Convergence theorems for common fixed points of asymptotically quasi-nonexpansive mappings in convex metric spaces

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ABSTRACT

In this paper, we consider an implicit iteration process to approximate the common fixed points of two finite families of asymptotically quasi-nonexpansive mappings in convex metric spaces. As a consequence of our result, we obtain some related convergence theorems. Our results generalize some recent results of Khan and Ahmed [4], Khan et al. [6], Sun [12], Wittmann [14] and Xu and Ori [15].

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1. Introduction and preliminaries

Throughout this paper, \( \mathbb{N} \) denotes the set of natural numbers and \( J = \{1, 2, \ldots, N\} \), the set of first \( N \) natural numbers. Denote by \( F(T) \) the set of fixed points of \( T \) and by \( F := \bigcap_{j \in J} F(T_j) \cap \bigcap_{j \in J} F(S_j) \) the set of common fixed points of two finite families of mappings \( \{T_j : j \in J\} \) and \( \{S_j : j \in J\} \).

Takahashi [13] introduced the notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in this setting. For convex metric spaces, Kirk [8] used the term “hyperbolic type spaces” and studied iteration processes for nonexpansive mappings in this abstract framework. Later on, many authors discussed the existence of the fixed point and the convergence of the iterative process for various mappings in convex metric spaces (see, for example, [1,4,7,9,13] and the references therein).

We recall some definitions in a metric space \((X, d)\).

Definition 1.1 [13]. A convex structure in a metric space \((X, d)\) is a mapping \( W : X \times X \times [0, 1] \rightarrow X \) satisfying, for all \( x, y, u \in X \) and all \( \lambda \in [0, 1] \),

\[
d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).\]

A metric space together with a convex structure is called a convex metric space.

A nonempty subset \( C \) of \( X \) is said to be convex if \( W(x, y; \lambda) \in C \) for all \( (x, y; \lambda) \in C \times C \times [0, 1] \).

Definition 1.2 [4]. Let \((X, d)\) be a metric space and \( q \) be a fixed element of \( X \). A \( q \)-starshaped structure in \( X \) is a mapping \( W : X \times X \times [0, 1] \rightarrow X \) satisfying, for all \( x, y \in X \) and all \( \lambda \in [0, 1] \),

\[
d(q, W(x, y; \lambda)) \leq \lambda d(q, x) + (1 - \lambda) d(q, y).\]

A metric space together with a \( q \)-starshaped structure is called a \( q \)-starshaped metric space.

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Clearly, a convex metric space is a $q$-starshaped metric space but the converse is not true in general.

We extend Definition 1.1 as follows: A mapping $W : X^3 \times [0,1]^3 \rightarrow X$ is said to be a convex structure on $X$, if it satisfies the following condition:

For any $(x,y,z; a,b,c) \in X^3 \times [0,1]^3$ with $a + b + c = 1$, and $u \in X$:

$$d(W(x,y,z; a,b,c), u) \leq ad(x,u) + bd(y,u) + cd(z,u).$$

If $(X,d)$ is a metric space with a convex structure $W$, then $(X,d)$ is called a convex metric space.

Let $(X,d)$ be a convex metric space. A nonempty subset $E$ of $X$ is said to be convex if $W(x,y,z; a,b,c) \in E$, $\forall (x,y,z) \in E^3$, $(a,b,c) \in [0,1]^3$ with $a + b + c = 1$.

**Remark 1.1.** Every normed space is a special convex metric space with a convex structure $W(x,y,z; a,b,c) = ax + by + cz$, for all $x, y, z \in X$ and $a, b, c \in (0,1)$ with $a + b + c = 1$. In fact,

$$d(u, W(x,y,z; a,b,c)) = ||u - (ax + by + cz)|| \leq ||u - x|| + ||y - u|| + ||z - u|| = ad(u,x) + bd(u,y) + cd(u,z), \forall u \in X.$$

However, there exist convex metric spaces which cannot be embedded into a normed space, see for example Takahashi [13].

**Definition 1.3.** A mapping $T : X \rightarrow X$ is called:

1. Nonexpansive if $d(Tx,Ty) \leq d(x,y) \forall x, y \in X$.
2. Quasi-nonexpansive if $d(Tx,p) \leq d(x,p) \forall x, y \in X, \forall p \in F(T)$.
3. Asymptotically nonexpansive [3] if there exists $k_n \in [0,\infty)$ for all $n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that $d(T^n x, T^n y) \leq (1 + k_n) d(x,y) \forall x, y \in X$.
4. Asymptotically quasi-nonexpansive if there exists $k_n \in [0,\infty)$ for all $n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} k_n = 0$ such that $d(T^n x, p) \leq (1 + k_n) d(x,p) \forall x \in X, \forall p \in F(T)$.

**Remark 1.2.** From Definition 1.3, the following implications are obvious:

Nonexpansiveness $\Rightarrow$ Quasi-nonexpansiveness.
Nonexpansiveness $\Rightarrow$ Asymptotically nonexpansiveness.
Quasi-nonexpansiveness $\Rightarrow$ Asymptotically quasi-nonexpansiveness.
Asymptotically nonexpansiveness $\Rightarrow$ Asymptotically quasi-nonexpansiveness.

The converses of these implications may not be true (see, for example, [3,10,16]).

In 2001, Xu and Ori [15] introduced the following implicit iteration process for common fixed points of a finite family of nonexpansive mappings $\{T_j : j \in J\}$ in Hilbert spaces:

$$x_n = x_0 x_{n-1} + (1 - x_0) T_n x_n, \quad n \in \mathbb{N},$$

(1.1)

where $T_n = T_n (\text{mod } N)$ and $\{a_n\}$ is a real sequence in $(0,1)$. They proved a weak convergence theorem using this process.

In 2003, Sun [12] extended the process (1.1) to the following process for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings $\{T_j : j \in J\}$ in uniformly convex Banach spaces:

$$x_n = x_0 x_{n-1} + (1 - x_0) T_j x_n, \quad n \in \mathbb{N},$$

(1.2)

where $n = (k - 1) N + j$, $j \in J$ and $\{a_n\}$ is a real sequence in $(0,1)$.

Sun [12] studied strong convergence of the process (1.2) for common fixed points of $\{T_j : j \in J\}$, requiring only one member of the family to be semicompact. The result of Sun [12] generalized and extended the corresponding results of Xu and Ori [15].

In 2008, Khan et al. [5] studied the following $n$-step iterative process for a finite family of mappings:

$$x_{n+1} = x_0 x_n + x_n T_{k-1} T_{k-2} \ldots T_{k-n} x_n$$

$$y_{k-1} = x_0 x_n + x_n T_{k-1} T_{k-2} \ldots T_{k-n} y_{k-2}$$

$$y_{k-2} = x_0 x_n + x_n T_{k-1} T_{k-2} \ldots T_{k-n} y_{k-3}$$

$$\vdots$$

$$y_2 = x_0 x_n + x_n T_{k-1} T_{k-2} y_1$$

$$y_1 = x_0 x_n + x_n T_{k-1} T_{k-2} y_0,$$

where $y_0 = x_n$ for all $n \in \mathbb{N} \cup \{0\}$. 
Khan and Ahmed [4] transformed this process into convex metric spaces as follows.

\[ x_{n+1} = W(T_n^j y_{(k-1)n}, x_n, \alpha_{3n}) \]  
\[ y_{(k-1)n} = W(T_{n-1}^j y_{(k-2)n}, x_n, \alpha_{(k-1)n}) \]  
\[ y_{(k-2)n} = W(T_{n-2}^j y_{(k-3)n}, x_n, \alpha_{(k-2)n}) \]  

\[ \vdots \]  
\[ y_{2n} = W(T_n^j y_{2n}, x_n, \alpha_{2n}) \]  
\[ y_{1n} = W(T_1^j y_{1n}, x_n, \alpha_{1n}) \],

where \( y_{kn} = x_n \) for all \( n \in \mathbb{N} \cup \{0\} \).

Recently, Khan et al. [6] introduced an implicit iteration process for two finite families of nonexpansive mappings as follows:

**Definition 1.4** [6]. Let \((E, \| \cdot \|)\) be Banach space and \(S_j, T_j : E \to E\) \((j \in J)\) be two finite families of nonexpansive mappings. For any given \(x_0 \in E\), define an iteration process \(\{x_n\}\) as

\[ x_n = \alpha_n x_{n-1} + \beta_n S_n x_n + \gamma_n T_n x_n, \quad n \in \mathbb{N}, \tag{1.4} \]

where \(T_n = T_{n(modN)}, S_n = S_{n(modN)}\) and \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are three sequences in \((0, 1)\) such that \(\alpha_n + \beta_n + \gamma_n = 1\) for all \(n \in \mathbb{N}\).

Recall that \(\{T_j, j \in J\}\) is called a finite family of asymptotically quasi-nonexpansive mappings if there exist \(p_j \in F(T_j)\) and sequences \(\{u_{jn}\} \subset [0, \infty)\) with \(\lim_{n \to \infty} u_{jn} = 0\) for each \(j \in J\) such that

\[ d(T_n^j x, p_j) \leq (1 + u_{jn})d(x, p_j) \]

for each \(x \in X\).

Now, we transform implicit iteration process (1.4) to the case of two families of asymptotically quasi-nonexpansive mappings in convex metric spaces as follows:

**Definition 1.5.** Let \((X, d, W)\) be a convex metric space with convex structure \(W\) and \(T_j, S_j : X \to X\) be two finite families of asymptotically quasi-nonexpansive mappings. For any given \(x_0 \in X\), we define iteration process \(\{x_n\}\) as follows.

\[ x_1 = W(x_0, S_1 x_1, T_1 x_1; \alpha_1, \beta_1, \gamma_1) \]
\[ x_2 = W(x_1, S_2 x_2, T_2 x_2; \alpha_2, \beta_2, \gamma_2) \]

\[ \vdots \]
\[ x_N = W(x_{N-1}, S_N x_N, T_N x_N; \alpha_N, \beta_N, \gamma_N) \]
\[ x_{N+1} = W(x_N, S^2_N x_{N+1}, T^2_N x_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}) \]

\[ \vdots \]
\[ x_{2N} = W(x_{2N-1}, S^2_{2N} x_{2N}, T^2_{2N} x_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}) \]
\[ x_{2N+1} = W(x_{2N}, S^2_{2N} x_{2N+1}, T^2_{2N} x_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}) \]

\[ \vdots \]

This iteration process can be rewritten in the following compact form:

\[ x_n = W(x_{n-1}, S^j_n x_n, T^j_n x_n; \alpha_n, \beta_n, \gamma_n), \quad n \in \mathbb{N}, \tag{1.5} \]

where \(n = (k - 1)N + j, j \in J\) and \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) are three sequences in \((0, 1)\) such that \(\alpha_n + \beta_n + \gamma_n = 1\) for all \(n \in \mathbb{N}\).

Note the difference between the iteration processes (1.3) and (1.5). The process (1.3) deals with one family and uses \(n\)-steps whereas (1.5) deals with two families and uses only one step. Hence our process is computationally simpler than that used by [4] and is able to deal with two families at the same time. The method of proof of our main result resembles with that of Kim et al. [7] but different from Khan and Ahmed [4].

The purpose of this paper is to study the convergence of a one-step implicit iteration process (1.5) for two finite families of asymptotically quasi-nonexpansive mappings in convex metric spaces. The main result of this paper is an extension and improvement of corresponding results in [4, 6, 12, 15].

The following proposition was proved by Khan and Ahmed [4] for one family of asymptotically quasi-nonexpansive mappings.
Proposition 1.1. Let $X$ be a convex metric space and \{${T_j: j \in J}$\} be a finite family of asymptotically quasi-nonexpansive mappings with $F := \bigcap_{j=1}^N F(T_j) \neq \emptyset$. Then, there exist a point $p \in F$ and a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n \to \infty} u_n = 0$ such that

$$d\left(T_j^j x, p\right) \leq (1 + u_n)d(x, p),$$

for all $x \in X$ and for each $j \in J$.

We extend the above proposition to the case of two finite families of asymptotically quasi-nonexpansive mappings as follows.

Proposition 1.2. Let $X$ be a convex metric space and \{${T_j: j \in J}$\}, \{${S_j: j \in J}$\} be two finite families of asymptotically quasi-nonexpansive mappings with $F := \bigcap_{j=1}^N (F(T_j) \cap F(S_j)) \neq \emptyset$. Then, there exist a point $p \in F$ and a sequence $\{w_n\} \subset (0, \infty)$ with $\lim_{n \to \infty} w_n = 0$ such that

$$d\left(T_j^j x, p\right) \leq (1 + w_n)d(x, p), \quad \text{and} \quad d\left(S_j^j x, p\right) \leq (1 + w_n)d(x, p)$$

for all $x \in X$ and for each $j \in J$.

Proof. Since $T_j, S_j : X \to X, j \in J$ are asymptotically quasi-nonexpansive mappings, therefore by Proposition 1.1, there exists a point $p \in F$ and two sequences $\{u_n\}, \{v_n\} \subset [0, \infty)$ with $\lim_{n \to \infty} u_n = \lim_{n \to \infty} v_n = 0$ such that $d\left(T_j^j x, p\right) \leq (1 + u_n)d(x, p)$ and $d\left(S_j^j x, p\right) \leq (1 + v_n)d(x, p)$ for all $x \in X$ and for each $j \in J$. Put $w_n = \sup \{u_n, v_n\}$ so that $d\left(T_j^j x, p\right) \leq (1 + w_n)d(x, p)$ and $d\left(S_j^j x, p\right) \leq (1 + w_n)d(x, p)$ for all $x \in X$ and for each $j \in J$. $\square$

In the sequel, we also need the following technical result.

Lemma 1.1 [11]. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying

$$\sum_{n=0}^{\infty} b_n < \infty, \quad \sum_{n=0}^{\infty} c_n < \infty, \quad a_{n+1} \leq (1 + b_n)a_n + c_n, \quad n \geq 0.$$

Then

(i) $\lim_{n \to \infty} a_n$ exists,

(ii) if $\lim \inf_{n \to \infty} a_n = 0$ then $\lim_{n \to \infty} a_n = 0$.

Remark 1.3. It is easy to verify that (ii) in Lemma 1.1 holds under the hypothesis $\lim \sup_{n \to \infty} a_n = 0$ as well. Therefore, the condition (ii) in Lemma 1.1 can be reformulated as follows:

(ii) If either $\lim \inf_{n \to \infty} a_n = 0$ or $\lim \sup_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} a_n = 0$.

2. Main results

We start by proving the following important result.

Theorem 2.1. Let $(X, d, W)$ be a convex metric space with convex structure $W$ and $T_j, S_j : X \to X$ be two finite families of asymptotically quasi-nonexpansive mappings. Suppose that $F \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset (s, 1 - s)$ for some $s \in (0, \frac{1}{2})$, $\sum_{k=1}^{\infty} w_k < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Suppose that $\{\alpha_n\}$ is as in (1.5). If $\lim_{n \to \infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{d(x, p) : p \in F\}$, then $\{x_n\}$ is a Cauchy sequence.

Proof.

We divide the proof into two parts. First we claim that there exists a constant $A > 0$ such that $d(x_{n+m}, p) \leq Ad(x_n, p)$ for all $n, m \in \mathbb{N}$ and for every $p \in F$ and then with the help of this, we prove that $\{x_n\}$ is a Cauchy sequence in $X$.

Let $p \in F$. From (1.5) and Proposition 1.2, it follows that

$$d(x_n, p) = d\left(W(x_{n-1}, S_j^j x_n, T_j^j x_n; \alpha_n, \beta_n, \gamma_n), p\right) \leq \alpha_n d(x_{n-1}, p) + \beta_n d(S_j^j x_n, p) + \gamma_n d(T_j^j x_n, p)$$

$$\leq \alpha_n d(x_{n-1}, p) + \beta_n (1 + w_n)d(x_n, p) + \gamma_n (1 + w_n)d(x_n, p) = \alpha_n d(x_{n-1}, p) + (\beta_n + \gamma_n + 2w_n)d(x_n, p)$$

(2.1)

for all $p \in F$. Since $\lim_{n \to \infty} \gamma_n = 0$, there exists a natural number $n_1$ such that for $n > n_1$, $\gamma_n \leq \frac{1}{2}$. Therefore

$$1 - \beta_n - \gamma_n \geq 1 - (1 - s) - \frac{s}{2} = \frac{s}{2}.$$
for \( n > n_1 \). Thus, from (2.1), we have
\[
(1 - \beta_n - \gamma_n) d(x_n, p) \leq \omega d(x_{n-1}, p) + 2w_n d(x_n, p)
\]
so that
\[
d(x_n, p) \leq \frac{\omega_n}{1 - \beta_n - \gamma_n} d(x_{n-1}, p) + \frac{2w_n}{1 - \beta_n - \gamma_n} d(x_n, p) \leq d(x_{n-1}, p) + \frac{4}{s} w_k d(x_n, p).
\]
(2.2)

Since \( \lim_{k \to \infty} w_k = 0 \), there exists natural number \( n_2 \) such that \( k \geq n_2 \) and
\[
w_k \leq \frac{s}{8}.
\]
(2.3)

From (2.2), we have
\[
\left(1 - \frac{4}{s} w_k\right) d(x_n, p) \leq d(x_{n-1}, p).
\]
That is,
\[
d(x_n, p) \leq \frac{s}{s - 4w_k} d(x_{n-1}, p).
\]
(2.4)

Let
\[
1 + \Delta_k = \frac{s}{s - 4w_k} = 1 + \frac{4w_k}{s - 4w_k}.
\]
But from (2.3), \( 4w_k \leq s/2 \), \( s - 4w_k \geq s - s/2 = s/2 \) so that \( \frac{1}{s - 4w_k} \leq \frac{2}{s} \) and so \( \Delta_k = \frac{4w_k}{s - 4w_k} \leq \frac{s}{s/2} = 2w_k \).

Thus
\[
\sum_{k=1}^{\infty} \Delta_k \leq \sum_{k=1}^{\infty} \frac{8}{s} w_k < \infty.
\]

Now by (2.4), we have
\[
d(x_n, p) \leq (1 + \Delta_k) d(x_{n-1}, p).
\]
(2.5)

Note that, when \( a > 0 \), \( 1 + a \leq e^a \). Thus from (2.5), we have
\[
d(x_{n+m}, p) \leq (1 + \Delta_k) d(x_{n+m-1}, p)
\leq (1 + \Delta_k)(1 + \Delta_k) d(x_{n+m-2}, p)]
\leq (1 + \Delta_k)^2 [(1 + \Delta_k) d(x_{n+m-3}, p)]
\vdots
\leq \exp \left(\sum_{k=1}^{\infty} \Delta_k\right) d(x_n, p)
\leq A d(x_n, p)
\]
for all \( p \in F, n, m \in \mathbb{N} \) and \( A = \exp \left\{ \sum_{k=1}^{\infty} \Delta_k \right\} < \infty \). That is,
\[
d(x_{n+m}, p) \leq A d(x_n, p).
\]
(2.7)

Now we use this to prove that \( \{x_n\} \) is a Cauchy sequence. From \( \lim_{n \to \infty} d(x_n, F) = 0 \), for each \( \varepsilon > 0 \) there exists \( n_1 \in \mathbb{N} \) such that
\[
d(x_n, F) < \frac{\varepsilon}{A + 1} \forall n \geq n_1.
\]
Thus, there exists \( q \in F \) such that
\[
d(x_n, q) < \frac{\varepsilon}{A + 1} \forall n \geq n_1.
\]
(2.8)

Using (2.7) and (2.8), we obtain
\[
d(x_{n+m}, x_n) \leq d(x_{n+m}, q) + d(x_n, q) \leq A d(x_{n}, q) + d(x_n, q) = (A + 1) d(x_n, q) < (A + 1) \left( \frac{\varepsilon}{A + 1} \right) = \varepsilon
\]
for all \( n, m \geq n_1 \). Therefore \( \{x_n\} \) is a Cauchy sequence. \( \square \)
Theorem 2.2. Let \((X,d,W)\) be a convex metric space with a convex structure \(W\) and \(T_j, S_j : X \rightarrow X\) be two finite families of asymptotically quasi-nonexpansive mappings. Suppose that \(F \neq \emptyset\) and that \(x_0 \in X\), \((\beta_n) \subset (s, 1 - s)\) for some \(s \in (0, \frac{1}{r})\), \(\sum_{k=1}^{\infty} w_k < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty\). Suppose that \(\{x_n\}\) is as in (1.5). Then

(i) \(\lim \inf_{n \to \infty} d(x_n, F) = \lim \sup_{n \to \infty} d(x_n, F) = 0\) if \(\{x_n\}\) converges to a unique point in \(F\).

(ii) \(\{x_n\}\) converges to a unique point in \(F\) if \(X\) is complete and either \(\lim \inf_{n \to \infty} d(x_n, F) = 0\) or \(\lim \sup_{n \to \infty} d(x_n, F) = 0\).

Proof

(i) Let \(p \in F\). Since \(\{x_n\}\) converges to \(p\), for a given \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[d(x_n, p) < \varepsilon \forall n \geq n_0.\]

Taking infimum over \(p \in F\), we have

\[d(x_n, F) < \varepsilon \forall n \geq n_0.\]

This means \(\lim_{n \to \infty} d(x_n, F) = 0\) so that \(\lim \inf_{n \to \infty} d(x_n, F) = \lim \sup_{n \to \infty} d(x_n, F) = 0\).

(ii) Suppose that \(X\) is complete and \(\lim \inf_{n \to \infty} d(x_n, F) = 0\) or \(\lim \sup_{n \to \infty} d(x_n, F) = 0\). Then, we have from (ii) in Lemma 1.1 and Remark 1.3 that \(\lim_{n \to \infty} d(x_n, F) = 0\). From the completeness of \(X\) and Theorem 2.1, we get that \(\lim_{n \to \infty} x_n\) exists and equals \(q \in X\), say. Moreover, since the set \(F\) of fixed points of asymptotically quasi-nonexpansive mappings is closed, \(q \in F\) from \(\lim_{n \to \infty} d(x_n, F) = 0\). That is, \(q\) is a common fixed point of \(\{T_j\} \ (j \in J)\) and \(\{S_j\} \ (j \in J)\). Hence \(\{x_n\}\) converges to a unique point in \(F\). \(\square\)

3. Applications

A couple of applications of Theorem 2.2 are grabbed in this section.

Theorem 3.1. Let \((X,d,W)\) be a complete convex metric space with a convex structure \(W\) and \(T_j, S_j : X \rightarrow X\) be two finite families of asymptotically quasi-nonexpansive mappings. Suppose that \(F \neq \emptyset\) and that \(x_0 \in X\), \((\beta_n) \subset (s, 1 - s)\) for some \(s \in (0, \frac{1}{r})\), \(\sum_{k=1}^{\infty} w_k < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty\). Suppose that \(\{x_n\}\) is as in (1.5). Assume that the following two conditions hold.

(i) \(\lim d(x_n, x_{n+1}) = 0\). \hfill (3.1)

(ii) the sequence \(\{y_n\}\) in \(X\) satisfying \(\lim \inf_{n \to \infty} d(y_n, y_{n+1}) = 0\) implies

\[\lim \inf_{n \to \infty} d(y_n, F) = 0 \quad \text{or} \quad \lim \sup_{n \to \infty} d(y_n, F) = 0.\] \hfill (3.2)

Then \(\{x_n\}\) converges to a unique point in \(F\).

Proof. From (3.1) and (3.2), we have that

\[\lim \inf_{n \to \infty} d(x_n, F) = 0 \quad \text{or} \quad \lim \sup_{n \to \infty} d(x_n, F) = 0.\]

Therefore, we obtain from (ii) in Theorem 2.2 that the sequence \(\{x_n\}\) converges to a unique point in \(F\). \(\square\)

Part (i) of the following Theorem is new in the present setting while part (ii) generalizes the corresponding result of Khan and Ahmed [4].

Theorem 3.2. Let \((X,d,W)\) be a complete convex metric space with a convex structure \(W\) and \(T_j, S_j : X \rightarrow X\) be two finite families of asymptotically quasi-nonexpansive mappings satisfying \(\lim_{n \to \infty} d(x_n, T_{\alpha n}) = \lim_{n \to \infty} d(x_n, S_{\alpha n}) = 0\). Suppose that \(F \neq \emptyset\) and that \(x_0 \in X\), \((\beta_n) \subset (s, 1 - s)\) for some \(s \in (0, \frac{1}{r})\), \(\sum_{k=1}^{\infty} w_k < \infty\) and \(\sum_{n=1}^{\infty} \gamma_n < \infty\). If either of the following is true, then the sequence \(\{x_n\}\) defined by (1.5) converges to a unique point in \(F\).

(i) If there exists a nondecreasing function \(g : [0, \infty) \rightarrow [0, \infty)\) with \(g(0) = 0\), \(g(r) > 0\) for all \(r \in (0, \infty)\) such that either \(d(x_n, T_{\alpha n}) \geq g(d(x_n, F))\) or \(d(x_n, S_{\alpha n}) \geq g(d(x_n, F))\) for all \(n \in \mathbb{N}\). (See Condition A of Khan and Fukhar-ud-din [2]).

(ii) There exists a function \(f : [0, \infty) \rightarrow [0, \infty)\) which is right continuous at 0, \(f(0) = 0\) and \(f(d(x_n, T_{\alpha n})) \geq d(x_n, F)\) or \(f(d(x_n, S_{\alpha n})) \geq d(x_n, F)\) for all \(n \in \mathbb{N}\).

Proof. First suppose that (i) holds. Then

\[\lim_{n \to \infty} g(d(x_n, F)) \leq \lim_{n \to \infty} d(x_n, T_{\alpha n}) = 0\]
or
\[ \lim_{n \to \infty} g(d(x_n, F)) \leq \lim_{n \to \infty} d(x_n, Sx_n) = 0. \]

In both the cases,
\[ \lim_{n \to \infty} g(d(x_n, F)) = 0. \]

Since \( g : [0, \infty) \to [0, \infty) \) is a nondecreasing function satisfying \( g(0) = 0, g(r) > 0 \) for all \( r \in (0, \infty) \), therefore we have
\[ \lim_{n \to \infty} d(x_n, F) = 0. \]

Now all the conditions of Theorem 2.2 are satisfied, therefore by its conclusion \( \{x_n\} \) converges to a point of \( F \).

Next, assume (ii). Then either
\[ \lim_{n \to \infty} d(x_n, F) \leq \lim_{n \to \infty} f(d(x_n, T_{j_0}x_n)) = f(\lim_{n \to \infty} d(x_n, T_{j_0}x_n)) = f(0) = 0 \]
or
\[ \lim_{n \to \infty} d(x_n, F) \leq \lim_{n \to \infty} f(d(x_n, S_{j_0}x_n)) = f(\lim_{n \to \infty} d(x_n, S_{j_0}x_n)) = f(0) = 0. \]

Again in both the cases, \( \lim_{n \to \infty} d(x_n, F) = 0. \) Thus, \( \liminf_{n \to \infty} d(x_n, F) = 0 \) or \( \limsup_{n \to \infty} d(x_n, F) = 0. \) By Theorem 2.2, \( \{x_n\} \) converges to a point in \( F \). \( \square \)

**Remark 3.3.** The results presented in this paper are extensions and improvements of the corresponding results of Khan and Ahmed [4], Khan et al. [6], Sun [12], Wittmann [14] and Xu and Ori [15] to two finite families of asymptotically quasi-nonexpansive mappings in convex metric spaces.

**Remark 3.4.** We can prove all the results obtained so far in the context of a \( q \)-starshaped metric space with suitable changes. We leave the details to the reader.

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**References**


