and \( t \in [m,n] \). If \( x, y \in \mu \), then \( \mu(x) \geq t \) and \( \mu(y) \geq t \). We have \( \mu(z) \lor m \geq \mu(x) \wedge \mu(y) \wedge n \geq t \wedge n = t \geq m \), for all \( z \in x+y \). Hence, \( \mu(z) \lor m \geq t \geq m \), for all \( z \in x+y \), which implies \( \mu(z) \geq t \), for all \( z \in x+y \), that is \( z \in \mu \), for all \( z \in x+y \). Now, if \( x, y \in \mu \), and \( r \in R \), then \( \mu(x) \geq t \). We have \( \mu(r) \lor m \geq \mu(x) \wedge n \geq t \geq m \). Hence, \( \mu(r) \lor m \geq t \geq m \), which implies \( \mu(r) \geq t \), that is \( r \in \mu \). Similarly, \( r \in \mu \). Therefore, \( \mu \) is a hyperideal of \( R \).

Conversely, let \( \mu \) be a fuzzy set in \( R \). If there exist \( x, y \in R \) such that \( \mu(x) \lor m \geq \mu(x) \wedge \mu(y) \wedge n = t \), for all \( z \in x+y \), then \( t \in (m,n] \), \( \mu(z) \geq t \), \( x \in \mu \) and \( y \in \mu \), for all \( z \in x+y \). Since \( \mu \) is a hyperideal of \( R \), we have \( z \in \mu \), for all \( z \in x+y \). Thus, \( z \subseteq \mu \), for all \( z \in x+y \). Hence, \( \mu(z) \geq t \) for all \( z \in x+y \), which is a contradiction. Therefore, for all \( x, y \in R \) we have \( \mu(x) \wedge \mu(y) \leq \mu(z) \lor m \), for all \( z \in x+y \). Also, if there exist \( r, x \in R \) such that \( \mu(r) \lor m \geq \mu(x) \wedge n \geq t \wedge n = t \), then \( t \in (m,n] \), \( \mu(r) \geq t \), which is a contradiction. Thus, for all \( r, x \in R \), we have \( \mu(x) \wedge n \leq \mu(x) \lor m \). Similarly, \( \mu(x) \wedge n \leq \mu(r) \lor m \), therefore, \( \mu \) is a fuzzy hyperideal with thresholds \((m,n)\) of \( R \).

Set theoretic multivalued logic is a special case of fuzzy logic such that the truth values are linguistic variables (or terms of the linguistic variables truth). By using extension principal some operators like \( \wedge, \lor, ?, \rightarrow \) can be applied in fuzzy logic. In fuzzy logic, \([P]\) means the truth value of fuzzy proposition \( P \). In the following, we show a correspondence between fuzzy logic and set-theoretical notions.

\[
\begin{align*}
x \in A & \Rightarrow A(x) \\
x \notin A & \Rightarrow \neg A(x) \\
[P \land Q] = \min([P],[Q]) \\
[P \lor Q] = \max([P],[Q]) \\
[P \rightarrow Q] = \min(1,1-[P]+[Q]) \\
[\forall x \in P(x)] = \inf(P(x)) 
\end{align*}
\]

?P if and only if \([P] = 1\) for all valuations.

We show some of important implication operators, where \( \alpha \) denotes the degree of membership of the premise and \( \beta \) is the degree of membership of the consequence and \( I \) the resulting degree of truth for the implication.

- Early Zadeh: \( I_{\text{EZ}}(\alpha,\beta) = \max\{1 - \alpha, \min(\alpha, \beta)\} \).
- Lukasiewicz: \( I_{\text{LU}}(\alpha,\beta) = \min\{1, \alpha + \beta\} \).
- Standard Star (Godel): \( I_{\text{G}}(\alpha,\beta) = \begin{cases} 1 & \text{if } 0 \leq \beta \\ \beta & \text{otherwise} \end{cases} \).
- Contraposition of (Godel): \( I_{\text{CG}}(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \alpha & \text{otherwise} \end{cases} \).
- Gaines-Rescher: \( I_{\text{GR}}(\alpha,\beta) = \begin{cases} 1 & \text{if } 0 \leq \beta \\ 0 & \text{otherwise} \end{cases} \).
- kleene-dienes: \( I_{\text{KD}}(\alpha,\beta) = \max\{1 - \alpha, \beta\} \).
Definition 5.3: A fuzzy set μ in R is called fuzzifying hyperideal of R, if and only if for all r,x,y∈R it satisfies:

1. \( I(\{x\} \land \{y\} \rightarrow \{z\}) \), for all \( z \in x+y \),
2. \( I(\{x\} \land \{rx\} \rightarrow \{xr\}) \).

Clearly, Definition 26 is equivalent to Definition 1. Therefore, a fuzzifying hyperideal is an ordinary fuzzy hyperideal. We have the notion of t-tautology. In fact \( ? \text{P} \), if and only if \( [P] \geq t \) [33].

Definition 5.4: A fuzzy set μ in R is said to be t-implication-based fuzzy left (resp. right) hyperideal of R with respect to the implication→if the following conditions hold for all r,x,y∈R:

1. \( \mu(\{x\} \land \{y\} \rightarrow \{z\}) \geq t \), for all \( z \in x+y \),
2. \( \mu(\{x\} \land \{rx\} \rightarrow \{xr\}) \).

A fuzzy set μ in R is said to be t-implication-based fuzzy hyperideal of R with respect to the implication→ if μ is both t-implication-based fuzzy left and right hyperideal of R with respect to the implication→.

Proposition 5.5: A fuzzy set μ of R is a t-implication-based fuzzy hyperideal of R with respect to the implication operator \( \rightarrow \) if and only if for all r,x,y∈R:

i) \( I(\mu(x) \land \mu(y), \mu(z)) \geq t \) for all \( z \in x+y \),
ii) \( I(\mu(x), \mu(rx)) \geq t \) and \( (I(\mu(x), \mu(xr)) \geq t \).

Proof: The proof is clear by considering the definitions.

Theorem 5.6

1. Let \( I = I_\text{fr} \) (Gaines). Then, μ is an 0.5-implication-based fuzzy hyperideal of R if and only if μ is a fuzzy hyperideal with thresholds \( m = 0 \) and \( n = 1 \) of R (or equivalent, μ is an ordinary fuzzy hyperideal of R).
2. Let \( I = I_\text{fr} \) (Godel). Then, μ is an 0.5-implication-based fuzzy hyperideal of R if and only if μ is a fuzzy hyperideal with thresholds \( m = 0 \) and \( n = 0.5 \) of R (or equivalent, μ is an \((e,e \lor q)\)-fuzzy hyperideal of R).
3. Let \( I = I_\text{cg} \) (Contraposition of Godel). Then, μ is an 0.5-implication-based fuzzy hyperideal of R if and only if μ is a fuzzy hyperideal with thresholds \( m = 0.5 \) and \( n = 1 \) of R (or equivalent, μ is an \((e,e \land q)\)-fuzzy hyperideal of R).

Proof: (1) Let μ be an 0.5-implication-based fuzzy hyperideal of R. Then \( I_\text{fr}(\mu(x) \land \mu(y), \mu(z)) \geq 0.5 \), for all \( z \in x+y \). Which implies \( \mu(z) \geq \mu(x) \land \mu(y) \), for all \( z \in x+y \). Also, \( I_\text{fr}(\mu(x), \mu(rx)) \geq 0.5 \), which implies \( \mu(rx) \geq \mu(x) \). Similarly, \( \mu(rx) \geq \mu(x) \). Therefore, μ is a fuzzy hyperideal with threshold m = 0 and n = 1 of R.

Conversely, let μ be a fuzzy hyperideal with threshold m = 0 and n = 1 of R. Then,

\[ \mu(z) \geq \mu(x) \land \mu(y), \text{ for all } z \in x+y, \]
\[ \mu(rx) \geq \mu(x), \mu(xr) \geq \mu(x), \text{ for all } r,x,y \in R. \]

Hence, \( I_\text{fr}(\mu(x) \land \mu(y), \mu(z)) = 1 \), for all \( z \in x+y \). \( I_\text{fr}(\mu(x), \mu(rx)) = 1 \). Thus, \( I_\text{fr}(\mu(x) \land \mu(y), \mu(z)) \geq 0.5 \), for all \( z \in x+y \), \( I_\text{fr}(\mu(x), \mu(rx)) \geq 0.5 \), and \( I_\text{fr}(\mu(x), \mu(xr)) \geq 0.5 \). Therefore, μ is a 0.5-implication-based fuzzy hyperideal of R.

(2) Let μ be an 0.5-implication-based fuzzy hyperideal of R. Then, for all r,x,y∈R we have \( I_\text{fr}(\mu(x) \land \mu(y), \mu(z)) \geq 0.5 \), for all \( z \in x+y \), \( I_\text{fr}(\mu(x), \mu(rx)) \geq 0.5 \) and \( I_\text{fr}(\mu(x), \mu(xr)) \geq 0.5 \). By the definition of \( I_\text{fr} \), we can consider the following cases:

(a) \( I_\text{fr}(\mu(x) \land \mu(y), \mu(z)) = 1 \), for all \( z \in x+y \), then \( \mu(x) \land \mu(y) \leq \mu(z) \), for all \( z \in x+y \), which implies \( \mu(x) \lor \mu(y) \geq 0.5 \), for all \( z \in x+y \).

(b) \( I_\text{fr}(\mu(x) \land \mu(y), \mu(x+y)) = 1 \), then \( \mu(z) = \mu(z) \), for all \( z \in x+y \) then \( \mu(z) \geq 0.5 \), for all \( z \in x+y \). Which implies \( \mu(x) \land \mu(y) \geq 0.5 \), for all \( z \in x+y \). Similarly, we can show that \( \mu(x) \lor 0.5 \leq \mu(rx) \) and \( \mu(x) \lor 0.5 \leq \mu(xr) \). Therefore, μ is a fuzzy hyperideal with thresholds m = 0 and n = 0.5 of R.

Conversely, let μ be a fuzzy hyperideal with thresholds m = 0 and n = 0.5 of R. Then, for all r,x,y∈R, by Definition Def \( \mu(x) \land \mu(y) \land 0.5 \leq \mu(z) \), for all \( z \in x+y \) and \( \mu(x) \land 0.5 \leq \mu(rx) \) and \( \mu(x) \land 0.5 \leq \mu(xr) \). Hence, in each case, \( I_\text{fr}(\mu(x) \land \mu(y), \mu(z)) \geq 0.5 \), for all \( z \in x+y \), \( I_\text{fr}(\mu(x), \mu(rx)) \geq 0.5 \), and \( I_\text{fr}(\mu(x), \mu(xr)) \geq 0.5 \). Therefore, μ is an 0.5-implication-based fuzzy hyperideal of R.

(3) Let μ be an 0.5-implication-based fuzzy hyperideal of R. Then, for all r,x,y∈R, we have for all \( z \in x+y \), \( I_\text{fr}(\mu(x), \mu(rx)) \geq 0.5 \) and \( I_\text{fr}(\mu(x), \mu(xr)) \geq 0.5 \). By
definition of $I_{cg}$, we can consider the following cases:

(a) If $I_{cg}(\mu(x) \land \mu(y), \mu(z)) = 1$, for all $z \in x+y$, then
\[ \mu(x) \land \mu(y) \leq \mu(z), \]
for all $z \in x+y$, which implies that
\[ \mu(x) \land \mu(y) \leq \mu(z) \vee 0.5, \]
for all $z \in x+y$.

(b) if $I_{cg}(\mu(x) \land \mu(y), \mu(z)) = 1 - (\mu(x) \land \mu(y))$, for all
\[ z \in x+y, \]
then $1 - (\mu(x) \land \mu(y)) \geq 0.5$, it implies that
\[ \mu(x) \land \mu(y) \leq 0.5 \]
and hence $\mu(x) \land \mu(y) \leq \mu(z) \vee 0.5$, for all $z \in x+y$. Similarly, we can show that
\[ \mu(x) \leq \mu(xr) \vee 0.5 \]
and $\mu(x) \leq \mu(xr) \vee 0.5$. Therefore, $\mu$ is a fuzzy hyperideal with threshold $m = 0.5$ and $n = 1$ of $R$.

Conversely, let $\mu$ be a fuzzy hyperideal with threshold $m = 0.5$ and $n = 1$ of $R$. Then, for all $x, y, z \in R$, we have
\[ \mu(x) \land \mu(y) \leq \mu(z) \vee 0.5, \]
for all $z \in x+y$, $\mu(x) \leq \mu(xr) \vee 0.5$ and $\mu(x) \leq \mu(xr) \vee 0.5$. Now, we can consider two following cases:

(a) $\mu(x) \land \mu(y) \leq \mu(z)$, for all $z \in x+y$, which implies
\[ I_{cg}(\mu(x) \land \mu(y), \mu(z)) = 1 \geq 0.5, \]
for all $z \in x+y$.

(b) if $\mu(x) \land \mu(y) > \mu(z)$, for all $z \in x+y$, which implies
\[ \mu(x) \land \mu(y) \geq 0.5. \]
Hence, $1 - (\mu(x) \land \mu(y)) \geq 0.5$. Thus,
\[ I_{cg}(\mu(x) \land \mu(y), \mu(z)) = 1 - (\mu(x) \land \mu(y)) \geq 0.5, \]
for all $z \in x+y$. Similarly, we can prove that
\[ I_{cg}(\mu(x), \mu(xr)) \geq 0.5 \]
and
\[ I_{cg}(\mu(x), \mu(xr)) \geq 0.5. \]
Therefore, $\mu$ is an 0.5-implication-based fuzzy hyperideal of $R$.

CONCLUSIONS

As a generalization of hyperideals in semihyperrings, we introduce and study new sorts of fuzzy hyperideal in semihyperrings and investigate related properties. In the notion of an $(\alpha, \beta)$-fuzzy hyperideal, we can consider twelve different types of such structures resulting from three choices of $\alpha$ and four choices of $\beta$. In this study, we focus on the types $(\epsilon, \epsilon)$, $(\epsilon q, \epsilon)$, $(\epsilon q, \epsilon q)$, $(\epsilon, \epsilon q)$, and their relationship to the implication operators. Future researches will focus on the following steps:

(1) Considering other types of fuzzy hyperideals of semihyperrings together with relations among them.

(2) Generalizations of the notion of quasi-coincidence of fuzzy points, such as: $q_k$ and $q_{(\lambda, \nu)}$.

(3) Investigating, generalized (interval valued) fuzzy prime and semiprime hyperideals of semihyperrings and their relationship to the triangular norms. (4) Introducing, $(\alpha, \beta)$-intuitionistic fuzzy (prime and semiprime) hyperideals of semihyperrings. (5) Using, the obtained results to solve some social networks problems.

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