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ORIGINAL ARTICLE

$(\in, \in \vee q)$-Intuitionistic Fuzzy Ideals of BG-algebra

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Abstract In this paper, we introduce the concept of $(\in, \in \vee q)$-intuitionistic fuzzy ideals of BG-algebra and investigate some of their basic properties.

Keywords BG-algebra · Fuzzy ideal · $(\in, \in \vee q)$-Fuzzy ideal · $(\in, \in \vee q)$-Intuitionistic fuzzy ideal · Homomorphism.

1. Introduction

In 1965, Zadeh [1] introduced the notion of a fuzzy subset of a set as a method of representing uncertainty in real physical world. The concept of intuitionistic fuzzy subset was introduced by Atanassov [2] in 1986, which is a generalization of the notion of fuzzy sets. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. In 1966, Imai and Iseki [3] introduced the two classes of abstract algebras, viz., BCK-algebras and BCI-algebras. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. Neggers and Kim [4] introduced a new concept, called B-algebras, which are related to several classes of algebras such as BCI/BCK-algebras. Kim and Kim [5] introduced the notion of BG-algebra which is a generalization of B-algebra. Zarandi and Saeid [6] developed intuitionistic fuzzy ideal of BG-algebra. Senapati, Bhowmik and Pal [7] studied

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intuitionistic fuzzy ideals in *BG*-algebras in 2012. Bhakat and Das [8] used the relation of “belongs to” and “quasi coincident with” between fuzzy point and fuzzy set to introduce the concept of $(\in, \in \vee q)$-fuzzy subgroup and $(\in, \in \vee q)$-fuzzy subring. Basnet and Singh [9] introduced $(\in, \in \vee q)$-fuzzy ideals of *BG*-algebra in 2011. Barbhuiya and Choudhury [10] introduced $(\in, \in \vee q)$-fuzzy ideals of *d*-algebra in 2014. Motivated by this, we introduce the notion of $(\in, \in \vee q)$-intuitionistic fuzzy ideals of *BG*-algebra and establish some of their basic properties.

2. Preliminaries

Definition 1 *A BG-algebra is a non-empty set X with a constant* 0 *and a binary operation* $*$ *satisfying the following axioms:*

(i) $x * x = 0$,

(ii) $x * 0 = x$,

(iii) $(x * y) * (0 * y) = x \ \forall \ x, y \in X$.

For brevity, we also call X a BG-algebra.

Example 1 Let $X = \{0, 1, 2, 3, 4\}$ with the following cayley table:

Table 1: Cayley table for *BG*-algebra.

*	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Then $(X, *, 0)$ is a *BG*-algebra.

Definition 2 *A non-empty subset S of a BG-algebra X is called a subalgebra of X if* $x * y \in S$ *for all* $x, y \in S$.

Definition 3 *A nonempty subset I of a BG-algebra X is called a BG-ideal of X if*

(i) $0 \in I$,

(ii) $x * y \in I, \ y \in I \Rightarrow x \in I \ \forall \ x, y \in X$.

Definition 4 *A fuzzy set* μ *in X is called a fuzzy BG-ideal of X if it satisfies the following conditions:*

(i) $\mu(0) \geq \mu(x)$,

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}\ \forall\ x, y \in X$.

Example 2 Consider a BG-algebra $X = \{0, 1, 2\}$ with the following cayley table:

Table 2: Example of fuzzy BG-ideal.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Define $\mu : X \to [0, 1]$ by $\mu(0) = 0.9, \mu(1) = 0.6, \mu(2) = 0.3$. Then it is easy to verify that μ is a fuzzy BG-ideal of X.

Definition 5 *An intuitionistic fuzzy set (IFS) A of a BG-algebra X is an object of the form* $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$, *where* $\mu_A : X \to [0, 1]$ *and* $\nu_A : X \to [0, 1]$ *with the condition* $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$. *The numbers* $\mu_A(x)$ *and* $\nu_A(x)$ *denote respectively the degree of membership and the degree of non-membership of the element x in set A. For the sake of simplicity, we shall use the symbol* $A = (\mu_A, \nu_A)$ *for the intuitionistic fuzzy set* $A = \{< x, \mu_A(x), \nu_A(x) > |\ x \in X\}$.

Definition 6 *If* $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ *and* $B = \{< x, \mu_B(x), \nu_B(x) > | x \in X\}$ *are any two IFS of a set X, then*

$A \subseteq B$ *if and only if for all* $x \in X, \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$,

$A = B$ *if and only if for all* $x \in X, \mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$,

$A \cap B = \{< x, (\mu_A \cap \mu_B)(x), (\nu_A \cap \nu_B)(x) > | x \in X\}$,

where $(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ *and* $(\nu_A \cap \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$,

$A \cup B = \{< x, (\mu_A \cup \mu_B)(x), (\nu_A \cup \nu_B)(x) > | x \in X\}$,

where $(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ *and* $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$.

Definition 7 *An intuitionistic fuzzy set A of a BG-algebra X is said to be an intuitionistic fuzzy BG-subalgebra of X if*

(i) $\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$,

(ii) $\nu_A(x * y) \leq \max\{\nu_A(x), \nu_A(y)\}\ \forall\ x, y \in X$.

Example 3 Consider a BG-algebra $X = \{0, 1, 2\}$ with the following cayley table:

Table 3: Example of intuitionistic fuzzy BG-subalgebra.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

The intuitionistic fuzzy subset $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ given by $\mu_A(0) = \mu_A(1) = 0.6, \mu_A(2) = 0.2$ and $\nu_A(0) = \nu_A(1) = 0.3, \nu_A(2) = 0.5$ is an intuitionistic fuzzy BG-subalgebra of X.

Definition 8 *An intuitionistic fuzzy set A of a BG-algebra X is said to be an intuitionistic fuzzy ideal (IFI) of X if*

(i) $\mu_A(0) \geq \mu_A(x)$,

(ii) $\nu_A(0) \leq \nu_A(x)$,

(iii) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$,

(iv) $\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\}\ \forall\ x, y \in X$.

Example 4 Consider a BG-algebra $X = \{0, 1, 2, 3\}$ with the following cayley table:

Table 4: Example of IFI.

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

The intuitionistic fuzzy subset $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ given by $\mu_A(0) = 1, \mu_A(1) = \mu_A(2) = \mu_A(3) = 0.3$, and $\nu_A(0) = 0, \nu_A(1) = \nu_A(2) = \nu_A(3) = 0.4$ is an intuitionistic fuzzy BG-subalgebra of X. Then A is an IFI of the BG-algebra X.

3. $(\in, \in \vee q)$-Intuitionistic Fuzzy Ideals of BG-algebra

Definition 9 *A fuzzy set μ of the form*

$$\mu(y) = \begin{cases} t, & if\ y = x,\ t \in (0, 1] \\ 0, & if\ y \neq x \end{cases}$$

is called a fuzzy point with support x and value t and is denoted by x_t.

Definition 10 *A fuzzy point x_t is said to belong to (respectively be quasi coincident with) a fuzzy set μ written as $x_t \in \mu$ (respectively $x_t q\mu$) if $\mu(x) \geq t$ (respectively $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q\mu$, then we write $x_t \in \vee q\mu$. (Note $\overline{\in \vee q}$ means $\in \vee q$ does not hold).*

Definition 11 *A fuzzy subset μ of a BG-algebra X is said to be an $(\in, \in \vee q)$-fuzzy ideal of X if*

$$(x * y)_t, y_s \in \mu \Rightarrow x_{m(t,s)} \in \vee q\mu.$$

Definition 12 *A fuzzy subset μ of a BG-algebra X is said to be an (α, β)-fuzzy ideal of X, if*

$$(x * y)_t, y_s \alpha\mu \Rightarrow x_{m(t,s)} \beta\mu \ \forall \ x, y \in X,$$

where $m(t, s) = \min\{t, s\}$ and $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$.

Definition 13 *A fuzzy point x_t is said to belong to (respectively be quasi coincident with) an intuitionistic fuzzy set $A = \{< x, \mu_A(x), \nu_A(x) > | \ x \in X\}$ written as $x_t \in A$ (respectively $x_t qA$), if $\mu_A(x) \geq t$ (respectively $\mu_A(x)+t > 1$) and $\nu_A(x) \leq t$ (respectively $\nu_A(x) + t < 1$). If $x_t \in A$ or $x_t qA$, then $x_t \in \vee qA$.*

Definition 14 *An intutionistic fuzzy subset $A = (\mu_A, \nu_A)$ in a BG-algebra X is said to be an $(\in, \in \vee q)$-IFI of X if it satisfies the following conditions:*

(i) $(x * y)_t, y_s \in \mu_A \Rightarrow x_{m(t,s)} \in \vee q\mu_A$,
i.e., $\mu_A(x*y) \geq t, \mu_A(y) \geq s \Rightarrow \mu_A(x) \geq m(t, s)$ *or* $\mu_A(x)+m(t, s) > 1, \forall x, y \in X$, *where* $m(t, s) = \min(t, s)$.

(ii) $(x * y)_t, y_s \in \nu_A \Rightarrow x_{M(t,s)} \in \vee q\nu_A$,
i.e., $\nu_A(x*y) \leq t, \nu_A(y) \leq s \Rightarrow \nu_A(x) \leq M(t, s)$ *or* $\nu_A(x)+M(t, s) < 1, \forall x, y \in X$, *where* $M(t, s) = \max(t, s)$.

Theorem 1 *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a BG-algebra X is an IFI of X iff A is an $(\in, \in)$-IFI of X.*

Proof Let $A = (\mu_A, \nu_A)$ be an IFI of X. Then

$$\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \tag{1}$$

and

$$\nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\} \ \forall \ x, y \in X. \tag{2}$$

Let $x, y \in X$ such that $(x * y)_t, y_s \in A$, where $t, s \in (0, 1)$. Then $\mu_A(x * y) \geq t, \mu_A(y) \geq s$ and $\nu_A(x * y) \leq t, \nu_A(y) \leq s$.

Now (1) $\Rightarrow \mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\} \geq \min\{t, s\} = m(t, s) \Rightarrow x_{m(t,s)} \in \mu_A$, and

(2) $\Rightarrow \nu_A(x) \leq \max\{\nu_A(x * y), \nu_A(y)\} \leq \max\{t, s\} = M(t, s) \Rightarrow x_{M(t,s)} \in \nu_A$.

Therefore, A is an $(\in, \in)$-IFI of X.

Conversely, let $A = (\mu_A, \nu_A)$ be an $(\in, \in)$-IFI of X. To prove that $A = (\mu_A, \nu_A)$ is an IFI of X.

Let $x, y \in X$ and $t = \mu_A(x * y)$, $s = \mu_A(y)$. Then

$$\begin{aligned}
&\mu_A(x * y) \geq t, \mu_A(y) \geq s \\
&\quad \Rightarrow (x * y)_t \in \mu_A, y_s \in \mu_A \\
&\quad \Rightarrow x_{m(t,s)} \in \mu_A \text{ [since } A = (\mu_A, \nu_A) \text{ be an } (\in, \in)\text{-IFI of } X] \\
&\quad \Rightarrow \mu_A(x) \geq m(t, s) \\
&\quad \Rightarrow \mu_A(x) \geq m\{\mu_A(x * y), \mu_A(y)\}. \qquad (3)
\end{aligned}$$

Again, let $x, y \in X$ and $t = \nu_A(x * y)$, $s = \nu_A(y)$. Then

$$\begin{aligned}
&\nu_A(x * y) \leq t, \mu_A(y) \leq s \\
&\quad \Rightarrow (x * y)_t \in \nu_A, y_s \in \nu_A \\
&\quad \Rightarrow x_{M(t,s)} \in \nu_A \text{ [since } A = (\mu_A, \nu_A) \text{ be an } (\in, \in)\text{-IFI of } X] \\
&\quad \Rightarrow \nu_A(x) \leq M(t, s) \\
&\quad \Rightarrow \nu_A(x) \leq M\{\nu_A(x * y), \nu_A(y)\}. \qquad (4)
\end{aligned}$$

Hence, from Eqs. (3) and (4), $A = (\mu_A, \nu_A)$ is an IFI of X.

Theorem 2 *If $A = (\mu_A, \nu_A)$ be a (q, q)-IFI of a BG-algebra X, then it is also an $(\in, \in)$-IFI of X.*

Proof Let $A = (\mu_A, \nu_A)$ be a (q, q)-IFI of a BG-algebra X. Let $x, y \in X$ such that $(x * y)_t, y_s \in \mu_A$. Then

$$\begin{aligned}
&\mu_A(x * y) \geq t \text{ and } \mu_A(y) \geq s \\
&\quad \Rightarrow \mu_A(x * y) + \delta > t \text{ and } \mu_A(y) + \delta > s,
\end{aligned}$$

[where δ is an arbitrary small positive number]

$$\begin{aligned}
&\quad \Rightarrow \mu_A(x * y) + \delta - t + 1 > 1 \text{ and } \mu_A(y) + \delta - s + 1 > 1 \\
&\quad \Rightarrow (x * y)_{\delta - t + 1} q \mu_A \text{ and } y_{\delta - s + 1} q \mu_A.
\end{aligned}$$

Since $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-IFI of X. Therefore, we have

$$\begin{aligned}
&x_{m(\delta - t + 1, \delta - s + 1)} q \mu_A \\
&\quad \Rightarrow \mu_A(x) + m(\delta - t + 1, \delta - s + 1) > 1 \\
&\quad \Rightarrow \mu_A(x) + \delta + 1 - \max(t, s) > 1 \\
&\quad \Rightarrow \mu_A(x) > M(t, s) - \delta \\
&\quad \Rightarrow \mu_A(x) > M(t, s) \text{ [since } \delta \text{ is arbitrary]} \\
&\quad \Rightarrow \mu_A(x) > M(t, s) > m(t, s) \\
&\quad \Rightarrow x_{m(t,s)} \in \mu_A.
\end{aligned}$$

Therefore,

$$(x * y)_t, y_s \in \mu_A \Rightarrow x_{m(t,s)} \in \mu_A. \qquad (5)$$

Again, let $x, y \in X$ such that $(x * y)_t, y_s \in \nu_A$. Then

$$\begin{aligned}
&\nu_A(x * y) \leq t \text{ and } \nu_A(y) \leq s \\
&\quad \Rightarrow \nu_A(x * y) - \delta < t \text{ and } \nu_A(y) - \delta < s,
\end{aligned}$$

[where δ is an arbitrary small positive number]

$\Rightarrow \nu_A(x*y) + 1 - \delta - t < 1$ and $\mu_A(y) + 1 - \delta - s < 1$
$\Rightarrow (x*y)_{1-\delta-t} q \nu_A$ and $(y)_{1-\delta-s} q \nu_A$
Since $A = (\mu_A, \nu_A)$ is a (q, q)-IFI of X. Therefore, we have
$x_{M(1-\delta-t,1-\delta-s)} q \nu_A$
$\Rightarrow \nu_A(x) + M(1 - \delta - t, 1 - \delta - s) < 1$
$\Rightarrow \nu_A(x) + 1 - \delta - m(t, s) < 1$
$\Rightarrow \nu_A(x) < m(t, s) + \delta$
$\Rightarrow \nu_A(x) < m(t, s)$[since δ is arbitrary]
$\Rightarrow \nu_A(x) < m(t, s) < M(t, s)$
$\Rightarrow x_{M(t,s)} \in \nu_A$.
Therefore,

$$(x*y)_t, y_s \in \nu_A \Rightarrow x_{M(t,s)} \in \nu_A. \tag{6}$$

Hence, from Eqs. (5) and (6), $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-IFI of X.

Remark 1 Converse of the above theorem is not true, i.e., every $(\in, \in)$-IFI is not a (q, q)-IFI.

Example 5 Consider a BG-algebra $X = \{0, 1, 2, 3\}$ with the following cayley table:

Table 5: Illustration of converse of Theorem 2.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let $A = (\mu_A, \nu_A)$ be an IFS in X defined as $\mu_A(0) = \mu_A(1) = 0.42, \mu_A(2) = \mu_A(3) = 0.35$, and $\nu_A(0) = \nu_A(1) = 0.53, \nu_A(2) = \nu_A(3) = 0.57$. Then $A = (\mu_A, \nu_A)$ is an $(\in, \in)$-IFI in X, but it is not a (q, q)-IFI, because if $x = 2, y = 1, t = 0.72, s = 0.62$, then $x * y = 2 * 1 = 3$. Here $\mu_A(x * y) + t = \mu_A(3) + 0.72 = 0.35 + 0.72 = 1.07 > 1$ and $\mu_A(y) + s = \mu_A(1) + 0.62 = 0.42 + 0.62 = 1.04 > 1$, i.e., $(x * y)_t q \mu_A$ and $y_s q \mu_A$, but $\mu_A(x) + m(t, s) = \mu_A(2) + m(0.72, 0.62) = 0.35 + 0.62 = 0.97 < 1$.

Theorem 3 *An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a BG-algebra X is an $(\in, \in \vee q)$-IFI of X iff*

(i) $\mu_A(x) \geq m(\mu_A(x * y), \mu_A(y), 0.5)$,

(ii) $\nu_A(x) \leq M(\nu_A(x * y), \nu_A(y), 0.5)$.

Proof (i) First, let $A = (\mu_A, \nu_A)$ be an $(\in, \in \vee q)$-IFI of X.

Case I Let $m(\mu_A(x*y),\mu_A(y)) < 0.5\ \forall x,y\in X$. Then

$$m(\mu_A(x*y),\mu_A(y),0.5) = m(\mu_A(x*y),\mu_A(y)).$$

If possible, let $\mu_A(x) < m(\mu_A(x*y),\mu_A(y))$. Choose a real number t such that $\mu_A(x) < t < m(\mu_A(x*y),\mu_A(y))$. Then $(x*y)_t,(y)_t\in\mu_A$.

But,

$$\mu_A(x) < t, \text{ i.e., } x_t\notin\mu_A \text{ and } \mu_A(x)+t<2t,$$

i.e., $\mu_A(x)+t < 2m(\mu_A(x*y),\mu_A(y)) < 2\times 0.5 = 1$

$$\Rightarrow \mu_A(x)+t<1 \Rightarrow x_t\overline{q}\mu_A.$$

which contradicts the fact that $A=(\mu_A,\nu_A)$ is an $(\in,\in\vee q)$-IFI of X. Therefore, $\mu_A(x)\geq m(\mu_A(x*y),\mu_A(y)) = m(\mu_A(x*y),\mu_A(y),0.5)$.

Case II Let $m(\mu_A(x*y),\mu_A(y))\geq 0.5$. Then $m(\mu_A(x*y),\mu_A(y)) = 0.5$. If possible, let $\mu_A(x) < m(\mu_A(x*y),\mu_A(y),0.5) = 0.5$. Then

$$\mu_A(x*y)\geq 0.5 \text{ and } \mu_A(y)\geq 0.5.$$

Therefore, $(x*y)_{0.5}, y_{0.5}\in\mu_A$.

But, $\mu_A(x) < 0.5$, therefore, $x_{0.5}\notin\mu_A$ and $\mu_A(x)+0.5 < 0.5+0.5 = 1$, i.e., $x_{0.5}\overline{\in\vee q}$, which is again a contradiction that $A=(\mu_A,\nu_A)$ is a $(\in,\in\vee q)$-IFI of X.

Hence, we must have $\mu_A(x)\geq 0.5 = m(\mu_A(x*y),\mu_A(y),0.5)$.

Converse Part:

$$\text{Let}\quad \mu_A(x)\geq m(\mu_A(x*y),\mu_A(y),0.5). \tag{7}$$

Let $x,y\in X$ such that $(x*y)_t, y_s\in\mu_A$. Then $\mu_A(x*y)\geq t$ and $\mu_A(y)\geq s$.

Therefore, $m(\mu_A(x*y),\mu_A(y))\geq m(t,s)$. By Eq. (7), $\mu_A(x)\geq m(t,s,0.5)$.

Now, if $m(t,s)\leq 0.5$, then $m(t,s,0.5) = m(t,s)$.

Therefore, $\mu_A(x)\geq m(t,s)$

$$\Rightarrow x_{m(t,s)}\in\mu_A \tag{8}$$

Again, if $m(t,s) > 0.5$, then $m(t,s,0.5) = 0.5$.

Therefore, $\mu_A(x)\geq m(t,s,0.5) = 0.5$, i.e., $\mu_A(x)+m(t,s) > 0.5+0.5 = 1$

$$\Rightarrow x_{m(t,s)}q\mu_A. \tag{9}$$

From Eqs. (8) and (9), we have

$$(x*y)_t, y_s\in\mu_A \Rightarrow x_{m(t,s)}\in\vee q\mu_A. \tag{10}$$

Therefore, μ_A is an $(\in,\in\vee q)$-IFI.

(ii) First, let $A=(\mu_A,\nu_A)$ be an $(\in,\in\vee q)$-IFI of X.

Case I Let $M(\nu_A(x*y),\nu_A(y)) > 0.5\ \forall x,y\in X$. Then

$$M(\nu_A(x*y),\nu_A(y),0.5) = M(\nu_A(x*y),\nu_A(y)).$$

If possible, let $\nu_A(x) > M(\nu_A(x*y),\nu_A(y))$. Choose a real number t such that

$$\nu_A(x) > t > M(\nu_A(x*y),\nu_A(y))$$

$\Rightarrow \nu_A(x*y) < t, \nu_A(y) < t$

$\Rightarrow (x*y)_t \in \nu_A, y_t \in \nu_A$.

But, $\nu_A(x) > t$

$\Rightarrow x_t \notin \nu_A$ and $\nu_A(x) + t > 2t$

$\Rightarrow \nu_A(x) + t > 2M(\nu_A(x*y), \nu_A(y)) > 2 \times 0.5 = 1$

$\Rightarrow \nu_A(x) + t > 1$,

which contradicts the fact that $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-IFI of X. Therefore, $\nu_A(x) \leq M(\nu_A(x*y), \nu_A(y)) = M(\nu_A(x*y), \nu_A(y), 0.5)$.

Case II Let $M(\nu_A(x*y), \nu_A(y)) \leq 0.5\ \forall x, y \in X$. Then $M(\nu_A(x*y), \nu_A(y)) = 0.5$. If possible, let $\nu_A(x) > M(\nu_A(x*y), \nu_A(y), 0.5) = 0.5$. Then

$\nu_A(x*y) \leq 0.5$ and $\nu_A(y) \leq 0.5$.

Therefore, $(x*y)_{0.5}, y_{0.5} \in \nu_A$. But $\nu_A(x) > 0.5$, therefore $x_{0.5} \notin \nu_A$ and $\nu_A(x)+0.5 > 0.5 + 0.5 = 1$, which is again a contradiction that $A = (\mu_A, \nu_A)$ is a $(\in, \in \vee q)$-IFI of X.

Hence, we must have $\nu_A(x) \leq 0.5 = M(\nu_A(x*y), \nu_A(y), 0.5)$.

Converse Part:

$$\text{Let} \quad \nu_A(x) \leq M(\nu_A(x*y), \nu_A(y), 0.5). \tag{11}$$

Let $x, y \in X$, such that $(x*y)_t, y_s \in \nu_A$. Then $\nu_A(x*y) \leq t$ and $\nu_A(y) \leq s$. Therefore $M(\nu_A(x*y), \nu_A(y)) \leq M(t, s)$ By (11), $\nu_A(x) \leq M(t, s, 0.5)$.

Now, if $M(t, s) \geq 0.5)$, then $M(t, s, 0.5) = M(t, s)$. Therefore,

$$\nu_A(x) \leq M(t, s)$$

$$\Rightarrow x_{M(t,s)} \in \nu_A. \tag{12}$$

Again, if $M(t, s) > 0.5$, then $M(t, s, 0.5) = 0.5$. Therefore,

$$\nu_A(x) \leq M(t, s, 0.5) = 0.5$$

$$\Rightarrow \nu_A(x) + M(t, s) < 0.5 + 0.5 = 1$$

$$\Rightarrow x_{M(t,s)} q \nu_A, \tag{13}$$

$$(12) \text{ and } (13) \Rightarrow (x*y)_t, y_s \in \nu_A \Rightarrow x_{M(t,s)} \in \vee q \nu_A. \tag{14}$$

(10) and (14)$\Rightarrow$ ν_A is an $(\in, \in \vee q)$-IFI.

Remark 2 An $(\in, \in)$-IFI is always an $(\in, \in \vee q)$-IFI of X, but not conversely and can be seen from the following example.

Example 6 Consider a BG-algebra $X = \{0, a, b, c\}$ with the following cayley table:

Table 6: Illustration of converse of Remark 2.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $A=(\mu_A,\nu_A)$ be an IFS in X defined as $\mu_A(0)=\mu_A(a)=\mu_A(c)=0.7, \mu_A(b)=0.55$, and $\nu_A(0)=\nu_A(a)=\nu_A(c)=0.42, \nu_A(b)=0.3$. Then $A=(\mu_A,\nu_A)$ is an $(\in,\in\vee q)$-IFI of X. By Theorem 3, it is not an $(\in,\in)$-IFI, since $c_{0.6}=(b*a)_{0.6}, a_{0.6}\in\mu_A$ but $b_{0.6}\notin\mu_A$.

Theorem 4 *An intuitionistic fuzzy subset $A=(\mu_A,\nu_A)$ of a BG-algebra X is an $(\in,\in\vee q)$-IFI of X and if $\mu_A(x)<0.5, \nu_A(x)>0.5\ \forall x,y\in X$, then $A=(\mu_A,\nu_A)$ is also an $(\in,\in)$ -IFI of X.*

Proof Let $A=(\mu_A,\nu_A)$ be an $(\in,\in\vee q)$-IFI of X and $\mu_A(x)<0.5$ and $\nu_A(x)>0.5\ \forall x,y\in X$. Let $(x*y)_t\in\mu_A, y_s\in\mu_A$. Then we have

$t\leq\mu_A(x*y)<0.5$ and $s\leq\mu_A(y)<0.5$.

Therefore $m(t,s)<0.5$ and also $\mu_A(x)<0.5$. Thus $\mu_A(x)+m(t,s)<0.5+0.5=1$.

Since μ_A is an $(\in,\in\vee q)$-IFI of X, therefore,

either $\mu_A(x)\geq m(t,s)$ or $\mu_A(x)+m(t,s)>1$.

So we must have $\mu_A(x)\geq m(t,s)\Rightarrow x_{m(t,s)}\in\mu_A$. Therefore,

$$(x*y)_t\in\mu_A, y_s\in\mu_A\Rightarrow x_{m(t,s)}\in\mu_A. \tag{15}$$

Thus, μ_A is $(\in,\in)$-IFI.

Again, let $(x*y)_t\in\nu_A, y_s\in\nu_A$. Then $0.5<\nu_A(x*y)\leq t$ and $0.5<\nu_A(y)\leq s$. Therefore, $M(t,s)>0.5$. Also $\nu_A(x)>0.5$. Thus, $\nu_A(x)+M(t,s)>0.5+0.5=1$.

Since ν_A is an $(\in,\in\vee q)$-IFI of X, we have

either $\nu_A(x)\leq m(t,s)$ or $\nu_A(x)+M(t,s)<1$.

So we must have, $\nu_A(x)\leq m(t,s)\Rightarrow x_{M(t,s)}\in\nu_A$. Therefore,

$$(x*y)_t\in\nu_A, y_s\in\nu_A\Rightarrow x_{M(t,s)}\in\nu_A. \tag{16}$$

Thus, ν_A is $(\in,\in)$-IFI.

Hence (15) and (16) $\Rightarrow$ $A=(\mu_A,\nu_A)$ is $(\in,\in)$-IFI of X.

Remark 3 Every $(\in,q)$-IFI of *BG*-algebra X is always a $(\in,\in\vee q)$-IFI of X.

Theorem 5 *An intuitionistic fuzzy subset $A=(\mu_A,\nu_A)$ of a BG-algebra X is an $(\in,\in\vee q)$-IFI of X iff the sets $(\mu_A)_t=\{x\mid\mu_A(x)\geq t$, where $t\in(0,0.5), \mu_A(0)\geq t\}$ and $(\nu_A)_s=\{x\mid\nu_A(x))<s$, where $s\in(0.5,1], \nu_A(0)<s\}$ are ideal of X.*

Proof Assume $A=(\mu_A,\nu_A)$ is an $(\in,\in\vee q)$-IFI of X. Clearly,

$0\in(\mu_A)_t, 0\in(\nu_A)_s$ [since $\mu_A(0)\geq t, \nu_A(0)\leq s$].

Let $x,y\in X$, such that $x*y, y\in(\mu_A)_t$ where $t\in(0,0.5]$. Therefore $\mu_A(x*y)\geq t, \mu_A(y)\geq s$.

Now by Theorem 3

$\mu_A(x)\geq m(\mu_A(x*y),\mu_A(y),0.5)\geq m(t,t,0.5)=t$

$\Rightarrow\mu_A(x)\geq t\Rightarrow x\in(\mu_A)_t.$

Therefore, $x*y, y\in(\mu_A)_t\Rightarrow x\in(\mu_A)_t.$

Hence $(\mu_A)_t$ is an ideal of X.

Again let $x,y\in X$ such that $x*y, y\in(\nu_A)_s$ where $s\in(0.5,1]$.

Therefore $\nu_A(x * y) < s, \nu_A(x * y) < s$.

Now by Theorem 3

$\nu_A(x) \leq M(\nu_A(x * y), \nu_A(y), 0.5) < M(s, s, 0.5) = s$

$\Rightarrow \nu_A(x) < s \Rightarrow x \in (\nu_A)_{s.}$

Therefore $x * y, y \in (\nu_A)_s \Rightarrow x \in (\nu_A)_{s.}$

Hence $(\nu_A)_s$ is an ideal of X.

Conversely, let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of X and the sets $(\mu_A)_t = \{x \mid \mu_A(x) \geq t$, where $t \in (0, 0.5)\}$ and $(\nu_A)_s = \{x \mid \nu_A(x)) < s$, where $s \in (0.5, 1]\}$ are ideal of X, to prove $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-IFI of X. Suppose $A = (\mu_A, \nu_A)$ is not an $(\in, \in \vee q)$-IFI of X, then there exist $a, b \in X$ such that at least one of $\mu_A(a) < m(\mu_A(a * b), \mu_A(b), 0.5)$ and $\nu_A(a) > M(\nu_A(a * b), \nu_A(b), 0.5)$ hold. Suppose $\mu_A(a) < m(\mu_A(a * b), \mu_A(b), 0.5)$ holds. Let $t = [\mu_A(a) + m(\mu_A(a * b), \mu_A(b), 0.5)]/2$. Then $t \in (0, 0.5)$ and

$$\mu_A(a) < t < m(\mu_A(a * b), \mu_A(b), 0.5) \tag{17}$$

$\Rightarrow \mu_A(a * b) > t, \mu_A(b) > t$

$\Rightarrow a * b \in (\mu_A)_t, b \in (\mu_A)_t$

$\Rightarrow a \in (\mu_A)_t$ [since $(\mu_A)_t$ is ideal].

Therefore $\mu_A(a) > t$, which contradicts (17). Hence we must have

$$\mu_A(x) \geq m(\mu_A(x * y), \mu_A(y), 0.5). \tag{18}$$

Next let $\nu_A(a) > M(\nu_A(a * b), \nu_A(b), 0.5)$ holds.

Let $s = [\nu_A(a) + M(\nu_A(a * b), \nu_A(b), 0.5)]/2$. Then $s \in (0.5, 1]$ and

$$\nu_A(a) > s > M(\nu_A(a * b), \nu_A(b), 0.5) \tag{19}$$

$\Rightarrow \nu_A(a * b) < s, \nu_A(b) < s$

$\Rightarrow a * b \in (\nu_A)_s, b \in (\nu_A)_s \Rightarrow a \in (\nu_A)_s$ [since $(\nu_A)_s$ is ideal].

Therefore $\nu_A(a) < s$, which contradicts (19). Hence we must have

$$\nu_A(x) = M(\nu_A(x * y), \nu_A(y), 0.5). \tag{20}$$

Hence (18) and (20) $\Rightarrow A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-IFI of X.

Theorem 6 *Let S be a subset of a BG-algebra X. Consider the IFS $A_S = (\mu_S, \nu_S)$ in X defined by*

$$\mu_S(x) = \begin{cases} 1, & if \quad x \in S, \\ 0, & otherwise, \end{cases} \qquad \nu_S(x) = \begin{cases} 0, & if \quad x \in S, \\ 1, & otherwise. \end{cases}$$

Then S is an ideal of X iff $A_S = (\mu_S, \nu_S)$ is an $(\in, \in \vee q)$-IFI X.

Proof Let S be an ideal of X. Now $(\mu_S)_t) = \{x \mid \mu_S(x) \geq t\} = S$, And $(\nu_S)_t) = \{x \mid \nu_S(x) < t\} = S$, which is an ideal. Hence by Theorem 5, $A_S = (\mu_S, \nu_S)$ is an

$(\in, \in \vee q)$-IFI X.

Conversely, assume that $A_S = (\mu_S, \nu_S)$ is an $(\in, \in \vee q)$-IFI X, to prove S is an ideal of X. Let $x * y, y \in S$. Then

$\mu_S(x) \geq m(\mu_S(x * y), \mu_S(y), 0.5) = m(1, 1, 0.5) = 0.5$
$\Rightarrow \mu_S(x) \geq 0.5 \Rightarrow \mu_S(x) = 1 \Rightarrow x \in S$

and $\nu_S(x) = M(\nu_S(x * y), \nu_S(y), 0.5) = M(0, 0, 0.5) = 0.5$
$\Rightarrow \nu_S(x) \leq 0.5 \Rightarrow \nu_S(x) = 0 \Rightarrow x \in S$.

Hence S is an ideal of X.

Theorem 7 *Let S be an ideal of X. Then there exists* $(\in, \in \vee q)$*-IFI* $A = (\mu_A, \nu_A)$ *of X such that* $(\mu_A)_t = (\nu_A)_s = S$ *for every* $t \in (0, 0.5)$ *and* $s \in (0.5, 1]$.

Proof Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in X defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in S, \\ u, & \text{otherwise}, \end{cases} \qquad \nu_A(x) = \begin{cases} 0, & \text{if } x \in S, \\ s, & \text{otherwise}, \end{cases}$$

where $u < t \in (0, 0.5]$. Therefore $(\mu_A)_t) = \{x : \mu_A(x) \geq t\} = S$, $(\nu_A)_t) = \{x : \nu_A(x) < t\} = S$, and hence $(\mu_A)_t = (\nu_A)_s = S$ is an ideal.

Now if $A = (\mu_A, \nu_A)$ is not an $(\in, \in \vee q)$-fuzzy ideal of X, then there exist $a, b \in X$ such that at least one of $\mu_A(a) < m(\mu_A(a * b), \mu_A(b), 0.5)$ and $\nu_A(a) > M(\nu_A(a * b), \nu_A(b), 0.5)$ hold. Suppose $\mu_A(a) < m(\mu_A(a * b), \mu_A(b), 0.5)$ holds, then choose a real number $t \in (0, 1)$ such that

$$\mu_A(a) < t < m(\mu_A(a * b), \mu_A(b), 0.5) \tag{21}$$

$\Rightarrow \mu_A(a * b) > t, \mu_A(b) > t$
$\Rightarrow a * b \in (\mu_A)_t, b \in (\mu_A)_t$
$\Rightarrow a \in (\mu_A)_t = S$ [since $(\mu_A)_t$ is ideal].

Therefore $(\mu_A)_t(a) = 1 > t$, which contradicts (21).

Hence we must have $\mu_A(x) < m(\mu_A(x * y), \mu_A(y), 0.5)$.

Again if $\nu_A(a) > M(\nu_A(a*b), \nu_A(b), 0.5)$ holds, then choose a real number $s \in (0, 1)$ such that

$$\nu_A(a) > s > M(\nu_A(a * b), \nu_A(b), 0.5) \tag{22}$$

$\Rightarrow \nu_A(a * b) < s, \nu_A(b) < s$
$\Rightarrow a * b \in (\nu_A)_s, , b \in (\nu_A)_s$
$\Rightarrow a \in (\nu_A)_s = S$ [since $(\nu_A)_s$ is ideal].

Therefore $\nu_A(a) = 0 < s$, which contradicts (22).

Hence we must have $\nu_A(x) \leq M(\nu_A(x * y), \nu_A(y), 0.5)$. Thus, $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-IFI X.

Definition 15 *Let* $A = (\mu_A, \nu_A)$ *be intuitionistic fuzzy subset of BG-algebra X and* $t \in (0, 1]$. *Then let*

$(\mu_A)_t = \{x \mid x_t \in \mu_A\} = \{x \mid \mu_A(x) \geq t\}$,

$< \mu_A >_t = \{x \mid x_t q \mu_A\} = \{x \mid \mu_A(x) + t > 1\}$,

$[\mu_A]_t = \{x \mid x_t \in \vee q \mu_A\} = \{x \mid\ < \mu_A(x) \geq t \text{ or } \mu_A(x) + t > 1\}$,

where $(\mu_A)_t$ *is called t level set of* $\mu_A, < \mu_A >_t$ *is called q level set of* μ_A *and* $[\mu_A]_t$ *is called* $\in \vee q$ *level set of* μ_A,

clearly,

$[\mu_A]_t =< \mu_A >_t \cup (\mu_A)_t$,

$(\nu_A)_t = \{x \mid x_t \in \nu_A\} = \{x \mid \nu_A(x) \leq t\}$,

$< \nu_A >_t = \{x \mid x_t q \nu_A\} = \{x \mid \nu_A(x) + t < 1\}$,

$[\nu_A]_t = \{x \mid x_t \in \vee q \nu_A\} = \{x \mid\ < \nu_A(x) \leq t \text{ or } \nu_A(x) + t < 1\}$,

where $(\nu_A)_t$ *is called t level set of* $\nu_A, < \nu_A >_t$ *is called q level set of* ν_A *and* $[\nu_A]_t$ *is called* $\in \vee q$ *level set of* ν_A,

clearly,

$[\nu_A]_t =< \nu_A >_t \cup (\nu_A)_t$.

Theorem 8 *Let* $A = (\mu_A, \nu_A)$ *be intuitionistic fuzzy subset of BG-algebra X. Then A is an* $(\in, \in \vee q)$*-IFI X iff* $[\mu_A]_t$ *and* $[\nu_A]_t$ *is an ideal of X for all* $t \in (0, 1]$. *We call* $[\mu_A]_t$ *and* $[\nu_A]_t$ *as* $\in \vee q$ *level ideals of* μ.

Proof Assume that A is an $(\in, \in \vee q)$-IFI of X, to prove $[\mu_A]_t$ and $[\nu_A]_t$ is an ideal of X. Let $x * y, y \in [\mu_A]_t$ for $t \in (0, 1]$. Then

$(x * y)_t \in \vee q \mu_A$ and $(y)_t \in \vee q \mu_A$,

i.e., $\mu_A(x * y) \geq t$ or $\mu_A(x * y) + t \geq 1$ and $\mu_A(y) \geq t$ or $\mu_A(y) + t \geq 1$.

Since A is an $(\in, \in \vee q)$-IFI of X,

$\mu_A(x) \geq m(\mu_A(x), \mu_A(y), 0.5)\ \forall\ x, y \in X.$

Now we have the following cases.

Case I $\mu_A(x * y) \geq t, \mu_A(y) \geq t$, let $t > 0.5$. Then

$\mu_A(x) \geq m(\mu_A(x), \mu_A(y), 0.5) = m(t, t, 0.5) = 0.5$,

$\Rightarrow \mu_A(x) \geq 0.5 \Rightarrow \mu_A(x) + t > 0.5 + 0.5 = 1 \Rightarrow x_t q \mu_A$.

Again if $t \leq 0.5$, then

$\mu_A(x) \geq m(\mu_A(x), \mu_A(y), 0.5) \geq m(t, t, 0.5) = t$,

$\Rightarrow \mu_A(x) \geq t \Rightarrow x_t \in \mu_A$.

Hence $(x)_t \in \vee q \mu_A \Rightarrow x_t \in [\mu_A]_t$.

Case II $\mu_A(x * y) \geq t, \mu_A(y) + t \geq 1$, let $t > 0.5$. Then

$\mu_A(x) \geq m(\mu_A(x), \mu_A(y), 0.5) > m(t, 1 - t, 0.5) = 1 - t$,

$\Rightarrow \mu_A(x) > 1 - t \Rightarrow \mu_A(x) + t > 1 \Rightarrow x_t q \mu_A$.

Again if $t \leq 0.5$, then

$\mu_A(x) \geq m(\mu_A(x), \mu_A(y), 0.5) = m(t, 1 - t, 0.5) = t$,

$\Rightarrow \mu_A(x) \geq t \Rightarrow x_t \in \mu_A$.

Hence $(x)_t \in \vee q \mu_A \Rightarrow x_t \in [\mu_A]_t$.

Case III $\mu_A(x * y) + t > 1, \mu_A(y) \geq t$.

This is similar to case II.

Case IV $\mu_A(x * y) + t \geq 1, \mu_A(y) + t \geq 1$, let $t > 0.5$. Then

$\mu_A(x) \geq m(\mu_A(x), \mu_A(y), 0.5) > m(1 - t, 1 - t, 0.5) = 1 - t$,

$\Rightarrow \mu_A(x) > 1 - t \Rightarrow \mu_A(x) + t > 1 \Rightarrow x_t q \mu_A$.

Again if $t \leq 0.5$, then $\mu_A(x) \geq m(\mu_A(x), \mu_A(y), 0.5) = m(1-t, 1-t, 0.5) = 0.5 \geq t$ $\Rightarrow \mu_A(x) \geq t \Rightarrow x_t \in \mu_A$.

Hence $(x)_t \in \vee q\mu_A \Rightarrow x_t \in [\mu_A]_t$.

Hence from above four cases $x * y, y \in [\mu_A]_t \Rightarrow x_t \in [\mu_A]_t$.

Hence $[\mu_A]_t$ is an ideal of X. Similarly, we can prove $[\nu_A]_t$ is an ideal of X.

Conversely, let $A = (\mu_A, \nu_A)$ be an IFS in X, such that $[\mu_A]_t$ and $[\mu_A]_t$ is an ideal of X for all $t \in (0, 1]$, to prove $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-IFI of X. Suppose A is not an $(\in, \in \vee q)$-IFI of X, then there exist $a, b \in X$ such that at least one of $\mu_A(a) < m(\mu_A(a * b), \mu_A(b), 0.5)$ and $\nu_A(a) > M(\nu_A(a * b), \nu_A(b), 0.5)$ hold. Suppose $\mu_A(a) < m(\mu_A(a * b), \mu_A(b), 0.5)$ is true, then choose $t \in (0, 1]$, such that

$$\mu_A(a) < t < m(\mu_A(a * b), \mu_A(b), 0.5). \tag{23}$$

Then $\mu_A(a * b) > t, \mu_A(b) > t \Rightarrow a * b, b \in (\mu_A)_t \subset [\mu_A]_t$ which is an ideal.
Therefore, $a \in [\mu_A]_t \Rightarrow \mu_A(a) \geq t$ or $\mu_A(a) + t > 1$ which contradict (23).

Again if $\nu_A(a) > M(\nu_A(a * b), \nu_A(b), 0.5)$ is true, then choose$t \in (0, 1]$, such that

$$\nu_A(a) > t > M(\nu_A(a * b), \nu_A(b), 0.5). \tag{24}$$

Then $\nu_A(a * b) < t, \nu_A(b) < t \Rightarrow a * b, b \in (\nu_A)_t \subset [\nu_A]_t$ which is an ideal.
Therefore, $a \in [\nu_A]_t \Rightarrow \nu_A(a) < t$ or $\nu_A(a) + t < 1$ which contradict (24).
Hence we must have

$$\mu_A(x) \geq m(\mu_A(x * y), \mu_A(y), 0.5),$$
$$\nu_A(x) \leq M(\nu_A(x * y), \nu_A(y), 0.5)\ \forall x, y \in X.$$

Hence $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-IFI of X.

Theorem 9 *Every* $(\in\vee q, \in\vee q)$*-IFI is an* $(\in, \in \vee q)$*-IFI.*

Proof It follows from definition.

Theorem 10 *Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be two* $(\in, \in \vee q)$*-IFIs of a BG-algebra* X*. Then* $A \cap B$ *is also an* $(\in, \in \vee q)$*-IFI of* X*.*

Proof Let $x, y \in X$. Now we have $A \cap B(x) = \{< x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) > | x \in X\}$,

$$\begin{aligned}(\mu_A \cap \mu_B)(x) &= m\{\mu_A(x), \mu_B(x)\}\\ &\geq m\{m\{\mu_A(x * y), \mu_A(y), 0.5\}, m\{\mu_B(x * y), \mu_B(y), 0.5\}\}\\ &\quad [\text{since } A \text{ is an } (\in, \in \vee q) - \text{IFI}]\\ &= m\{m\{\mu_A(x * y), \mu_B(x * y)\}, m\{\mu_A(y), \mu_A(y)\}, 0.5\}\\ &= m\{(\mu_A \cap \mu_B)(x * y), (\mu_A \cap \mu_B)(y), 0.5\},\end{aligned} \tag{25}$$

$$\begin{aligned}(\nu_A \cap \nu_B)(x) &= M\{\nu_A(x), \nu_B(x)\}\\ &\leq M\{M\{\nu_A(x * y), \nu_A(y), 0.5\}, M\{\nu_B(x * y), \nu_B(y), 0.5\}\}\\ &\quad [\text{since } A \text{ is an } (\in, \in \vee q) - \text{IFI}] \qquad (26)\\ &= M\{M\{\nu_A(x * y), \nu_B(x * y)\}, M\{\nu_A(y), \nu_A(y)\}, 0.5\}\\ &= M\{(\nu_A \cup \nu_B)(x * y), (\nu_A \cup \nu_B)(y), 0.5\}.\end{aligned}$$

(25) and (26) $\Rightarrow (A \bigcap B)$ $(\in, \in \vee q)$-IFI of X.

The above theorem can be generalized as

Theorem 11 *Let* $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i = 1, 2, 3, \cdots\}$ *be a family of* $(\in, \in \vee q)$*-IFIs of a BG-algebra X. Then* $\bigcap_i^n A_i$ *is also an* $(\in, \in \vee q)$*-IFI of X, where* $\bigcap_i^n A_i(x) = \{< x, m\{\mu_{A_i}(x) \mid i = 1, 2, 3, \cdots\}, M\{\nu_{A_i}(x) \mid i = 1, 2, 3, \cdots\} >: x \in X\}$.

4. Cartesian Product of *BG*-algebras and Their $(\in, \in \vee q)$-Intuitionistic Fuzzy Ideals

Theorem 12 *Let X ,Y be two BG-algebras. Then their cartesian product* $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ *is also a BG-algebra under the binary operation* $*$ *defined in* $X \times Y$ *by* $(x, y) * (p, q) = (x * p, y * q)$ *for all* $(x, y), (p, q) \in X \times Y$.

Proof Clearly, $0 \in X, 0 \in Y$, therefore $(0, 0) \in X \times Y$.

Let $(x, y), (p, q) \in X \times Y$. Now

(i) $(x, y) * (x, y) = (x * x, y * y) = (0, 0) \in X \times Y,$

(ii) $(x, y) * (0, 0) = (x * 0, y * 0) = (x, x) \in X \times Y,$

(iii) $$\begin{aligned}((x, y) * (p, q)) * ((0, 0)) * (p, q)) &= (x * p, y * p) * (0 * p, 0 * q)\\ &= ((x * p) * (0 * p), (y * p) * (0 * q))\\ &= (x, y) \text{ for all } (x, y), (p, q) \in X \times Y,\end{aligned}$$

which shows that $(X \times Y, (0, 0), *)$ is a *BG*-algebra.

Definition 16 *Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be two* $(\in, \in \vee q)$*-IFIs of a BG-algebra X. Then their Cartesian product* $A \times B$ *is defined by* $(A \times B)(x, y) = \{< (x, y), m\{\mu_A(x), \mu_B(y)\}, M\{\nu_A(x), \nu_B(y)\} >: x, y \in X\}$ *where* $\mu_A, \mu_B : X \to [0, 1]$ *and* $\nu_A, \nu_B : X \to [0, 1]$ $\forall x, y \in X$.

Theorem 13 *Let* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be two* $(\in, \in \vee q)$*-IFIs of a BG-algebra X. Then* $A \times B$ *is also an* $(\in, \in \vee q)$*-IFI of X.*

Proof Similar to Theorem 10.

5. Homomorphism of *BG*-algebras and Intuitionistic Fuzzy Ideals

Definition 17 *Let X and X′ be two BG-algebras. Then a mapping $f : X \to X'$ is said to be homomorphism if $f(x * y) = f(x) * f(y) \forall x, y \in X$.*

Theorem 14 *Let X and X′ be two BG-algebras and $f : X \to X'$ be homomorphism. If $A = (\mu_A, \nu_A)$ is an $(\in, \in \vee q)$-IFI of X', then $f^{-1}(A)$ is $(\in, \in \vee q)$-IFI of X.*

Proof $f^{-1}(A) = f^{-1}(\mu_A, \nu_A)(x)$ is defined as $f^{-1}(\mu_A, \nu_A)(x) = (\mu_A, \nu_A)(f(x)) \forall x \in X$. Let $A = (\mu_A, \nu_A)$ be an $(\in, \in \vee q)$-IFI of X', let $x, y \in X$ such that $(x * y)_t, y_s \in f^{-1}(A) = f^{-1}(\mu_A, \nu_A) = (f^{-1}\mu_A, f^{-1}\nu_A)$. Then $(x * y)_t, y_s \in f^{-1}(\mu_A)$ and $(x * y)_t, y_s \in f^{-1}(\nu_A)$.

Case I Let $(x * y)_t, y_s \in f^{-1}(\mu_A)$
$\Rightarrow f^{-1}(\mu_A)(x * y) \geq t$ and $f^{-1}(\mu_A)(y) \geq s$
$\Rightarrow \mu_A f(x * y) \geq t$ and $\mu_A f(y) \geq s$
$\Rightarrow (f(x * y))_t \in \mu_A$ and $(f(y))_s \in \mu_A$
$\Rightarrow (f(x) * f(y))_t \in \mu_A$ and $(f(y))_s \in \mu_A$ [since f is homomorphism]
$\Rightarrow (f(x))_{m(t,s)} \in \mu_A$
$\Rightarrow \mu_A(f(x)) \geq m(t, s)$ or $\mu_A(f(x)) + m(t, s) > 1$
$\Rightarrow f^{-1}(\mu_A)(x)) \geq m(t, s)$ or $f^{-1}(\mu_A)(x)) + m(t, s) > 1$
$\Rightarrow x_{m(t,s)} \in f^{-1}(\mu_A)$ or $x_{m(t,s)} \in q f^{-1}(\mu_A)$
$\Rightarrow x_{m(t,s)} \in \vee q f^{-1}(\mu_A)$.
Therefore,

$$(x * y)_t, y_s \in f^{-1}(\mu_A) \Rightarrow x_{m(t,s)} \in \vee q f^{-1}(\mu_A). \tag{27}$$

Case II Let $(x * y)_t, y_s \in f^{-1}(\nu_A)$
$\Rightarrow f^{-1}(\nu_A)(x * y) \leq t$ and $f^{-1}(\nu_A)(y) \leq s$
$\Rightarrow \nu_A f(x * y) \leq t$ and $\nu_A f(y) \leq s$
$\Rightarrow (f(x * y))_t \in \nu_A$ and $(f(y))_s \in \nu_A$
$\Rightarrow (f(x) * f(y))_t \in \nu_A$ and $(f(y))_s \in \nu_A$ [since f is homomorphism]
$\Rightarrow (f(x))_{M(t,s)} \in \nu_A$
$\Rightarrow \nu_A(f(x)) \leq M(t, s)$ or $\nu_A(f(x)) + M(t, s) < 1$
$\Rightarrow f^{-1}(\nu_A)(x)) \leq M(t, s)$ or $f^{-1}(\nu_A)(x)) + M(t, s) < 1$
$\Rightarrow x_{M(t,s)} \in f^{-1}(\nu_A)$ or $x_{M(t,s)} \in q f^{-1}(\nu_A)$
$\Rightarrow x_{M(t,s)} \in \vee q f^{-1}(\nu_A)$.
Therefore,

$$(x * y)_t, y_s \in f^{-1}(\nu_A) \Rightarrow x_{M(t,s)} \in \vee q f^{-1}(\nu_A). \tag{28}$$

(27) and (28) $\Rightarrow$ $f^{-1}(A) = f^{-1}(\mu_A, \nu_A) = (f^{-1}\mu_A, f^{-1}\nu_A)$ is an $(\in, \in \vee q)$-IFI of X.

Theorem 15 *Let X and X′ be two BG-algebras and $f : X \to X'$ be an onto homomorphism. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subset of X′ such that $f^{-1}(A)$ is an $(\in, \in \vee q)$-IFI of X, then A is also an $(\in, \in \vee q)$-IFI of X.*

Proof Let $x', y' \in X'$ such that $(x' * y')_t, y'_s \in A = (\mu_A, \nu_A)$ where $t, s \in [01]$, that is $(x' * y')_t, y'_s \in \mu_A$ and $(x' * y')_t, y'_s \in \nu_A$. Then $\mu_A(x' * y') \geq t$ and $\mu_A(y') \geq s$ and

$\nu_A(x' * y') \leq t$ and $\nu_A(y') \leq s$. Since f is onto, so there exists $x, y \in X$ such that $f(x) = x', f(y) = y'$, also f is homomorphism so $f(x * y) = f(x) * f(y) = x' * y'$.

Now $(x' * y')_t, y'_s \in \mu_A$

$\Rightarrow \mu_A(f(x * y)) \geq t$ and $\mu_A((f(y)) \geq s$

$\Rightarrow f^{-1}(\mu_A)(x * y) \geq t$ and $f^{-1}(\mu_A)(y) \geq s$

$\Rightarrow (x * y)_t \in f^{-1}(\mu_A)$ and $(y)_s \in f^{-1}(\mu_A)$

$\Rightarrow (x)_{m(t,s)} \in \vee q f^{-1}(\mu_A)$

[since $f^{-1}(\mu_A)$ is a $(\in, \in \vee q)$ intuitionistic fuzzy ideal of X]

$\Rightarrow f^{-1}(\mu_A)(x) \geq m(t, s)$ or $f^{-1}(\mu_A)(x) + m(t, s) > 1$

$\Rightarrow \mu_A(f(x)) \geq m(t, s)$ or $\mu_A(f(x)) + m(t, s) > 1$

$\Rightarrow \mu_A(x') \geq m(t, s)$ or $\mu_A(x') + m(t, s) > 1$

$\Rightarrow x'_{m(t,s)} \in \vee q \mu_A$.

Therefore,

$$(x' * y')_t, y'_s \in \mu_A \Rightarrow x'_{m(t,s)} \in \vee q \mu_A. \tag{29}$$

Again $(x' * y')_t, y'_s \in \nu_A$

$\Rightarrow \nu_A(x' * y') \leq t$ and $\nu_A(y') \leq s$

$\Rightarrow \nu_A(f(x * y)) \leq t$ and $\nu_A(f(y)) \leq s$

$\Rightarrow f^{-1}(\nu_A)(x * y) \leq t$ and $f^{-1}(\nu_A)(y) \leq s$

$\Rightarrow (x * y)_t \in f^{-1}(\nu_A)$ and $(y)_s \in f^{-1}(\nu_A)$

$\Rightarrow (x)_{M(t,s)} \in \vee q f^{-1}(\nu_A)$ [since $f^{-1}(\nu_A)$ is a $(\in, \in \vee q)$-IFI of X]

$\Rightarrow f^{-1}(\nu_A)(x) \leq M(t, s)$ or $f^{-1}(\nu_A)(x * y) + M(t, s) < 1$

$\Rightarrow (\nu_A)f(x) \leq M(t, s)$ or $(\nu_A)f(x) + M(t, s) < 1$

$\Rightarrow (\nu_A)(f(x)) \leq M(t, s)$ or $(\nu_A)(f(x)) + M(t, s) < 1$

$\Rightarrow (\nu_A)(x') \leq M(t, s)$ or $(\nu_A)(x') + M(t, s) < 1$

$\Rightarrow (x')_{M(t,s)} \in \nu_A$ or $(x')_{M(t,s)} q \nu_A$.

Therefore,

$$(x' * y')_t, y'_s \in \nu_A \Rightarrow (x')_{M(t,s)} \in \vee q \nu_A. \tag{30}$$

(29) and (30) $\Rightarrow$ A is an $(\in, \in \vee q)$-IFI of X.

6. Conclusion

In this paper, we introduce the concept of $(\in, \in \vee q)$-IFIs of BG-algebra and investigate some of their useful properties. In my opinion, these definitions and results can be extended to other algebraic systems also. In the notions of (α, β)-fuzzy ideals, we can define twelve different types of ideals by three choices of α and four choices of β. In the present paper, we mainly discuss $(\in, \in \vee q)$ type fuzzy ideal. In the future, the following studies may be carried out: 1) $(\in, \in \vee q)$-IFIs of d-algebra, 2) $(\in, \in \vee q)$-doubt fuzzy ideals of BG-algebra.

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