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System of Nonlinear Regularized Nonconvex Variational Inequalities in Banach Spaces

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Abstract

The aim of this communication is to suggest a *system of nonlinear regularized nonconvex variational inequalities* in q-uniformly smooth Banach spaces and established an equivalence relation between this system and fixed point problems. By using the equivalence relation we construct a new perturbed projection iterative algorithms with mixed errors for finding a solution set of *system of nonlinear regularized nonconvex variational inequalities.* Also proved the convergence of the suggested iterative sequences generated by the algorithm.

Keywords: System of nonlinear regularized nonconvex variational inequalities, uniformly r-prox regular sets, q-uniformly smooth Banach spaces, iterative sequences, convergence analysis, mixed errors.

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1 Introduction

Variational inequalities and variational inclusions, which have been extended and generalized in different directions by using novel, innovative techniques and ideas, and provide a mathematical models to some problems arising in economics, mechanics, engineering sciences and other pure and applied sciences [8]. Recently some systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years because of their closed relation to Nash equilibrium problems. Huang and Fang [10] introduced a system of order complementarity problems and established some existence results for these using fixed point theory. Verma [18] introduced and studied some system of variational inequalities and developed some iterative algorithm for approximating the solutions of system of variational inequalities. We remark that the almost all results concerning the system of solutions of iterative scheme for solving the system of variational inequalities and related problems are being considered in the setting of convex sets. Consequently the techniques are based on the projections of operator over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the unified prox-regular sets are nonconvex and included the convex sets as special cases, *see* [4, 12, 20]. Inspired by the recent work going on this fields *see* [1, 2, 3, 6, 7, 9, 11, 13, 15, 17], in this paper, we introduced and studied a *system of nonlinear regularized nonconvex variational inequalities* in q-uniformly smooth Banach spaces. We established the equivalence between the *system of nonlinear regularized nonconvex variational inequalities* and some fixed point problems. By using the equivalence relation, we construct a perturbed projection iterative algorithms with mixed errors for finding a solution set of the aforementioned system. Also we proved the convergence of the defined iterative algorithms under suitable assumptions.

2 Preliminaries

Let $\mathcal{X}$ be a real Banach space with dual space $\mathcal{X}^*$, $\langle \cdot, \cdot \rangle$ be the dual pairing between $\mathcal{X}$ and $\mathcal{X}^*$ and $CB(\mathcal{X}^*)$ denote the family of all nonempty closed bounded subsets of $\mathcal{X}$. The generalized duality mapping $J_q : \mathcal{X} \longrightarrow 2^{\mathcal{X}^*}$ is defined by

$$J_q(x) = \{f^* \in \mathcal{X}^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\} \ \forall x \in \mathcal{X}$$

where $q > 1$ is a constant. In particular J_2 is a usual normalized duality mapping. It is known that in general $J_q(x) = \|x\|^{q-1} J_2(x)$ for all $x \neq 0$ and J_q is single valued if $\mathcal{X}^*$ is strictly convex. In the sequel, we always assume that $\mathcal{X}$ is a real Banach space such that J_q is a single valued. If $\mathcal{X}$ is a Hilbert space then J_q becomes the identity mapping on $\mathcal{X}$. The modulus of smoothness of $\mathcal{X}$ is the function $\rho_{\mathcal{X}} : [0, \infty) \longrightarrow [0, \infty)$ is defined by

$$\rho_{\mathcal{X}}(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space $\mathcal{X}$ is called uniformly smooth if

$$\lim_{t \longrightarrow 0} \frac{\rho_{\mathcal{X}}(t)}{t} = 0.$$

$\mathcal{X}$ is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_{\mathcal{X}}(t) < ct^q, q > 1.$$

Note that J_q is single valued if $\mathcal{X}$ is uniformly smooth. It is known that

$$L_p(\text{or } l_p) \text{ or } W_m^p = \begin{cases} p - \text{uniformly smooth} & \text{if } 1 < p < 0, \\ 2 - \text{uniformly smooth} & \text{if } p \geq 2. \end{cases}$$

A Banach space $\mathcal{X}$ is said to be uniformly convex if given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathcal{X}$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$. It is well known that L_p, l_p and Sobolev spaces $W_m^p(1 < p < \infty)$ are uniformly convex. Note that J_q is a single valued if $\mathcal{X}$ is uniformly smooth.

Definition 2.1. The proximal normal cone of $\mathcal{K}$ at a point $u \in \mathcal{X}$ is given by

$$N_{\mathcal{K}}^P(u) = \{\zeta \in \mathcal{X} : u \in P_{\mathcal{K}}(u + \alpha\zeta)\},$$

where $\alpha > 0$ is a constant and $P_{\mathcal{K}}$ is the projection operator of $\mathcal{X}$ onto $\mathcal{K}$, that is,

$$P_{\mathcal{K}}(u) = \{v \in \mathcal{K} : d_{\mathcal{K}}(u) = \|u - v\|\},$$

where $d_{\mathcal{K}}(u)$ is the usual distance function to the subset $\mathcal{K}$, that is,

$$d_{\mathcal{K}}(u) = \inf_{v \in \mathcal{K}} \|u - v\|.$$

Lemma 2.2. Let $\mathcal{K}$ be a nonempty closed subset of $\mathcal{X}$. Then $\zeta \in N_{\mathcal{K}}^P(u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle \zeta, j_q(v - u) \rangle \leq \alpha \|v - u\|^q \quad \forall v \in \mathcal{K}.$$

Definition 2.3. The Clarke normal cone, denoted by $N_{\mathcal{K}}^C(u)$, is defined as

$$N_{\mathcal{K}}^C(u) = \overline{co}[N_{\mathcal{K}}^P(u)],$$

where $\overline{co}\mathcal{A}$ means the closure of the convex hull of $\mathcal{A}$. It is clear that $N_{\mathcal{K}}^P(x) \subseteq N_{\mathcal{K}}^C(x)$. The converse is not true in general. Note that $N_{\mathcal{K}}^C(x)$ is closed and convex, but $N_{\mathcal{K}}^P(x)$ is convex, which may not be closed, (*see* [5, 16, 19]).

Definition 2.4. [4] For any $r \in (0, +\infty]$, a subset $\mathcal{K}_r$ of $\mathcal{H}$ is said the normalized uniformly prox-regular (or uniformly r-prox-regular) if every nonzero proximal normal to $\mathcal{K}_r$ can be realized by an r-ball. This means that for all $\overline{x} \in \mathcal{K}_r$ and all $0 \neq \zeta \in N_{\mathcal{K}_r}^P(\overline{x})$ with $\|\zeta\| = 1$,

$$\langle \zeta, j_q(x - \overline{x}) \rangle \leq \frac{1}{2r} \|x - \overline{x}\|^q, \quad x \in \mathcal{K}.$$

Lemma 2.5. [5] A closed set $\mathcal{K} \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for every $r > 0$.

Proposition 2.6. [16] Let $r > 0$ and let $\mathcal{K}_r$ be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set

$$\mathcal{U}(r) = \{u \in \mathcal{X} : 0 \leq d_{\mathcal{K}_r}(u) < r\}.$$

Then the following statements are hold:

(a) for all $x \in \mathcal{U}(r)$, $P_{\mathcal{K}_r}(x) \neq \emptyset$;

(b) for all $r' \in (0, r)$, is Lipschitz continuous mapping with constant $\frac{r}{r-r'}$ on

$$\mathcal{U}(r') = \{u \in \mathcal{H} : 0 \leq d_{\mathcal{K}_r}(u) < r'\};$$

(c) the proximal normal cone is closed as a set-valued mapping.

From Proposition 2.6 (c), we have $N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$. Therefore we define $N_{\mathcal{K}_r}(x) = N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$ for a class of sets.

Definition 2.7. The single-valued mapping $h : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be

(i) accretive if

$$\langle h(x) - h(y), j_q(x - y)\rangle \geq 0, \ \forall x, y \in \mathcal{X},$$

(ii) β-strongly accretive if there exists a constant $\beta > 0$ such that

$$\langle h(x) - h(y), j_q(x - y)\rangle \geq \beta\|x - y\|^q, \ \forall x, y \in \mathcal{X},$$

(iii) σ-Lipschitz continuous mapping if there exists a constant $\sigma > 0$ such that

$$\|h(x) - h(y)\| \leq \sigma\|x - y\|, \ \forall x, y \in \mathcal{X}.$$

Definition 2.8. Let $g : \mathcal{X} \longrightarrow \mathcal{X}$ be a single-valued mapping and let $Q : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ be a single valued mapping. Then Q is said to be

(i) accretive if

$$\langle Q(x, y) - Q(x', y), j_q(x - x')\rangle \geq 0, \quad \forall x, y \in \mathcal{X},$$

(ii) $\kappa - g$-strongly accretive with respect to g and the first variable of Q if there exists a constant $\kappa > 0$ such that

$$\langle Q(x, y) - Q(x', y), j_q(g(x) - g(y))\rangle \geq \kappa\|g(x) - g(x')\|^q, \ \forall x, y \in \mathcal{X}.$$

Definition 2.9. A two-variable set-valued mapping $T : \mathcal{X} \times \mathcal{X} \longrightarrow 2^{\mathcal{X}}$ is $\xi - \widehat{\mathcal{D}}$-Lipschitz continuous in the first variable, if there exists a constant $\xi > o$ such that, for all $x, x' \in \mathcal{X}$,

$$\widehat{\mathcal{D}}(T(x, y), T(x', y')) \leq \xi\|x - x'\|, \quad \forall y, y' \in \mathcal{X},$$

where $\widehat{\mathcal{D}}$ is the Hausdorff pseudo metric, that is, for any two nonempty subsets A and B of $\mathcal{X}$

$$\widehat{\mathcal{D}}(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}.$$

Lemma 2.10. [21] The real Banach space $\mathcal{X}$ is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in \mathcal{X}$

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

3 System of Nonlinear Regularized Nonconvex Variational Inequalities

In this section, we introduce a new *system of nonlinear regularized nonconvex variational inequalities* in Banach space and investigated their relations.

Let $Q_i : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ be the single-valued mappings, $T_i, : \mathcal{X} \times \mathcal{X} \longrightarrow CB(\mathcal{X})$ be nonlinear set-valued mappings and $g_i, h_i : \mathcal{X} \longrightarrow \mathcal{X}$ be nonlinear single valued mappings such that $\mathcal{K}_r \subseteq g_i(\mathcal{X})$, $(i = 1, \dots, N)$. For any constants $\eta_i > 0, (i = 1, \dots, N)$, we consider the problem of finding $x_i \in \mathcal{X}(i = 1, \dots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \dots, N-1)$, $u_N \in T_N(x_1, x_N)$ such that $h_i(x_i) \in \mathcal{K}_r(i = 1, \dots, N)$ and

$$\begin{cases} \langle \eta_i Q_i(x_{i+1}, u_i) + h_i(x_i) - g_i(x_{i+1}), j_q(g_i(x) - h_i(x_i)\rangle + \frac{1}{2r}\|g_i(x) - h_i(x_i)\|^q \geq 0, (i = 1, \dots, N) \\ \langle \eta_N Q_N(x_1, u_N) + h_N(x_N) - g_N(x_1), j_q(g_N(x) - h_N(x_N))\rangle + \frac{1}{2r}\|g_N(x) - h_N(x_N)\|^q \geq 0, \\ \quad \forall x \in \mathcal{K}_r : g_1(x), \dots, g_N(x) \in \mathcal{K}_r. \end{cases} \tag{3.1}$$

The problem (3.1) is called the *system of nonlinear regularized nonconvex variational inequalities.*

Lemma 3.1. Let $\mathcal{K}_r$ be uniformly r-prox-regular set then the problem (3.1) is equivalent to finding $x_i \in \mathcal{X}(i = 1, \dots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \dots, N-1)$, $u_N \in T_N(x_1, x_N)$ such that

$$\begin{cases} 0 \in \eta_i Q_i(x_{i+1}, u_i) + h_i(x_i) - g_i(x_{i+1}) + N^P_{\mathcal{K}_r}(h_i(x_i))(i = 1, \dots, N-1), \\ 0 \in \eta_N Q_N(x_1, u_N) + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{3.2}$$

where $N^P_{\mathcal{K}_r}(s)$ denotes the P-normal cone of $\mathcal{K}_r$ at s in the sense of nonconvex analysis.

Proof. Let $(x_1, \dots, x_N, u_1, \dots, u_N)$ with $x_i \in \mathcal{X}$, $h_i(x_i) \in \mathcal{K}_r(i = 1, \dots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \dots, N-1)$, $u_N \in T_N(x_1, x_N)$ be solution sets of the system (3.1). If

$$\eta_1 Q_1(x_2, u_1) + h_1(x_1) - g_1(x_2) = 0$$

because the vector zero always belongs to any normal cone, then

$$0 \in \eta_1 Q_1(x_2, u_1) + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)).$$

If

$$\eta_1 Q_1(x_2, u_1) + h_1(x_1) - g_1(x_2) \neq 0$$

then for all $x \in \mathcal{X}$ with $g_1(x) \in \mathcal{K}_r$

$$\langle -(\eta_1 Q_1(x_2, u_1) + h_1(x_1) - g_1(x_2)), j_q(g_1(x) - h_1(x_1))\rangle \leq \frac{1}{2r}\|g_1(x) - h_1(x_1)\|^q. \tag{3.3}$$

From Lemma 2.2 we have

$$-(\eta_1 Q_1(x_2, u_1) + h_1(x_1) - g_1(x_2)) \in N^P_{\mathcal{K}_r}(h_1(x_1))$$

and

$$\begin{cases} 0 \in \eta_i Q_i(x_{i+1}, u_i) + h_i(x_i) - g_i(x_{i+1}) + N^P_{\mathcal{K}_r}(h_i(x_i))(i = 1, \ldots, N-1), \\ 0 \in \eta_N Q_N(x_1, u_N) + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{3.4}$$

Conversely if $(x_1, \ldots, x_N, u_1, \ldots, u_N)$ with $x_i \in \mathcal{X}(i = 1, \ldots, N)$, $h_i(x_i) \in \mathcal{K}_r(i = 1, \ldots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \ldots, N-1)$, $u_N \in T_N(x_1, x_N)$ are solution sets of the system (3.2) then from Definition 2.4, $x_i \in \mathcal{X}(i = 1, \ldots, N)$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \ldots, N-1)$, $u_N \in T_N(x_1, x_N)$ with $h_i(x_i) \in \mathcal{K}_r(i = 1, \ldots, N)$ are solution sets of the system (3.1). □

The problem (3.2) is called *system of nonlinear regularized nonconvex variational inclusions* in real Banach spaces.

4 Main results

Lemma 4.1. Let $Q_1, \cdots, Q_N, T_1, \ldots, T_N, g_1, \ldots, g_N, h_1, \ldots, h_N, \eta_1, \ldots, \eta_N$ be the same as in the system (3.1). Then $(x_1, \ldots, x_N, u_1, \ldots, u_N)$ with $x_i \in \mathcal{X}, h_i(x_i) \in \mathcal{K}_r$ for all $i = 1, \ldots, N$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, $\ldots, u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ are solution sets of the system (3.1) if and only if

$$\begin{cases} h_i(x_i) = P_{\mathcal{K}_r}[g_i(x_{i+1}) - \eta_i Q_i(x_{i+1}, u_i)](i = 1, \ldots, N-1), \\ h_N(x_N) = P_{\mathcal{K}_r}[g_N(x_1) - \eta_N Q_N(x_1, u_N)], \end{cases} \tag{4.1}$$

where $P_{\mathcal{K}_r}$ is the projection of $\mathcal{X}$ onto the uniformly r-prox-regular set $\mathcal{K}_r$.

Proof. Let $(x_1, \ldots, x_N, u_1, \ldots, u_N)$ with $x_i \in \mathcal{X}, h_i(x_i) \in \mathcal{K}_r$ for all $i = 1, \ldots, N$ and $u_i \in T_i(x_{i+1}, x_i)(i = 1, \ldots, N-1)$, $u_N \in T_N(x_1, x_N)$ are solution sets of the system (3.1). Then from Lemma 3.1 we have

$$\begin{cases} 0 \in \eta_i Q_i(x_{i+1}, u_i) + h_i(x_i) - g_i(x_{i+1}) + N^P_{\mathcal{K}_r}(h_i(x_i))(i = 1, \ldots, N-1), \\ 0 \in \eta_N Q_N(x_1, u_N) + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{4.2}$$

$$\Leftrightarrow \begin{cases} g_i(x_{i+1}) - \eta_i Q_i(x_{i+1}, u_i) \in (I + N^P_{\mathcal{K}_r})(h_i(x_i))(i = 1, \ldots, N-1), \\ g_N(x_1) - \eta_N Q_N(x_1, u_N) \in (I + N^P_{\mathcal{K}_r})(h_N(x_N)), \end{cases} \tag{4.3}$$

$$\Leftrightarrow \begin{cases} h_i(x_i) = P_{\mathcal{K}_r}[g_i(x_{i+1}) - \eta_i Q_i(x_{i+1}, u_i)](i = 1, \ldots, N-1), \\ h_N(x_N) = P_{\mathcal{K}_r}[g_N(x_1) - \eta_N Q_N(x_1, u_N)], \end{cases} \tag{4.4}$$

where I is an identity mapping and $P_{\mathcal{K}_r} = (I + N^P_{\mathcal{K}_r})^{-1}$. □

Remark 4.2. The inequality (4.1) can be written as follows

$$\begin{cases} p_i = g_i(x_{i+1}) - \eta_i Q_i(x_{i+1}, u_i), & h_i(x_i) = P_{\mathcal{K}_r}[p_i] (i = 1, \dots, N-1), \\ p_N = g_N(x_1) - \eta_N Q_N(x_1, u_N), & h_N(x_N) = P_{\mathcal{K}_r}[p_N], \end{cases} \tag{4.5}$$

where $\eta_i > 0, i = 1, \dots, N$ are constants.

The fixed point formulation (4.5) enables us to construct the following perturbed iterative algorithms with mixed errors.

Algorithm 4.3. Let $Q_1, \cdots, Q_N, T_1, \dots, T_N, g_1, \dots, g_N, h_1, \dots, h_N, \eta_1, \dots, \eta_N$ be the same as in the system (3.1) such that $h_1, \dots, h_N : \mathcal{X} \longrightarrow \mathcal{X}$ be onto operators. Let $e_1^0, \dots, e_N^0, r_1^0, \dots, r_N^0 \in \mathcal{X}$, $\alpha_0 \in \mathbb{R}$ and $\eta_0 > 0$. For given $p_1^0, \dots, p_N^0 \in \mathcal{X}$, we let $x_1^0, \dots, x_N^0 \in \mathcal{X}$, $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, ...,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ such that

$$\begin{cases} h_i(x_i^0) = P_{\mathcal{K}_r}(p_i^0);\ p_i^1 = (1-\alpha_0)p_i^0 + \alpha_0(g_i(x_{i+1}) - \eta_0 Q_i(x_{i+1}^0, u_i^0) + e_i^0) + r_i^0 (i = 1, \dots, N-1), \\ h_N(x_N^0) = P_{\mathcal{K}_r}(p_N^0);\ p_N^1 = (1-\alpha_0)p_N^0 + \alpha_0(g_N(x_1) - \eta_0 Q_N(x_1^0, u_N^0) + e_N^0) + r_N^0. \end{cases} \tag{4.6}$$

We Choose $x_1^1, \dots, x_1^N \in \mathcal{X}$ such that $h_1(x_1^1) = P_{\mathcal{K}_r}(p_1^1)$,...,$h_N(x_1^N) = P_{\mathcal{K}_r}(p_N^1)$. By Nadler theorem [14], there exists

$$\begin{cases} \|Q_i(x_{i+1}^0, u_i^0) - Q_i(x_{i+1}^1, u_i^1)\| \le \varsigma_i \|x_{i+1}^0 - x_{i+1}^1\| + \varrho_i \|u_i^0 - u_i^1\| (i = 1, \dots, N-1), \\ u_i^1 \in T_i(x_{i+1}^0, x_i^0);\ \|u_i^0 - u_i^1\| \le (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_i(x_{i+1}^0, x_i^0), T_i(x_{i+1}^1, x_i^1)) (i = 1, \dots, N-1), \\ \|Q_N(x_1^0, u_N^0) - Q_N(x_1^1, u_N^1)\| \le \varsigma_N \|x_1^0 - x_1^1\| + \varrho_N \|u_N^0 - u_N^1\|, \\ u_N^1 \in T_N(x_1^0, x_N^0);\ \|u_N^0 - u_N^1\| \le (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^0, x_N^0), T_N(x_1^1, x_N^1)). \end{cases} \tag{4.7}$$

Continuing the above process inductively, we can obtain the sequences $\{x_1^n\}_{n=0}^\infty, \dots, \{x_N^n\}_{n=0}^\infty$, $\{u_1^n\}_{n=0}^\infty, \dots, \{u_N^n\}_{n=0}^\infty$ by using

$$\begin{cases} h_i(x_i^n) = P_{\mathcal{K}_r}(p_i^n);\ p_i^{n+1} = (1-\alpha_n)p_i^n + \alpha_n(g_i(x_{i+1}^n) - \eta_i Q_i(x_{i+1}^n, u_i^n) + e_i^n) + r_i^n, \\ (i = 1, \dots, N-1) \\ h_N(x_N^n) = P_{\mathcal{K}_r}(p_N^n);\ p_N^{n+1} = (1-\alpha_n)p_N^n + \alpha_n(g_N(x_1^n) - \eta_N Q_N(x_1^n, u_N^n) + e_N^n) + r_N^n, \end{cases} \tag{4.8}$$

and

$$\begin{cases} \|Q_i(x_{i+1}^n, u_i^n) - Q_i(x_{i+1}^{n+1}, u_i^{n+1})\| \le \varsigma_i \|x_{i+1}^n - x_{i+1}^{n+1}\| + \varrho_i \|u_i^n - u_i^{n+1}\| (i = 1, \dots, N-1), \\ u_i^n \in T_i(x_{i+1}^n, x_i^n);\ \|u_i^n - u_i^{n+1}\| \le (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_i(x_{i+1}^n, x_i^n), T_i(x_{i+1}^{n+1}, x_i^{n+1})), \\ \|Q_N(x_1^n, u_N^n) - Q_N(x_1^{n+1}, u_N^{n+1})\| \le \varsigma_N \|x_1^n - x_1^{n+1}\| + \varrho_N \|u_N^n - u_N^{n+1}\|, \\ u_N^n \in T_N(x_1^n, x_N^n);\ \|u_N^n - u_N^{n+1}\| \le (1 + (1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n, x_N^n), T_N(x_1^{n+1}, x_N^{n+1})), \end{cases} \tag{4.9}$$

where $0 \le \alpha_n \le 1$ is a parameter and $\{e_1^n\}_{n=0}^\infty, \dots, \{e_N^n\}_{n=0}^\infty$, $\{r_1^n\}_{n=0}^\infty, \dots, \{r_N^n\}_{n=0}^\infty$ are sequences in $\mathcal{X}$ to take into account of a possible inexact computation of the resolvent operator satisfying the following conditions:

$$\lim_{n \longrightarrow \infty} e_i^n = \lim_{n \longrightarrow \infty} r_i^n = 0;$$

$$\sum_{n=1}^{\infty}\|e_i^n - e_i^{n-1}\| < \infty, \quad \sum_{n=1}^{\infty}\|r_i^n - r_i^{n-1}\| < \infty, \text{ for all } i = 1,\dots,N. \tag{4.10}$$

Theorem 4.4. Let $Q_i, T_i, g_i, h_i, \eta_i$, for $i = 1,\dots,N$ be the same as in the system (3.1) such that, for each $i = 1,\dots,N$,

(i) Q_i is $\kappa_i - g_i$-strongly accretive with respect to first variable of Q_i and T_i is $\xi_i - \widehat{\mathcal{D}}$-Lipschitz continuous with first variables;

(ii) h_i is β_i-strongly accretive and σ_i-Lipschitz continuous;

(iii) g_i is μ_i-Lipschitz continuous;

(iii) Q_i is ζ_i-Lipschitz continuous with first variable with constant $\zeta_i > 0$ and ϱ_i-Lipschitz continuous with second variable with constant $\varrho_i > 0$ *respectively.*

If the constants $\eta_i > 0$ satisfying the following conditions:

$$\sqrt[q]{\mu_i^q - q\eta_i\kappa_i\mu_i^q + c_q\eta_i^q\zeta_i^q} < (r - r')(1 - \pi_1)r^{-1} - \varrho_i\xi_i, \tag{4.11}$$

$$\Omega_i = \sqrt[q]{\mu_i^q - q\eta_i\kappa_i\mu_i^q + c_q\eta_i^q\zeta_i^q} + \varrho_i\xi_i \quad \text{and} \quad \pi_i = \sqrt[q]{1 - q\beta_i + c_q\sigma_i^q}\ (i = 1,\dots,N), \tag{4.12}$$

where $r' \in (0, r)$ and c_q is constant, then there exists $x_1^*,\dots,x_N^* \in \mathcal{X}$ with $h_1(x_1^*),\dots,h_N(x_N^*) \in \mathcal{K}_r$ and $u_1^* \in T_1(x_2^*, x_1^*)$, $u_2^* \in T_2(x_3^*, x_2^*)$, ..., $u_{N-1}^* \in T_{N-1}(x_N^*, x_{N-1}^*)$, $u_N^* \in T_N(x_1^*, x_N^*)$ such that $(x_1^*,\dots,x_N^*, u_1^*,\dots,u_N^*)$ is a solution set of (3.1) and sequences $\{(x_1^*,\dots,x_N^*, u_1^*,\dots,u_N^*)\}_{n=0}^{\infty}$ suggested by Algorithm 4.3 converges strongly to $(x_1^*,\dots,x_N^*, u_1^*,\dots,u_N^*)$.

Proof. From (4.8), we have

$$\begin{aligned}\|p_1^{n+1} - p_1^n\| &\le (1-\alpha_n)\|p_1^n - p_1^{n-1}\| + \alpha_n\|g_1(x_2^n) - g_1(x_2^{n-1}) - \eta_1(Q_1(x_2^n, u_1^n) \\ &\quad - Q_1(x_2^{n-1}, u_1^{n-1}))\| + \alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\| \\ &\le (1-\alpha_n)\|p_1^n - p_1^{n-1}\| + \alpha_n\|g_1(x_2^n) - g_1(x_2^{n-1}) - \eta_1(Q_1(x_2^n, u_1^n) - Q_1(x_2^{n-1}, u_1^n))\| \\ &\quad + \alpha_n\eta_1\|(Q_1(x_2^{n-1}, u_1^n) - Q_1(x_2^{n-1}, u_1^{n-1}))\| + \alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\|\end{aligned} \tag{4.13}$$

Since Q_1 is ζ_1-Lipschitz continuous with first variable and ϱ_1-Lipschitz continuous with second variable and T_1 is $\xi_1 - \widehat{\mathcal{D}}$-Lipschitz continuous with first variables, we have

$$\|Q_1(x_2^n, u_1^n) - Q_1(x_2^{n-1}, u_1^n)\| \le \zeta_1\|x_2^n - x_2^{n-1}\|, \tag{4.14}$$

$$\begin{aligned}\|Q_1(x_2^{n-1}, u_1^n) - Q_1(x_2^{n-1}, u_1^{n-1})\| &\le \varrho_1\|u_1^n - u_1^{n-1}\| \\ &\le \varrho_1(1 + \frac{1}{n})\widehat{\mathcal{D}}(T(x_2^n, x_1^n), T(x_2^{n-1}, x_1^{n-1})) \\ &\le \varrho_1\xi_1(1 + \frac{1}{n})\|x_2^n - x_2^{n-1}\|.\end{aligned} \tag{4.15}$$

Since Q_1 is $\kappa_1 - g_1$-strongly accretive with respect to first variable of Q_1 and g_1 is μ_1-Lipschitz continuous, we get

$$\begin{aligned}
\|g_1(x_2^n) - g_1(x_2^{n-1}) - \eta_1(Q_1(x_2^n, u_1^n) - Q_1(x_2^{n-1}, u_1^n))\|^q = {} & \|g_1(x_2^n) - g_1(x_2^{n-1})\|^q \\
& - q\eta_1\langle Q_1(x_2^n, u_1^n) - Q_1(x_2^{n-1}u_1^n), j_q(g_1(x_2^n) - g_1(x_2^{n-1}))\rangle \\
& + c_q\eta_1^q\|Q_1(x_2^n, u_1^n) - Q_1(x_2^{n-1}, u_1^n)\|^q \\
\leq {} & \mu_1^q\|x_2^n - x_2^{n-1}\|^q - q\eta_1\kappa_1\|g_1(x_2^n) - g_1(x_2^{n-1})\|^q + c_q\eta_1^q\zeta_1^q\|x_2^n - x_2^{n-1}\|^q \\
\leq {} & \mu_1^q\|x_2^n - x_2^{n-1}\|^q - q\eta_1\kappa_1\mu_1^q\|x_2^n - x_2^{n-1}\|^q + c_q\eta_1^q\zeta_1^q\|x_2^n - x_2^{n-1}\|^q \\
= {} & \left(\mu_1^q - q\eta_1\kappa_1\mu_1^q + c_q\eta_1^q\zeta_1^q\right)\|x_2^n - x_2^{n-1}\|^q. \qquad (4.16)
\end{aligned}$$

It follows from (4.13) and (4.16), we obtain that

$$\begin{aligned}
\|p_i^{n+1} - p_i^n\| \leq {} & (1-\alpha_n)\|p_i^n - p_i^{n-1}\| + \alpha_n\left(\sqrt[q]{\mu_i^q - q\eta_i\kappa_i\mu_i^q + c_q\eta_i^q\zeta_i^q} + \varrho_i\xi_i(1+\frac{1}{n})\right)\|x_{i+1}^n - x_{i+1}^{n-1}\| \\
& + \alpha_n\|e_i^n - e_i^{n-1}\| + \|r_i^n - r_i^{n-1}\|(i = 1, \ldots, N-1), \\
\|p_N^{n+1} - p_N^n\| \leq {} & (1-\alpha_n)\|p_N^n - p_N^{n-1}\| + \alpha_n\Big(\sqrt[q]{\mu_N^q - q\eta_N\kappa_N\mu_N^q + c_q\eta_N^q\zeta_N^q} \\
& + \varrho_N\xi_N(1+\frac{1}{n})\Big)\|x_1^n - x_1^{n-1}\| + \alpha_n\|e_N^n - e_N^{n-1}\| + \|r_N^n - r_N^{n-1}\|. \qquad (4.17)
\end{aligned}$$

By using (4.8), we get that

$$\begin{aligned}
\|x_1^n - x_1^{n-1}\| \leq {} & \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|h_1(x_1^n) - h_1(x_1^{n-1})\| \\
= {} & \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|P_{\mathcal{K}_r}(p_1^n) - P_{\mathcal{K}_r}(p_1^{n-1})\| \\
\leq {} & \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \frac{r}{r-r'}\|p_1^n - p_1^{n-1}\|. \qquad (4.18)
\end{aligned}$$

Since h_1 is β_1-strongly accretive and σ_1-Lipschitz continuous, we have

$$\begin{aligned}
\|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\|^q = {} & \|x_1^n - x_1^{n-1}\|^q - q\langle h_1(x_1^n) - h_1(x_1^{n-1}), j_q(x_1^n - x_1^{n-1})\rangle \\
& + c_q\|h_1(x_1^n) - h_1(x_1^{n-1})\|^q \\
\leq {} & \|x_1^n - x_1^{n-1}\|^q - q\beta_1\|x_1^n - x_1^{n-1}\|^q + c_q\sigma_1^q\|x_1^n - x_1^{n-1}\|^q \\
= {} & (1 - q\beta_1 + c_q\sigma_1^q)\|x_1^n - x_1^{n-1}\|^q. \qquad (4.19)
\end{aligned}$$

By (4.18) and (4.19), we obtain that

$$\|x_1^n - x_1^{n-1}\| \leq \sqrt[q]{1 - q\beta_1 + c_q\sigma_1^q}\|x_1^n - x_1^{n-1}\| + \frac{r}{r-r'}\|p_1^n - p_1^{n-1}\|$$

that is

$$\|x_1^n - x_1^{n-1}\| \leq \frac{r}{(r-r')(1 - \sqrt[q]{1 - q\beta_1 + c_q\sigma_1^q})}\|p_1^n - p_1^{n-1}\|. \qquad (4.20)$$

Similarly, we can prove that

$$\begin{aligned}
\|x_i^n - x_i^{n-1}\| \leq {} & \frac{r}{(r-r')(1 - \sqrt[q]{1 - q\beta_i + c_q\sigma_i^q})}\|p_i^n - p_i^{n-1}\|(i = 1, \ldots, N-1), \\
\|x_N^n - x_N^{n-1}\| \leq {} & \frac{r}{(r-r')(1 - \sqrt[q]{1 - q\beta_N + c_q\sigma_N^q})}\|p_N^n - p_N^{n-1}\|. \qquad (4.21)
\end{aligned}$$

It follows from (4.13), (4.15), (4.16), (4.20) and (4.21) that

$$\begin{aligned} \|p_i^{n+1} - p_i^n\| &\leq (1-\alpha_n)\|p_i^n - p_i^{n-1}\| + \alpha_n \frac{r\Omega_i(n)}{(r-r')(1-\pi_{i+1})}\|p_{i+1}^n - p_{i+1}^{n-1}\| \\ &\quad +\alpha_n\|e_i^n - e_i^{n-1}\| + \|r_i^n - r_i^{n-1}\|(i=1,\dots,N-1), \\ \|p_N^{n+1} - p_N^n\| &\leq (1-\alpha_n)\|p_N^n - p_N^{n-1}\| + \alpha_n \frac{r\Omega_N(n)}{(r-r')(1-\pi_1)}\|p_1^n - p_1^{n-1}\| \\ &\quad +\alpha_n\|e_N^n - e_N^{n-1}\| + \|r_N^n - r_N^{n-1}\|, \end{aligned} \tag{4.22}$$

where, for all $i = 1, 2, \dots, N$

$$\Omega_i(n) = \sqrt[q]{\mu_i^q - q\eta_i\kappa_i\mu_i^q + c_q\eta_i^q\zeta_i^q} + \varrho_i\xi_i(1+\frac{1}{n}), \text{ and } \pi_i = \sqrt[q]{1 - q\beta_i + c_q\sigma_i^q}.$$

Now we define $\|\cdot\|_*$ on $\underbrace{\mathcal{X}\times\ldots\times\mathcal{X}}_{N-\text{times}}$ by

$$\|(x_1,\dots,x_N)\|_* = \|x_1\| + \ldots + \|x_N\|, \quad \text{for all } (x_1,\dots,x_N) \in \underbrace{\mathcal{X}\times\ldots\times\mathcal{X}}_{N-\text{times}}.$$

It is obvious that $(\underbrace{\mathcal{X}\times\ldots\times\mathcal{X}}_{N-\text{times}}, \|\cdot\|_*)$ is a Banach space, applying (4.22) we have

$$\begin{aligned} &\|(p_1^{n+1},\dots,p_N^{n+1}) - (p_1^n,\dots,p_N^n)\|_* \leq (1-\alpha_n)\|(p_1^n,\dots,p_N^n) - (p_1^{n-1},\dots,p_N^{n-1})\|_* \\ &+\alpha_n\Theta(n)\|(p_1^n,\dots,p_N^n) - (p_1^{n-1},\dots,p_N^{n-1})\|_* + \alpha_n\|(e_1^n,\dots,e_N^n) - (e_1^{n-1},\dots,e_N^{n-1})\|_* \\ &+\|(r_1^n,\dots,r_N^n) - (r_1^{n-1},\dots,r_N^{n-1})\|_*. \end{aligned} \tag{4.23}$$

Put

$$\Theta(n) = \max\left\{\frac{r\Omega_1(n)}{(r-r')(1-\pi_2)}, \dots, \frac{r\Omega_N(n)}{(r-r')(1-\pi_1)}\right\}. \tag{4.24}$$

Let $\Theta(n) \longrightarrow \Theta$, as $n \longrightarrow \infty$

$$\Theta = \max\left\{\frac{r\Omega_1}{(r-r')(1-\pi_2)}, \dots, \frac{r\Omega_N}{(r-r')(1-\pi_1)}\right\}. \tag{4.25}$$

By (4.11), we know that $0 \leq \Theta < 1$. For $\Theta = \frac{1}{2}(\Theta+1) \in (\Theta, 1)$ there exists $n_0 \geq 1$ such that

$\Theta(n) = \widehat{\Theta}$ for each $n \geq n_0$. So it follows from (4.23) that, for each $n \geq n_0$,

$$\begin{aligned}
&\|(p_1^{n+1},\dots,p_N^{n+1}) - (p_1^n,\dots,p_N^n)\|_* \leq (1-\alpha_n)\|(p_1^n,\dots,p_N^n) - (p_1^{n-1},\dots,p_N^{n-1})\|_* \\
&\quad +\alpha_n\widehat{\Theta}\|(p_1^n,\dots,p_N^n) - (p_1^{n-1},\dots,p_N^{n-1})\|_* + \alpha_n\|(e_1^n,\dots,e_N^n) - (e_1^{n-1},\dots,e_N^{n-1})\|_* \\
&\quad +\|(r_1^n,\dots,r_N^n) - (r_1^{n-1},\dots,r_N^{n-1})\|_* \\
&= (1-\alpha_n(1-\widehat{\Theta}))\|(p_1^n,\dots,p_N^n) - (p_1^{n-1},\dots,p_N^{n-1})\|_* + \alpha_n\|(e_1^n,\dots,e_N^n) - (e_1^{n-1},\dots,e_N^{n-1})\|_* \\
&\quad +\|(r_1^n,\dots,r_N^n) - (r_1^{n-1},\dots,r_N^{n-1})\|_* \\
&\leq (1-\alpha_n(1-\widehat{\Theta}))\Big((1-\alpha_n(1-\widehat{\Theta}))\|(p_1^{n-1},\dots,p_N^{n-1}) - (p_1^{n-2},\dots,p_N^{n-2})\|_* \\
&\quad +\alpha_n\|(e_1^{n-1},\dots,e_N^{n-1}) - (e_1^{n-2},\dots,e_N^{n-2})\|_* + \|(r_1^{n-1},\dots,r_N^{n-1}) - (r_1^{n-2},\dots,r_N^{n-2})\|_*\Big) \\
&\quad +\alpha_n\|(e_1^n,\dots,e_N^n) - (e_1^{n-1},\dots,e_N^{n-1})\|_* + \|(r_1^n,\dots,r_N^n) - (r_1^{n-1},\dots,r_N^{n-1})\|_* \\
&= (1-\alpha_n(1-\widehat{\Theta}))^2\|(p_1^{n-1},\dots,p_N^{n-1}) - (p_1^{n-2},\dots,p_N^{n-2})\|_* + \alpha_n\Big((1-\alpha_n(1-\widehat{\Theta})) \\
&\quad \times\|(e_1^{n-1},\dots,e_N^{n-1}) - (e_1^{n-2},\dots,e_N^{n-2})\|_* + \|(e_1^n,\dots,e_N^n) - (e_1^{n-1},\dots,e_N^{n-1})\|_*\Big) \\
&\quad +(1-\alpha_n(1-\widehat{\Theta}))\|(r_1^{n-1},\dots,r_N^{n-1}) - (r_1^{n-2},\dots,r_N^{n-2})\|_* + \|(r_1^n,\dots,r_N^n) - (r_1^{n-1},\dots,r_N^{n-1})\|_* \\
&\leq \\
&\ \vdots \\
&\leq (1-\alpha_n(1-\widehat{\Theta}))^{n-n_0}\|(p_1^{n_0+1},\dots,p_N^{n_0+1}) - (p_1^{n_0},\dots,p_N^{n_0})\|_* \\
&\quad +\alpha_n\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(e_1^{n-(i-1)},\dots,e_N^{n-(i-1)}) - (e_1^{n-i},\dots,e_N^{n-i})\|_* \\
&\quad +\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(r_1^{n-(i-1)},\dots,r_N^{n-(i-1)}) - (r_1^{n-i},\dots,r_N^{n-i})\|_*.
\end{aligned} \tag{4.26}$$

Thus, for any $m \geq n > n_0$, we get that

$$\begin{aligned}
&\|(p_1^m,\dots,p_N^m) - (p_1^n,\dots,p_N^n)\|_* \leq \sum_{j=n}^{m-1}\|(p_1^{j+1},\dots,p_N^{j+1}) - (p_1^j,\dots,p_N^j)\|_* \\
&\leq \sum_{j=n}^{m-1}(1-\alpha_n(1-\widehat{\Theta}))^{j-n_0}\|(p_1^{n_0+1},\dots,p_N^{n_0+1}) - (p_1^{n_0},\dots,p_N^{n_0})\|_* \\
&\quad +\alpha_n\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(e_1^{n-(i-1)},\dots,e_N^{n-(i-1)}) - (e_1^{n-i},\dots,e_N^{n-i})\|_* \\
&\quad +\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(r_1^{n-(i-1)},\dots,r_N^{n-(i-1)}) - (r_1^{n-i},\dots,r_N^{n-i})\|_*.
\end{aligned} \tag{4.27}$$

Since $(1-\alpha_n(1-\widehat{\Theta})) \in (0,1)$, it follows from (4.10) and (4.30) that

$$\|(p_1^m,\dots,p_N^m) - (p_1^n,\dots,p_N^n)\|_* = \|p_1^m - p_1^n\| + \ldots + \|p_N^m - p_N^n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

So $\{p_1^n\},\ldots,\{p_N^n\}$ are Cauchy sequences in $\mathcal{X}$ and thus there exists $p_1^*,\ldots,p_N^*\in\mathcal{X}$ such that $p_1^n\longrightarrow p_1^*,\ldots, p_N^n\longrightarrow p_N^*$ as $n\longrightarrow\infty$. By (4.22) and (4.23), it follows that the sequences $\{x_1^n\},\ldots,\{x_N^n\}$ are also Cauchy sequences in $\mathcal{X}$. Hence there exists $x_1^*,\ldots,x_N^*\in\mathcal{X}$ such that $x_1^n\longrightarrow x_1^*,\ldots,x_N^n\longrightarrow x_N^*$ as $n\longrightarrow\infty$. Since for each $i=1,\ldots,N$, T_i are $\xi_i-\widehat{\mathcal{D}}$-Lipschitz continuous in the first variable, therefore it follow from (4.7) that

$$\begin{aligned}
\|u_i^n-u_i^{n+1}\| &\le (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_i(x_{i+1}^n,x_i^n),T_i(x_{i+1}^{n+1},x_i^{n+1}))\\
&\le (1+(1+n)^{-1})\xi_i\|x_{i+1}^n-x_{i+1}^{n+1}\|\longrightarrow 0(i=1,2,\ldots,N-1),\\
\|u_N^n-u_N^{n+1}\| &\le (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n,x_N^n),T_1(x_1^{n+1},x_N^{n+1}))\\
&\le (1+(1+n)^{-1})\xi_N\|x_1^n-x_1^{n+1}\|\longrightarrow 0,\quad \text{as } n\longrightarrow\infty. \qquad (4.28)
\end{aligned}$$

Hence $\{u_1^n\},\ldots,\{u_N^n\}$ are Cauchy sequences in $\mathcal{X}$ and so there exists $u_1^*,\ldots,u_N^*\in\mathcal{X}$ such that $x_1^n\longrightarrow x_1^*,\ldots,x_N^n\longrightarrow x_N^*$ as $n\longrightarrow\infty$. Further $u_1^n\in T_1(x_2^n,x_1^n)$ we have

$$\begin{aligned}
d(u_1^*,T_1(x_2^*,x_1^*)) &= \inf\{\|u_1^*-t\|:t\in T_1(x_2^*,x_1^*)\}\\
&\le \|u_1^*-u_1^n\|+d(u_1^n,T_1(x_2^*,x_1^*))\\
&\le \|u_1^*-u_1^n\|+\widehat{\mathcal{D}}(T_1(x_2^n,x_1^n),T_1(x_2^{n+1},x_1^{n+1}))\\
&\le \|u_1^*-u_1^n\|+\|x_2^n-x_2^*\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \qquad (4.29)
\end{aligned}$$

Hence $d(u_1^*,T_1(x_2^*,x_1^*))=0$ and so $u_1^n\longrightarrow u_1^*\in T_1(x_2^*,x_1^*)$.
By the same method, we can prove that

$$\begin{aligned}
d(u_i^*,T_i(x_{i+1}^*,x_i^*)) &\le \|u_i^*-u_i^n\|+\|x_{i+1}^n-x_{i+1}^*\|\longrightarrow 0(i=1,2,\ldots,N-1),\\
d(u_N^*,T_N(x_1^*,x_N^*)) &\le \|u_N^*-u_N^n\|+\|x_1^n-x_1^*\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \qquad (4.30)
\end{aligned}$$

Therefore $u_2^*\in T_2(x_3^*,x_2^*),\ldots,u_N^*\in T_N(x_1^*,x_N^*)$. Since g_i for $i=1,\ldots,N$ is continuous, it follows from (4.8) and (4.10) that

$$\begin{aligned}
p_i^* &= g_i(x_{i+1}^*)-\eta_iQ_i(x_{i+1}^*,u_i^*)(i=1,2,\ldots,N-1),\\
p_N^* &= g_N(x_1^*)-\eta_NQ_N(x_1^*,u_N^*). \qquad (4.31)
\end{aligned}$$

Since $Q_1,\cdots,Q_N,h_1,\ldots,h_N$ and $P_{\mathcal{K}_r}$ are continuous, it follows from (4.8) and (4.31) that

$$\begin{aligned}
h_i(x_i^*) &= P_{\mathcal{K}_r}(p_i^*)=P_{\mathcal{K}_r}(g_i(x_{i+1}^*)-\eta_iQ_i(x_{i+1}^*,u_i^*))(i=1,2,\ldots,N-1),\\
h_N(x_N^*) &= P_{\mathcal{K}_r}(p_N^*)=P_{\mathcal{K}_r}(g_N(x_1^*)-\eta_NQ_N(x_1^*,u_N^*)). \qquad (4.32)
\end{aligned}$$

Now Lemma 4.1, guarantees that $(x_1^*,\ldots,x_N^*,u_1^*,\ldots,u_N^*)$ is a solution set of system (3.1). □

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