On the solution of the coupled Schrödinger–KdV equation by the decomposition method

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Abstract

In this Letter, we consider a coupled Schrödinger–Korteweg–de Vries equation (or Sch–KdV) equation with appropriate initial values using the Adomian’s decomposition method (or ADM). In this method, the solution is calculated in the form of a convergent power series with easily computable components. The method does not need linearization, weak nonlinearity assumptions or perturbation theory. The convergence of the method as applied to Sch–KdV is illustrated numerically.

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1. Introduction

The KdV and Schrödinger equations were sharing with another partial differential equations such as KdV–mKdV, KdV–Euler–Darboux equation, Korteweg–de Vries–Zakharov–Kuznetsov (KdV–ZK), KdV–Burgers equation and Schrödinger–KdV equation [1–8]. In this Letter we consider one of them which is Schrödinger–KdV equation. The equation has the following form:

\[ i u_t = u_{xx} + uv, \quad v_t = -6uv_x - v_{xxx} + \left( |u|^2 \right)_x, \]

with initial data

\[ u(x, 0) = g_1(x), \quad v(x, 0) = g_2(x). \]

Nonlinear phenomena play a crucial role in applied mathematics and physics. Calculating exact and numerical solutions, in particular, traveling wave solutions, of nonlinear equations in mathematical physics play an important role.
role in soliton theory [9,10]. Many explicit exact methods have been introduced in literature [11–15]. Some of them are: Backlund transformation, Generalized Miura Transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine–cosine method, Painlevé method, homogeneous balance method, similarity reduction method and so on. Recently, Fan et al. [8] applied extended tanh method and found some new explicit solutions for the Sch–KdV equation (1).

In recent years a lot of attention has been devoted to the study of Adomian’s decomposition method to investigate various scientific models. The Adomian decomposition method, which accurately computes the series solution, is of great interest to applied sciences [10,16,17]. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The nonlinear equation is solved easily and elegantly without linearizing the problem by using the ADM, see [18–27].

In the present Letter, the Adomian’s decomposition method will be applied for computing solutions to a Sch–KdV equation. A useful attraction of this method is that it has proved to be reliable and effective. The method provide the solutions in the form of a power series with easily computable terms, see [16,17]. In the literature [10,21–26] and others showed that the power series obtained converges rapidly and introduces the exact solution in a closed form. For concrete problems, we usually use a truncated series of the power series solution. The accuracy and rapid convergence of the solutions are demonstrated through some numerical examples.

2. Description of the Adomian’s decomposition method

We first consider the Sch–KdV equation (1) written in an operator form

\[ L_t u = -iL_{xx}u - iN(u, v), \quad L_t v = -6M(u, v) - L_{xxx}v + R(u, v), \] (2)

where the notation \( L_t = \frac{\partial}{\partial t} \), \( L_{xx} = \frac{\partial^2}{\partial x^2} \) and \( L_{xxx} = \frac{\partial^3}{\partial x^3} \) symbolize the linear differential operators, the notations \( N(u, u) = uv, \ M(u, v) = uv_x \) and \( R(u, v) = (|u|^2)_x \) symbolize the nonlinear operators. Applying the inverse operator \( L_t^{-1} = \int_0^t \cdots \, dt \) to the system (2) yields

\[ u(x, t) = g_1(x) - iL_t^{-1}[L_{xx}u + N(u, v)], \quad v(x, t) = g_2(x) - L_t^{-1}[6M(u, v) + L_{xxx}v - R(u, v)], \] (3)

where \( g_1(x) = u(x, 0) \) and \( g_2(x) = v(x, 0) \) are given functions for initial conditions. The Adomian decomposition method [16,17] assumes an infinite series solution for unknown functions \( u(x, t) \) and \( v(x, t) \) in the form

\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad v(x, t) = \sum_{n=0}^{\infty} v_n(x, t), \] (4)

and nonlinear operators \( N(u, u) = uv, \ M(u, v) = uv_x \) and \( R(u, v) = (|u|^2)_x \) by the infinite series of Adomian polynomials given by

\[ N(u, u) = \sum_{n=0}^{\infty} A_n, \quad M(u, v) = \sum_{n=0}^{\infty} B_n, \quad R(u, v) = \sum_{n=0}^{\infty} C_n, \] (5)

where \( A_n, B_n \) and \( C_n \) are the appropriate Adomian’s polynomials which are generated according to algorithms determined in [28,29]. For nonlinear operator \( M_1(u, v) \) these polynomials can be defined by

\[ A_n(u_0, \ldots, u_n; v_0, \ldots, v_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} N \left[ \sum_{k=0}^{n} \lambda^k u_k, \sum_{k=0}^{n} \lambda^k v_k \right]_{\lambda=0}^n, \quad n \geq 0. \] (6)

This formulae is easy to be set in a computer code to get as many polynomial as we need in the calculation of the numerical as well as explicit solutions. For a detailed explanation of Adomian decomposition method and other general formula of Adomian polynomials, we refer the reader to [10,16,17,28,29].
Following the decomposition method, the nonlinear system (3) is constructed in a form of the recursive relations given by

\begin{align*}
    u_0(x, t) & = 0, \\
    u_1(x, t) & = g_1(x) - i \mathcal{L}^{-1}[L_{xx} u_0 + A_0], \\
    u_{n+1}(x, t) & = -i \mathcal{L}^{-1}[L_{xx} u_n + A_n], \\
    v_0(x, t) & = 0, \\
    v_1(x, t) & = g_2(x) - \mathcal{L}^{-1}[6B_0 + L_{xxx} v_0 - C_0], \\
    v_{n+1}(x, t) & = -\mathcal{L}^{-1}[6B_n + L_{xxx} v_n - C_n],
\end{align*}

(7)

where \( n \geq 1 \), the functions \( g_1(x) \) and \( g_2(x) \) are getting from the initial conditions. It is worth noting that the zeroth components \( u_0 \) and \( v_0 \) are defined then the remaining components \( u_n \), and \( v_n, n \geq 1 \), can be completely determined such that each terms are computed by using the previous terms. As a result, the components \( u_0, u_1, u_2, \ldots \) and \( v_0, v_1, v_2, \ldots \) are identified and the series solutions thus entirely determined. However, in many cases the exact solution in a closed form may be obtained.

For numerical comparisons purposes, we construct the solution \( u(x, t) \) and \( v(x, t) \)

\begin{align*}
    \lim_{n \to \infty} \phi_n(x, t) & = u(x, t), \\
    \lim_{n \to \infty} \psi_n(x, t) & = v(x, t),
\end{align*}

(8)

where

\begin{align*}
    \phi_n(x, t) & = \sum_{k=0}^{n-1} u_k(x, t), \\
    \psi_n(x, t) & = \sum_{k=0}^{n-1} v_k(x, t), \\
    n & \geq 1,
\end{align*}

and the recurrence relation is given as in (6). Moreover, the decomposition series (4) solutions are generally converged very rapidly in real physical problems [10,16,17]. The convergence of the decomposition series have investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the literature [30–34]. They obtained some results about the speed of convergence of this method providing us to solve linear and nonlinear functional equations. In recent work of Ngarhasta et al. [35] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM.

3. Implementation of the Adomian’s method

We first consider the application of the decomposition method to the Sch–KdV equation (1) with the initial conditions

\begin{align*}
    u(x, 0) & = 6 \sqrt{2} e^{i\alpha x} k^2 \text{sech}^2(kx), \\
    v(x, 0) & = \frac{\alpha + 16k^2}{3} - 6k^2 \text{tanh}^2(kx).
\end{align*}

(9)

where \( \alpha \) and \( k \) are arbitrary constants.

Using (7) with (6) for the functional coupled equation (1) and initial conditions (9) gives

\begin{align*}
    u_0 &= 0, \\
    u_1 &= 6 \sqrt{2} e^{i\alpha x} k^2 \text{sech}^2(kx)^2, \\
    v_1 &= \frac{\alpha + 16k^2}{3} - 6k^2 \text{tanh}(kx)^2, \\
    u_2 &= 6i \left[ \left( \sqrt{2} \alpha e^{i\alpha x} k^2 \text{sech}^2(kx) - 2 \sqrt{2} e^{i\alpha x} k^4 \text{sech}^4(kx) - 4i \sqrt{2} \alpha e^{i\alpha x} k^3 \text{sech}^2(kx) \text{tanh}(kx) \\
    & \quad + 4 \sqrt{2} e^{i\alpha x} k^4 \text{sech}^2(kx) \text{tanh}^3(kx) \right) \right], \\
    v_2 &= -48i (2k^5 \text{sech}(kx)^4 \text{tanh}(kx) - k^5 \text{sech}(kx)^2 \text{tanh}(kx)^3),
\end{align*}

(10) (11) (12) (13)
The numerical results when $\alpha = 0.05$ and $k = 0.05$ for the solution of Eq. (1)

<table>
<thead>
<tr>
<th>$(x_i, t_i)$</th>
<th>$(0.1, 0.1)$</th>
<th>$(0.2, 0.2)$</th>
<th>$(0.3, 0.3)$</th>
<th>$(0.4, 0.4)$</th>
<th>$(0.5, 0.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_i, t_i)$</td>
<td>$(0.1, 0.1)$</td>
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<td>$(0.3, 0.3)$</td>
<td>$(0.4, 0.4)$</td>
<td>$(0.5, 0.5)$</td>
</tr>
<tr>
<td>$u(x, t) - \phi_3(x, t)$</td>
<td>$1.16783 \times 10^{-6}$</td>
<td>$2.0514 \times 10^{-6}$</td>
<td>$2.9342 \times 10^{-6}$</td>
<td>$3.81699 \times 10^{-6}$</td>
<td>$4.85984 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

$u_3 = -i\left[\sqrt{2}\alpha^4 \kappa^2 \kappa^2 \sinh(2x) - 2\kappa^2 \sinh(2x) \right] + \frac{3\alpha^4 \kappa^2 \kappa^2 \sinh(2x)}{4\sqrt{2}} \left[ -3i\alpha^4 - 72i\alpha^2 k^2 - 528i\kappa^4 - 4i\alpha^4 \cosh(2kx) - 48i\alpha^2 k^2 \cosh(2kx) + 4i\alpha^4 \cosh(2kx) - 24i\alpha^2 k^2 \cosh(2kx) - 16i\alpha^4 \cosh(2kx) + 16\alpha^3 k \sinh(2kx) + 320\alpha^3 k \sinh(2kx) + 8\alpha^3 k \sinh(2kx) - 32\alpha^3 k \sinh(2kx) \right]$, (14)
Fig. 1. The numerical results for $\phi_2(x,t)$ and $\psi_2(x,t)$: (b), (d) in comparison with the analytical solutions $u(x,t)$ and $v(x,t)$: (a), (c) $\alpha = 0.05$ and $k = 0.05$, for the solitary wave solutions with the initial conditions (9) of Eq. (1), respectively.

\[ v_3 = -6k^8 t^2 \left( 1208 - 1191 \cosh(2kx) + 120 \cosh(4kx) - \cosh(6kx) \right) \text{sech}^8(kx) \\
- 144 e^{2i\alpha x} k^4 t \text{sech}^5(kx) \left( -i\alpha \cosh(kx) + 2k \sinh(kx) \right), \]  
\hspace{1cm} (15)

and so on, the other components of the decomposition series (4) can be determined in a similar way. Substituting Eqs. (10)–(15) into (4) and using the decomposition series (4) which is a Taylor series, we obtain the closed form solutions

\[ u(x,t) = 6\sqrt{2} e^{i\theta} k^2 \text{sech}^2(k\xi), \quad v(x,t) = \frac{\alpha + 16k^2}{3} - 6k^2 \tanh^2(k\xi), \]  
\hspace{1cm} (16)

where

\[ \theta = \left( \frac{\alpha t}{3} + \alpha^2 t - \frac{10k^2 t}{3} + \alpha x \right), \quad \xi = x + 2\alpha t \]

and $\alpha$, $k$ are arbitrary constants. These solutions are constructed by Fan et al. [11].

4. Discussion and conclusions

The Adomian’s decomposition method was used for finding the exact and approximate traveling-waves solutions of the Sch–KdV equation. The method can be also easy to be extended to other nonlinear evaluation equations, with the aid of MATHEMATICA (or MATLAB, MAPLE, REDUCE, etc.), the course of solving nonlinear evaluation equations can be carried out in computer. Some coupled Sch–KdV equations with initial conditions are discussed as demonstrations. It may be concluded that the Adomian methodology is very powerful and efficient technique in finding exact solutions for wide classes of nonlinear problems. It is also worth noting to point out that the advantage of the decomposition methodology is the fast convergence of the solutions.
Fig. 2. The numerical results for $\phi_4(x, t)$ and $\psi_4(x, t)$: (b), (d) in comparison with the analytical solutions $u(x, t)$ and $v(x, t)$: (a), (c). $\alpha = 0.05$ and $k = 0.05$, for the solitary wave solutions with the initial conditions (9) of Eq. (1), respectively.

Fig. 3. The numerical results for $\phi_4(x, t)$ and $\psi_4(x, t)$: (b), (d) in comparison with the analytical solutions $u(x, t)$ and $v(x, t)$: (a), (c). $\alpha = 0.05$ and $k = 0.05$, for the solitary wave solutions with the initial conditions (9) of Eq. (1), respectively.
Furthermore, as the decomposition method does not require discretization of the variables, i.e., time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the decomposition method for the coupled Sch–KdV equation controllable and absolute errors are very small with the present choice of $t$ and $x$ which are given in Table 1. It show that the implemented method achieves an accuracy of minimum six and maximum eight significant figures for Eq. (1) using a reasonable small number of $n$ in the formulae (8). Both the exact results and the approximate solutions obtained as $n = 2, 4, 6$ by using the formulae (8) are plotted in Figs. 1, 2 and 3. There is no visible difference in the two solutions. It is also evident that when compute more terms for the decomposition series the numerical results are getting much more closer to the corresponding exact solutions with the initial conditions (9) of Eq. (1).

Clearly, the series solution methodology can also be applied to many other nonlinear problems. However, as we have seen in the previous sections, the decomposition method does not require linearization or perturbation for obtaining closed form solutions. Additionally, it does not need any discretization to get numerical solutions. The computational size has been reduced and the rapid convergence has been guaranteed. The decomposition method introduces a significant improvement in this field over existing techniques.

References