The Structurally Dynamic Cellular Network and Quantum Graphity Approaches to Quantum Gravity - A Review and Comparison

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(Dated: January 5, 2015)

Starting from the working hypothesis that both physics and the corresponding mathematics have to be described by means of discrete concepts on the Planck-scale, one of the many problems one has to face in this enterprise is to find the discrete protoforms of the building blocks of our ordinary continuum physics and mathematics. We regard these continuum concepts and continuum spacetime in particular as being emergent, coarse-grained and derived relative to an underlying erratic and disordered microscopic substratum which is expected to play by quite different rules. A central role in our analysis is played by a geometric renormalization group which creates (among other things) a kind of sparse translocal network of correlations between the points in classical continuous space-time and underlies, in our view, such mysterious phenomena as holography and the black hole entropy-area law. The same point of view holds for quantum theory which we also regard as a low-energy, coarse-grained continuum theory, being emergent from something more fundamental. In this paper we review our approach and compare it to the quantum graphity framework.

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I. INTRODUCTION

Among the various approaches to quantum gravity (QG) or quantum space-time physics, the most prominent being string theory and loop quantum gravity, there are some which assume that physics and in particular space-time (S-T) itself are basically discrete on the primordial (Planck) scale. One can as well call this primordial state pregeometry, thus indicating that a S-T in the ordinary sense does not yet exist. In some of our papers we sometimes dubbed this state quantum space or QX. This working philosophy typically considers gravity, continuous space-time and/or quantum theory as being emergent or derived structures, living over a microscopic primordial substratum which presumably plays by quite different rules as compared to the above coarse grained concepts existing on a more macroscopic scale.

In the classification of Carlip these approaches are called type II ([1]). While some are openly (e.g. dynamical triangulation) or implicitly (spin-networks) inspired by more geometrical ideas, there are others which rely on quite different sources of inspiration. We would like to take the opportunity to develop this other strand of ideas and concepts and exhibit their roots because whereas some of the aforementioned frameworks are looking superficially quite similar at first glance, under closer inspection it becomes clear that they are drawing their ideas from quite different sources.

To give an example of the latter class we mention the so-called quantum graphity approach (see for example [2],[3],[4]) which we want to compare to our framework being introduced in the following and which may be called Structurally Dynamic Cellular Networks (SDCN) or automata (SDCA), the latter being a slightly simpler version.

To put it in a nutshell, there exist in our view more or less three subclasses of this type II classification:

1. Model theories which concentrate more on the geometric side of the presumed primordial substratum with the quantum matter degrees of freedom (DoF) playing no central role (e.g. dynamical triangulation, quantum graphity).
2. The cellular automaton interpretation of quantum mechanics (most prominently figuring the work of 't Hooft). From the many papers he published we mention only quite a few; the recent review [3] and some of the earlier ones, e.g. [6] and [7].

The characteristic of 't Hooft’s approach is a relatively immediate (formal) quantization of classical states (called by him primordial quantization) and Hilbert space construction while the static geometric background is the typical regular lattice structure of the CA-framework. That means, each of the classical states is promoted to a (basis) vector in some Hilbert space, while the important superposition principle is more or less decreed to hold by definition.

3. The SDCN approach undertakes to treat the geometric and quantum matter DoF as coevolving structures. The central aim is to create a model of the quantum vacuum as the carrier of all the emergent higher and derived structures (as was, for example, the working philosophy of Wheeler (cf. [8]).

The first published papers developing this working philosophy were [9], [10] and [11]. The important ingredient was that both the matter states (located at the vertices of the network) and the geometric DoF (sitting at the edges or links) coevolve dynamically with links being allowed to be created and/or deleted in this process. This implies that a geometric unfolding of the whole network structure becomes possible together with geometric phase transitions which may simulate the situation, for example, prevailing in the big bang and/or inflation era.

We want to mention another important point of the SDCN-approach which becomes apparent when we undertake to reconstruct continuum S-T via a coarse graining or renormalization process (cf. [12], [13]) from some underlying microscopic network substratum. This process leads almost automatically to the expected near-order structure of ordinary continuous space-time physics (locality of the physical processes) on one hand, and to a so-called far-order structure on the other hand. The latter means that there always do exist a sparse network of translocal links between regions of S-T which are far apart with respect to the distance metric of the ordinary continuous space.

We expect that these non-local aspects underlie for example the quantum entanglement, the holographic principle and the black hole entropy-area law (see e.g. [14], [15], [16]).

Before going into the technical details of the network approach we think, it is a good idea to make some brief remarks on the previous history of the discrete network approach to quantum gravity and/or quantum theory as this point of view is, perhaps unjustifiably, a little bit brushed under the carpet as a consequence of the claim that for example string theory is the only game in town whereas the latter theory is fully continuous and assumes quantum mechanics to hold unaltered all the way down to the Planck scale.

There do in fact exist various discrete approaches to fundamental physics in the recent and not so recent past. For example, already Born and Jordan experimented with discrete difference equations before the advent of the correct quantum theory in the twenties. For the sake of brevity we want to begin our sketchy historical remarks with the conceptualization of CA in this context of fundamental physics.

A fairly complete discussion of the history can for example be found in the book by Ilachinski ([17]) together with a more exhaustive bibliography, see in particular chapter 12 where the question is raised: “Is Nature, underneath it all, a CA?” Such ideas have in fact been around for quite some time (cf. [18], [19], [20], [21]). There was even a whole conference on Physics of Computation, held at MIT in 1982, devoted to such aspects (collected in Vol.21 of the Int.J.Theor.Phys.)

What makes CA so fascinating is the observation that various model classes are capable of universal computing (like e.g. the famous Game of Life being invented by Conway; see e.g. [22]). The creation of life on a computer was one of the buzz-words of the emerging field of Artificial Life (to mention just a few sources, see e.g. [23], [24], [25]. After all, CA or SDCN have the potential to realize an important meta idea of modern science, i.e., to generate complex behavior, starting from a, at first glance, relatively simple dynamical (discrete) system which is governed by algorithmic rules. If it is really the case that at the bottom everything is information (in an admittedly vague sense), this approach has its special merits in the competition with the (perhaps better known) rivals mentioned above.

However, to model the physical universe on some primordial scale like the Planck scale, we have to satisfy a couple of severe constraints which come from the quantum world and the necessity to unite the quantum world with general relativity. After all, we have to implement both quantum entanglement and the emergence of space-time together with its seething sea of vacuum fluctuations. We think this is more than can be expected from a CA living on a static rigid lattice.

Remark 1.1 It should be emphasized that it is not our aim to merely somehow reproduce certain aspects of, say, quantum theory on a computer. What is actually needed is a structural equivalence of the respective network model and, say, quantum space-time with all of its intricate properties (cf. our introductory remarks in [22] or [24]).

Let us make a final remark concerning the various mathematical fields which are involved in our enterprise as this variety of connections is frequently also invoked as a special merit on the side of string theory. To mention just a few fields: advanced graph theory (e.g. clique graphs, random graphs), Connes’ noncommutative analysis and
geometry, operator theory on discrete spaces (graph Laplacians, Dirac operators, their eigenvalues), generalizations of dimensional concepts, leading even to connections with geometric group theory (via the so-called Cayley graphs), Gromov-Haussdorff Limit of irregular spaces, Small World Networks etc.

We begin our investigation in section II by introducing our underlying network model and the necessary concepts and notions. We then proceed in section III with the derivation of various concepts and tools of discrete (functional) analysis and operator theory. In a next step in section IV we introduce the concept of generalized dimension on such discrete and irregular spaces. Then follows the ambitious enterprise to define a geometric renormalization process with classical space-time emerging as some coarse grained limit in section VI.

We will then presents some arguments in section VII on how these steps will lead to a better understanding of various crucial concepts of modern physics (in particular concerning quantum gravity). Examples being e.g. the mysterious phenomenon of holography, quantum entanglement, the black hole entropy-area law etc. In the last two sections, VIII and IX we first make a brief review of another bottom-up model of emergent spacetime called quantum graphity and then try to comment on differences and similarities of our model (SDCN) and the quantum graphity, and make some concluding remarks about our model.

II. THE MICROSCOPIC SDCN-SUBSTRATUM

Our networks are defined on general graphs, $G$, with $V(G)$ the set of its vertices (sites or nodes) and $E(G)$ its set of edges (links or bonds).

**Definition II.1** Here are some graph-theoretical notions and concepts (for more details see e.g. [22]).

1. We write the simple, or directed labeled graph as $G := (V, E)$ where $V$ is the countable set of vertices $\{n_i\}$ and $E$ the set of edges. The graph is called simple if there do not exist elementary loops and multiple edges. An elementary loop is an edge which starts and ends on the same vertex (or connect the same vertex to itself). Multiple edges happen when there is more than one edge directly connecting two vertices. (We could of course also discuss more general graphs). Furthermore, for simplicity, we assume the graph to be connected, i.e. two arbitrary vertices are connected by a sequence of consecutive edges called an edge sequence or walk. A minimal edge sequence, that is one with each intermediate vertex occurring only once, is called a path.

2. For convenience we assume the graph to be locally finite, which means that the vertex degree, $v_i$, or the number of edges incident on each vertex $n_i$ is finite. Sometimes it is useful to make the stronger assumption that $v_i$ is globally bounded away from $\infty$.

3. One can give the edges both an orientation and a direction (these two slightly different geometric concepts are frequently intermixed in the literature). In an undirected graph the edges $e_{ij}$ correspond to unordered pairs of vertices $\{n_i, n_j\}$ while in a directed graph, the edges have a direction represented by an ordered pair of vertices $d_{ij} := (n_i, n_j)$, that is, the edge points from $n_i$ to $n_j$. In this work, we adopt the convention that if two vertices $n_i, n_k$ are connected by an edge in an undirected graph, we interpret is as follows: There exists a directed edge, $d_{ik}$, pointing from $n_i$ to $n_k$ and a directed edge, $d_{ki}$, pointing in the opposite direction. In an algebraic sense, which will become clear below, we call their superposition

$$b_{ik} := d_{ik} - d_{ki} = -b_{ki} \quad (2.1)$$

the corresponding oriented edge (for obvious reasons; the directions are fixed while the orientation can change sign). In a sense the above reflects the equivalence of an undirected graph with a directed multi-graph having two directed edges pointing in opposite directions for each undirected edge.

We now associate states $s_i$ and $J_{ik}$ with the vertices and edges $n_i$ and $e_{ik}$. The local vertex states $s_i$ can assume values in a certain discrete set. In the examples we have studied, we follow the philosophy that the network should be allowed to find its typical range of states via the imposed dynamics. That is, we allow the $s_i$ to vary in principle over the set $q \cdot \mathbb{Z}$, with $q$ a certain discrete quantum of information, energy or some other relevant physical quantity. The edge states can assume the values $J_{ik} \in \{-1, 0, +1\}$. Viewed geometrically we associate the states $J_{ik} = +1, -1, 0$ with directed edges pointing from vertex $n_i$ to $n_k$ (in $+1$ case) or in the opposite direction (in $-1$ case) or, in the case $J_{ik} = 0$, with an empty edge. Our network will be updated after each discrete step of the evolution parameter $t \cdot \tau$, $t \in \mathbb{Z}$ (we may call it somewhat sloppily clock time as is done for example in computer science) according to the dynamics introduced below.

**Remark II.2** Note that this is exactly the same procedure as in CA.
We then have as underlying substratum a clock time dependent directed graph, $G_t$. Our physical idea is that at each clock time step, an elementary quantum $q$ is transported along each existing directed edge in the indicated direction between the two vertices which are connected by this edge.

To implement our general working philosophy of mutual interaction of overall vertex states and network geometry, we now describe some particular network laws, which we investigated in greater detail in [26] together with a lot of numerical simulation and analysis. We mainly consider two different classes of evolution laws for vertex and edge states (for reasons of simplicity we choose units so that $q$ is set equal to one):

- Network type I

\[
s_i(t + 1) = s_i(t) + \sum_k J_{ki}(t) \tag{2.2}
\]

\[
J_{ik}(t + 1) = \begin{cases} 
\text{sign}(\Delta s_{ik}), & \text{If } |\Delta s_{ik}| \geq \lambda_2 \lor (|\Delta s_{ik}| \geq \lambda_1 \land J_{ik}(t) \neq 0) \\
0, & \text{otherwise} \end{cases} \tag{2.3}
\]

- Network type II

\[
s_i(t + 1) = s_i(t) + \sum_k J_{ki}(t) \tag{2.4}
\]

\[
J_{ik}(t + 1) = \begin{cases} 
\text{sign}(\Delta s_{ik}), & \text{If } 0 < |\Delta s_{ik}| < \lambda_1 \lor (0 < |\Delta s_{ik}| < \lambda_2 \land J_{ik}(t) \neq 0) \\
J_{ik}(t), & \text{If } \Delta s_{ik} = 0 \\
0, & \text{otherwise} \end{cases} \tag{2.5}
\]

where $\Delta s_{ik} = s_i(t) - s_k(t)$ and $\lambda_2 \geq \lambda_1 \geq 0$ ($\lor, \land$ meaning or, and, respectively). We see that in type I, vertices are connected that have very different internal states, leading to large local fluctuations, while for the type II, vertices with similar internal states are connected.

**Remark II.3** The role of the $\lambda$ parameters is the following. They define kind of a hysteresis interval regulating the switching off and on of edges. We hope that they can be tuned such that the network can perform phase transitions. We studied the $\lambda$ dependence of various network properties in the computer simulations being discussed in [26].

We proceed by making some remarks in order to put our approach into the appropriate context.

**Remark II.4**

1. It is important that, generically, laws as introduced above, do not lead to a reversible time evolution, i.e., there will typically be attractors or state-cycles in the total phase space (the overall configuration space of the vertex and edge states). On the other hand, there exist strategies (in the context of cellular automata!) to design particular reversible network laws (cf. e.g. [30]) which are however, typically of second order in clock time. Usually the existence of attractors is considered to be important for pattern formation. On the other hand, it may suffice that the fraction of the phase space, occupied by the system, shrinks in the course of evolution. That is, one has a flow into smaller subvolumes of phase space.

2. In the above class of laws, a direct edge-edge interaction is not yet implemented. Note that it would imply a direct nonlinear action of geometry on itself, similar to the interaction of pure gauge fields in gauge theory. We are prepared to incorporate such a (possibly important) contribution in a next step if it turns out to be necessary. In any case there are not so many ways to do this in a sensible way. Stated differently, the class of possible, physically sensible interactions is perhaps not so large.

3. We would like to emphasize that the (nondynamical) clock-time, $t$, should not be confused with the notion of physical time, i.e., the time operationally employed on much coarser scales. The latter is rather supposed to be a collective variable and is expected (or hoped!) to emerge via a cooperative effect. Clock-time may be an ideal element, i.e., a notion which comes from outside, so to speak, but – at least for the time being – we have to introduce some mechanism which allows us to label consecutive states or describe change or evolution.

We make the following observation because it is relevant if one follows the general spirit of modern high energy physics.
Observation II.5 (Gauge Invariance) The above dynamical law depends nowhere on the absolute values of the vertex charges \( s_i(t) \) but only on their relative differences. By the same token, charge is nowhere created or destroyed. We have

\[
\Delta \left( \sum_{i \in I} s_i \right) = 0
\]

where, for simplicity, we represent the set of sites by their set of indices \( I \), and \( \Delta \) denotes the difference between consecutive clock-time steps. Put differently, we have conservation of the global vertex charge. To avoid artificial ambiguities we can, e.g., choose a fixed reference level and take the constraint

\[
\sum_{i \in I} s_i = 0
\]

as our initial condition.

Summarizing the main steps of our working philosophy:

Remark II.6 Irrespective of the technical details of the dynamical evolution law under discussion, the following, in our view crucial, principles should be emulated in order to match fundamental requirements concerning the capability of emergent and complex behavior.

1. As is the case with, say, gauge theory or general relativity, our evolution law on the surmised primordial level should implement the mutual interaction of two fundamental substructures. Put sloppily: “geometry” acting on “matter” and vice versa, where in our context “geometry” is assumed to correspond in a loose sense with the local and/or global edge states and “matter” with the structure of the vertex states.

2. By the same token, the alluded self-referential dynamical circuitry of positive feedback structure is expected to favor a kind of undulating behavior or self-excitation in contrast to a return to some uninteresting ‘equilibrium state’ as is frequently the case in systems consisting of a single component which directly feeds back on itself. This propensity for the autonomous generation of undulation patterns is in our view an essential prerequisite for some form of “protoquantum behavior” which we hope to recover on some coarse grained and less primordial level of the network dynamics.

3. In the same sense we expect the overall pattern of switched-on and -off edges to generate a kind of “protogravity”.

We want to comment on a particular intriguing result from our numerical simulation performed in [26], that is the phenomenon of limit cycles. Because of the finite phase space of the CA (technically it is infinite, but the vertex states only fill a finite interval of \( \mathbb{Z} \) due to the nature of the network laws), network states will eventually repeat, which leads to a limit cycle because of the memory-less dynamics. We tested for the appearance of such limit cycles for different network size \( n \) (number of vertices) and to our surprise, networks of type I had with very few exceptions extremely short limit cycles of period 6. The exceptions we were able to find, had periods of a multiple of 6, the longest found (in a network with \( n = 810 \) vertices) was 36. The prevalence of such short limit cycles is still an open question and beyond this work. We note in this context that already S. Kauffmann observed such short cycles in his investigation of switching nets ([23], [29]) and found it very amazing.

Remark II.7 Our computer simulations employed the following initial conditions. We started with a maximal complete graph, i.e. each pair of vertices is connected by an undirected edge. The vertex states were chosen from a uniform distribution scattered over some interval of integers (we tried also other distributions but did not find any significantly different results). The initial values of the edge states were chosen from the set \( \{+1, -1\} \) with equal weight. By this choice we wanted to simulate the initial condition prevalent in the big bang era.

These phenomena of short limit cycles are remarkable in the face of the huge accessible phase spaces of typical models, and points to some hidden ordering tendencies in these model classes. What is even more startling is that this phenomenon prevails also in our network type 1 when we introduce a further element of possible disorder by allowing edges to be dynamically created and deleted. We formulate the following hypothesis.

Conjecture II.8 We conjecture that this important phenomenon has its roots in the self-referential structure (feedback mechanisms) of many of the used model systems.
It is instructive to observe the emergence of such short cycles in very small models on paper, setting for example $\lambda_1 = \lambda_2 = 0$, i.e., no switching-off of edges and taking $n = 2, 3$ or 4. Taking, e.g., $n = 2$ and starting from $s_2(0) = s_1(0) \mod 2$, the network will eventually reach a state $s_1(t_0) = s_2(t_0)$. Without loss of generality we can assume $s_1(t_0) = s_2(t_0) = 0$ and $J_{12}(t_1) = 1$. This state develops into a cycle of length 6 as illustrated in table 2a(1) of [26]. For $s_1(0) = s_2(0) + 1 \mod 2$ the state eventually becomes $s_1(t_1) = s_2(t_1) + 1$, without loss of generality $s_1(t_1) = 1$, $s_2(t_1) = 0$, $J_{12}(t_1) = 1$, resulting in the dynamics in table 2a(2) of [26]. Again, the length of the cycle is 6. Hence, 6 is a good candidate for a short cycle length, which – of course – does not explain why such a short length should appear at all.

The transients (i.e., the clock time interval between some random initial state; cf. [24] and the clock time the system arrives on an attractor) in networks of type I are also rather short and grow slowly with the network size. On the other hand, networks of type II have much longer limit cycles and transients. Because of numerical limitations we were only able to determine cycle lengths for small networks. We observed that the typical transient and cycle lengths both grow approximately exponentially (cf. table 2b of [26]).

### III. DIFFERENTIAL AND OPERATOR CALCULUS ON GRAPHS

In the following section we introduce differential and operator calculus on graphs. To some extent this topic carries the flavor of our own ideas (that is, we surmise that not everything we introduce below can be found in the standard mathematical literature. We note for example that, as a minor point, the use of matrices instead of operators is widespread in the literature). A classical text is for example [31] and a nice more recent source is [32]. Our own framework can be found in the papers [9], [33], [34] where more references are given. In a first step we introduce the vertex and edge Hilbert spaces for directed and undirected graphs (for reasons of mathematical simplicity we restrict ourselves to locally finite graphs; the more general situation can of course also be dealt with. For a directed graph we then have ingoing and outgoing edges at each vertex.

**Definition III.1** We denote the in-vertex degree at vertex $x_i$ by $v_i^{in}$, the out-vertex degree by $v_i^{out}$ and the local vertex degree by $v_i = v_i^{in} + v_i^{out}$.

For such a graph we can introduce two Hilbert spaces: a vertex Hilbert space, $\mathcal{H}_v$, and an edge Hilbert space, $\mathcal{H}_e$, with orthonormal bases being the set of vertices, $x_i$, and the set of existing directed edges, $d_{ij}$. This means that we introduce a formal scalar product on $\mathcal{H}_v$ and $\mathcal{H}_e$ respectively with

$$\langle x_i, x_j \rangle = \delta_{ij}, \quad \langle d_{ij}, d_{lm} \rangle = \delta_{il} \delta_{jm}. \quad (3.1)$$

Then the vectors in these spaces can be written as the formal sums

$$f = \sum_{i=1}^{\infty} f_i x_i, \quad g = \sum_{i,j=1}^{\infty} g_{ij} d_{ij} \text{ with } f_i, g_{ij} \in \mathbb{C} \quad (3.2)$$

with $\sum |f_i|^2 < \infty$ and $\sum |g_{ij}|^2 < \infty$.

**Remark III.2** We treat the vertices and edges as abstract basis elements (in a way similar to the group algebra of a group). One can of course consider the abstract vectors equally well as discrete functions over the vertex- or edge-set respectively, and the basis vectors as elementary indicator functions. Therefore we replace from now on the vertices $n_i$ by the corresponding indicator functions $x_i$, having the value one at the respective vertex under discussion and zero elsewhere.

If we deal with an undirected but orientable graph we find it convenient to introduce the superposition

$$b_{ij} := d_{ij} - d_{ji} = -b_{ji} \quad (3.3)$$

and relate it to an undirected but orientable edge. We now introduce two operators, interpolating between $\mathcal{H}_v$ and $\mathcal{H}_e$. We define them on the basis vectors:

$$d : \mathcal{H}_v \to \mathcal{H}_d \quad (3.4)$$

$$d(x_i) := \sum_k d_{ki} - \sum_{k'} d_{k'i} \quad (3.5)$$
with the first sum running over the ingoing edges relative to \( x_i \) and the second sum running over the outgoing ones.

In the case of a symmetric (or undirected graph) we have
\[
d(x_i) := \sum_k (d_{ki} - d_{ik}) = \sum_k b_{ki}.
\]

This operator is closely related to a sort of non-commutative discrete differential calculus on graphs as we have
\[
df = \sum_{i,k} (f_k - f_i) d_{ik}.
\]

A simple calculation shows that the adjoint, \( d^* : \mathcal{H}_d \to \mathcal{H}_v \), acts on the basis vectors of \( \mathcal{H}_d \) as follows:
\[
d^*(d_{ik}) = x_k - x_i
\]

**Remark III.3** Note that these operators are closely related to the boundary and coboundary operator in algebraic topology.

In algebraic graph theory (finite graphs) the so-called *incidence matrix*, \( B \), is introduced, having the entry 1 if vertex \( x_i \) is the positive end of a certain (ingoing) edge, and having a \(-1\) if it is the negative end (outgoing edge) (see for example [32]). This matrix corresponds to our operator \( d^* \).

Another important operator is the *adjacency matrix*, \( A \), being a map from \( \mathcal{H}_v \) to \( \mathcal{H}_v \) and having (in ordinary graph theory of (un)oriented graphs) a \(+1\) at entry \((i,j)\) if \( x_i \) and \( x_j \) are connected by an edge. This matrix is a symmetric operator, \( a_{ij} = a_{ji} \). In our more general context (which includes however the ordinary situation as a special case) of directed graphs one can introduce the *in-adjacency* matrix, \( A^\text{in} \), and the *out-adjacency* matrix, \( A^\text{out} \), with \( A = A^\text{in} + A^\text{out} \). \( A^\text{in} \) and \( A^\text{out} \) having a \(+1\) at entry \((k,i)\) if \( x_k \) and \( x_i \) are connected by a directed edge \( d_{ki} \) and correspondingly for \( A^\text{out} \).

**Remark III.4** Note that our \( A \) differs slightly from the classical \( A \). The classical \( A \) for an undirected graph equals our \( A^\text{in} \) or \( A^\text{out} \) in that case. Our operators apply to more general situations with our \( A \) being even symmetric for arbitrary directed graphs.

In our (operator)-notation they are given by
\[
A x_i = \sum_{k \sim i} \epsilon_{ki} x_k, \quad A^\text{in} x_i = \sum_{k \rightarrow i} x_k, \quad A^\text{out} x_i = \sum_{i \rightarrow k'} x_{k'}
\]

with \( \sim \) designating the unordered pair \( \{x_i, x_k\} \), \( k \rightarrow i \) the ordered pair \((k,i)\) and \( \epsilon_{k,i} \) is either 1 or 2, depending on the two possible cases of one directed edge between vertex \( x_i \) and vertex \( x_k \) or two directed edges, pointing in opposite directions.

These operators can be built up from more elementary operators (cf. [33] or [34]) \( d_I : \mathcal{H}_v \to \mathcal{H}_d, d_I^* : \mathcal{H}_d \to \mathcal{H}_v \) where \( I = 1,2 \)
\[
d_1 x_i = \sum_k d_{ki}, \quad d_2 x_i = \sum_{k'} d_{ik'} \quad (3.10)
\]
\[
d_1^* d_{ik} = x_k, \quad d_2^* d_{ik} = x_i \quad (3.11)
\]
\[
d = d_1 - d_2, \quad d^* = d_1^* - d_2^* \quad (3.12)
\]
\[
d_1^* d_1 x_i = v^\text{in}_i \cdot x_i, \quad d_2^* d_2 x_i = v^\text{out}_i \cdot x_i \quad (3.13)
\]
\[
d_1^* d_2 x_i = \sum_{i \sim k'} x_{k'}, \quad d_2^* d_1 x_i = \sum_{k \rightarrow i} x_{k'} \quad (3.14)
\]

where \( v^\text{in}_i, v^\text{out}_i \) is the in-, out-degree of vertex \( x_i \) respectively. We hence have

**Lemma III.5** The in-, out-vertex degree matrices (having the respective vertex degrees on the diagonal) read
\[
V^\text{in} = d_1^* d_1, \quad V^\text{out} = d_2^* d_2.
\]

The in-, out-adjacency matrices read
\[
A^\text{in} = d_2^* d_1, \quad A^\text{out} = d_1^* d_2
\]

and \( A = A^\text{in} + A^\text{out} \) is symmetric.
Proposition III.6 The so-called graph Laplacian is the following positive operator

\[-\Delta := d^* d = (V^{in} + V^{out}) - (A^{in} + A^{out}) = V - A.\] (3.17)

Note that for an undirected graph (i.e. both \(d_{ik}\) and \(d_{ki}\) being present) the above Laplacian is simply twice the classical Laplacian (matrix).

The reason to call this operator a Laplacian stems from the observation that it acts like a second order partial difference operator on functions of \(H_0\).

\[-\Delta f = \sum_i f_i \left( v^{in}_i x_i + v^{out}_i x_i - \sum_{k \rightarrow i} x_k - \sum_{i \rightarrow k} x_i \right)\] (3.18)

and after a simple relabeling of indices

\[-\Delta f = -\sum_i \left( \sum_{k \rightarrow i} f_k + \sum_{i \rightarrow k} f_i - v^{in}_i f_i - v^{out}_i f_i \right) x_i = -\sum_i \left( \sum_{k \rightarrow i} (f_k - f_i) + \sum_{i \rightarrow k} (f_k - f_i) \right) x_i = -\sum_i \left( \sum_{k \rightarrow i} \epsilon_{ki} (f_k - f_i) \right) x_i\] (3.19)

which reduces to the ordinary expression in the undirected case.

Forming now the direct sum \(H := H_0 \oplus H_d\), we can introduce yet another important graph operator which closely entangles geometric and functional analytic properties of graphs (and similar structures); see [33] and [34].

Definition III.7 We define the graph Dirac operator as follows

\[D : H \rightarrow H \quad \text{with} \quad D := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix}, \quad H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}.\] (3.20)

Observation III.8 We have

\[D^2 = DD = \begin{pmatrix} d^* d & 0 \\ 0 & dd^* \end{pmatrix}\] (3.21)

with \(d^* d = -\Delta\).

The action of \(dd^*\) on a basis vector \(d_{ik}\) reads

\[dd^* d_{ik} = d(x_k - x_i) = \sum_{k'} d_{k'k} - \sum_{k''} d_{kk''} - \sum_{l} d_{li} + \sum_{l'} d_{l'i}\] (3.22)

which, after some relabeling and introduction of the Kronecker delta symbol, can be written as

\[dd^* d_{ik} = \sum_{m,j} (d_{m,j} \delta_{jk} - d_{j,m} \delta_{jk} - d_{m,j} \delta_{ij} + d_{j,m} \delta_{ij})\] (3.23)

For a function \(g = \sum g_{ik} d_{ik}\) we hence get

\[dd^* g = \sum_{l,m} \left( \sum_{i} g_{il} - g_{mi} - g_{ii} + g_{ii} \right) d_{lm}.\] (3.24)

There is a pendant in the calculus of differential forms on general Riemannian manifolds where, with the help of the Hodge-star operation, we can construct a dual, \(\delta\), to the ordinary exterior derivative. The generalized Laplacian then reads

\[-\Delta = \delta d + d\delta\] (3.25)
with $\delta$ (modulo certain combinatorial prefactors) corresponding to our $d^*$ (see for example [33] or [34]).

With the help of the machinery we have introduced above further properties of the graph Laplacian and Dirac operator can be derived, some of them being directly related to geometric and/or wiring properties of the graph under discussion. Of particular relevance are spectral properties of $-\Delta$ and $D$. In most of the graph literature the graphs are assumed to be finite, hence the corresponding operators are automatically bounded and $-\Delta$ and $D$ are self adjoint. A fortiori the spectrum is discrete as the corresponding Hilbert spaces are finite dimensional. This is the reason why most of the graph literature used the matrix calculus (which is in our view a little bit clumsy). Investigation of the unbounded case are less numerous. We discussed this more general case in [35] and [34] where also more references are mentioned. A particular result is for example:

**Theorem III.9** For a globally bounded vertex degree, $-\Delta$ and $D$ are bounded self adjoint operators with their bounds being explicitly computable. (For a proof see [33] and/or [34].)

It is noteworthy that with the help of the tools we have introduced and developed above we can successfully deal with various interesting modern topics of mathematical physics in this particular context of irregular discrete spaces. To mention a few cases, in [9],[33],[34] we treated graphs as models of noncommutative geometry and supersymmetry. Among other things we introduced an example of Connes’ spectral triple, and calculated the so-called Connes distance metric on graphs. That our networks/graphs carry automatically and naturally a supersymmetric structure may perhaps be a further hint that this approach and its continuum limit has something to do with the real physical world of high-energy physics.

### IV. DIMENSIONAL CONCEPTS ON GRAPHS OR NETWORKS

There exist a variety of concepts in modern mathematics which extend the ordinary or naive notion of dimension one is accustomed to in e.g. differential topology or linear algebra. In fact, topological dimension and related concepts are notions which are perhaps even closer to the underlying intuition (cf. e.g. [37]).

Apart from the purely geometric concept, there is also an important physical role being played by something like dimension, having pronounced effects on the behavior of, say, many-body-systems near their phase transition points or in the critical region.

But even in the case of e.g. lattice systems they are usually treated as being embedded in an ambient continuous background space (typically euclidean) which supplies the concept of ordinary dimension so that the intrinsic dimension of the discrete array itself does not usually openly enter the considerations.

Anyway, it is worthwhile even in this relatively transparent situation to have a closer look on the situations where attributes of something like dimension really enter the physical stage. Properties of models of, say, statistical mechanics are typically derived from the structure of the microscopic interactions of their constituents. This then is more or less the only place where dimensional aspects enter the calculations.

Naive reasoning might suggest that it is something like the number of nearest neighbors (in e.g. lattice systems) which encodes the dimension of the underlying space and influences via that way the dynamics of the system. However, this surmise, as we will show in the following, does not reflect the crucial point which is more subtle.

This holds the more so for systems which cannot be considered as being embedded in a smooth regular background and hence do not inherit their dimension from the embedding space. A case in point is our primordial network in which Planck-scale-physics is assumed to take place. In our approach, it is in fact exactly the other way round: smooth space-time is assumed to emerge via a (geometric) phase transition or a certain cooperative behavior and after some “coarse graining” from this more fundamental structure. That is, our task is to formulate an intrinsic notion of dimension for model theories living on quite irregular spaces without making recourse to the dimension of some continuous embedding space.

In a first step we will show that graphs or networks as introduced in the preceding sections carry a natural metric structure. We have already introduced a certain neighborhood structure in a graph with the help of the minimal number of consecutive edges connecting two given vertices. In a connected graph any two vertices can be connected by a sequence of edges. Without loss of generality one can restrict oneself to paths. One can then define the length of a path (or sequence of edges) by the number $l$ of consecutive edges making up the path.

**Remark IV.1** Unless otherwise stated we deal in the following with undirected graphs, that is we regard our networks or graphs as metric spaces.

**Observation IV.2** Among the paths connecting two arbitrary vertices $x_i$, $x_k$ there exists at least one with minimal length which we denote by $d(x_i, x_k)$. We apologize for using again the letter $d$ for distance but this is a standard
notation for a distance metric. Furthermore, the differential operator $d$ will not appear again in the following so that there should not arise any confusion. This $d$ has the properties of a metric, i.e:

$$d(x_i, x_i) = 0 \quad (4.1)$$

$$d(x_i, x_k) = d(x_k, x_i) \quad (4.2)$$

$$d(x_i, x_l) \leq d(x_i, x_k) + d(x_k, x_l). \quad (4.3)$$

(The proof is more or less evident).

**Corollary IV.3** With the help of the metric, one gets a natural neighborhood structure around any given vertex, where $B_m(x_i)$ comprises all the vertices, $x_k$, with $d(x_i, x_k) \leq m$, and $\partial B_m(x_i)$ the vertices with $d(x_i, x_k) = m$.

This natural neighborhood structure enables us to develop the concept of an intrinsic dimension on graphs and networks. To this, one first has to realize what property really matters physically (e.g. dynamically), independent of the kind of model or embedding space.

**Observation IV.4** The crucial and characteristic property of, say, a graph or network which may be associated with something like dimension is the number of “new vertices” in $B_{m+1}$ compared to $B_m$ for $m$ sufficiently large or $m \rightarrow \infty$. The deeper meaning of this quantity is that it measures the kind of “wiring” or “connectivity” in the network and is therefore a “topological invariant”.

Regarding the graph as an example of a metric space we shall replace the discrete value $m$ by the continuous parameter $r$, hence writing the ball-neighborhoods as $B(x, r)$. We then define the growth function and spherical growth function on $G$ relative to some arbitrary but fixed vertex $x$. (We use here the notion more common in e.g. geometric group theory. In other fields it is also called the distance degree sequence, cf. [38]).

**Definition IV.5** The growth function $\beta(G, x, r)$ is defined by

$$\beta(G, x, r) = |B(x, r)| \quad (4.4)$$

with $|B(x, r)|$ denoting the number of vertices in $B(x, r)$.

Correspondingly we define

$$\partial \beta(G, x, k) := \beta(G, x, k) - \beta(G, x, k - 1). \quad (4.5)$$

With the help of the limiting behavior of $\beta$ and $\partial \beta$, we introduce two dimensional concepts.

**Definition IV.6** The (upper, lower) internal scaling dimension with respect to the vertex $x$ is given by

$$D_{s}(x) := \limsup_{r \rightarrow \infty} (\ln \beta(G, x, r) / \ln r), \quad D_{c}(x) := \liminf_{r \rightarrow \infty} (\ln \beta(G, x, r) / \ln r) \quad (4.6)$$

where

$$\beta(G, x, k) := \sum_{i=1}^{k} d(x_i, x_l). \quad (4.7)$$

The (upper,lower) connectivity dimension is defined correspondingly as

$$D_{u}(x) := \limsup_{k \rightarrow \infty} (\ln \partial \beta(G, x, k) / \ln k) + 1, \quad D_{l}(x) := \liminf_{k \rightarrow \infty} (\ln \partial \beta(G, x, k) / \ln k) + 1. \quad (4.8)$$

Remark IV.7

1. The two notions are not entirely the same in general, whereas they coincide for many models (this is quite similar to the many different fractal dimensions).

2. For regular lattices, both concepts yield the expected result, i.e. the embedding dimension. In general however, the upper and lower limits are different and non-integer. Similarities to fractal dimensions are not accidental. For a more thorough discussion of all these points see [11].

Some historical remarks are perhaps in order. We developed and investigated this concepts in [11] almost from scratch. We later observed that there existed some scattered remarks in the literature using similar concepts but, as to our knowledge, nowhere were the extremely interesting properties of this concept studied in detail in the physics literature (cf. the remarks and references in [11]). On the other hand we later (roughly at the time of writing [13]) came upon similar concepts employed in a completely different context, that is, a field of pure mathematics called geometric group theory (see e.g. [29]). We come back to this point below when we introduce the renormalization and coarse graining process on graphs or networks.

It is important that these notions display a marked rigidity against all sorts of deformations of the underlying graph and are independent of the reference vertex for locally finite graphs. We mention only two properties in this direction.
Observation IV.8 If the vertex-degree of the graph is locally finite, the numerical values of the above quantities are independent of the reference vertex.
(The simple proof can be found in [11]).

In the following theorem we prove stability of the graph dimension under local perturbations of the wiring of graphs. In the first step we add edges in the \( k \)-neighborhoods of each vertex. In the second part of the theorem the local deformations are slightly more complicated.

Definition IV.9 We pass over from a graph \( G \) to a new graph \( G' \), having the same vertex set, by means of a number of edge deletions. These edge deletions are called local of order \( k \) if only edges between pairs of vertices \( x, y \) are deleted which have a distance \( d(x, y) \) in \( G' \) being globally bounded by \( k \) (note that the distance metrics in \( G \) and \( G' \) will differ in general!).

Theorem IV.10 With \( G \) locally finite the following holds:
1. Insertions of arbitrarily many edges within a \( k \)-neighborhood of any vertex do not alter the above dimensional notions.
(The proof and a slight generalization can be found in Lemma 4.10 of [11]).

2. Edge deletions fulfilling the above described property also do not alter the dimensional notions on the graph.
(This can be proved by reversing the process, i.e., we pass from the graph \( G' \) to \( G \) by \( k \)-local edge insertions; cf. Theorem 6.8 in [11]).

Such deformation results are very useful because it turns out to be surprisingly difficult to construct sufficiently irregular large graphs with prescribed properties, for example, having a prescribed (possibly noninteger) dimension. The above theorem guarantees that irregular graphs which can be constructed via appropriate deformations starting from e.g. regular graphs will have the same dimension. For more results in this direction see the following sections (or [11] and [13]).

We want to conclude this section with a remark concerning the nature of the above defined graph dimension. At a first glance, it may remind the reader of the various fractal dimensions (see e.g. [40]) but this impression is not entirely correct. In a sense it is just the opposite of a fractal dimension. While fractal dimension is related to the infinitesimal structure of (irregular) sets, it is in our case the large distances which matter. Therefore the notion of growth degree is a better description. The reason for this duality stems from our working philosophy to construct the continuum from some discrete irregular underlying structure by performing a continuum limit which includes an essential ingredient a rescaling process. This will become clearer in the following and is just the opposite of going into the infinitesimal small as in fractal geometry.

It is a characteristic of our construction that we go to large distances on the underlying graphs (as is the case in the lattice approximations in quantum field theory). After all, to arrive at a rigorous definition, the graphs have to be infinite. For large but finite graphs we can of course use the concept in an approximate way. Going to large and at the end finite distances is also crucial when we take the continuum limit in order to reconstruct a corresponding continuum theory. In this process we rescale the original graph distance metric, that is we go over from the original distance \( d(x, y) \) to \( \lambda \cdot d(x, y) \) with \( \lambda \rightarrow 0 \). Consequently points which lie very far apart in the underlying graph or network \( G \) become infinitesimal neighbors in the continuum limit. That is, the growth degree characterizes in the end the infinitesimal neighborhood of points in the continuum which is a property of the notion of dimension in the continuum.

As a last remark we want to mention another concept of dimension which is frequently employed in the physics of statistical and critical systems on irregular geometric structures. It is called the spectral dimension. As far as we are aware of, early attempts can be found in [41]. A careful mathematical analysis is made in [42]. A nice paper, relating the spectral dimension to our scaling dimension, is [42]. Another investigation of these dimensions is made in [44] (we emphasize that our list is quite incomplete). The spectral dimension is closely related to diffusion processes on the underlying networks and the return probability of random walks. Both types of dimensions encode certain geometric properties of the network which can be associated with something like “geometric” or “physical”? dimension. There are some connections between these notions but generically they are numerically different.

V. DYNAMICAL NETWORKS AS RANDOM GRAPHS

A. The Statistical Hypothesis

As we are dealing with very large graphs, which are a fortiori constantly changing their shape, that is, their distribution of (active) edges, we expect the dynamics to be sufficiently stochastic so that a point of view may be
appropriate which reminds of the working philosophy of statistical mechanics. This, however, does not imply that our evolving network is nothing but a simple random graph as introduced below (cf. the remarks at the end of this section). It rather means that some of its geometric characteristics can, or should, be studied within this well-developed context.

Visualizing the characteristics and patterns being prevalent in large and “typical” graphs was already a notorious problem in combinatorial graph theory and led to the invention of the random graph framework (see the more complete discussion in [42]). The guiding idea is to deal with graphs of a certain type in a probabilistic sense. This turns out to be particularly fruitful as many graph characteristics (or their absence) tend to occur with almost certainty in a probabilistic sense (as has been first observed by Erdös and Rényi). The standard source is [46] (for further references see [45]).

Another strand of ideas stems from the theory of dynamical systems and cellular automata, where corresponding statistical and ensemble concepts are regularly employed. Typically, we are looking for attractors in phase space, which are assumed to correspond to large scale (that is, after coarse graining and rescaling), quasi continuous or macroscopic patterns of the system. Experience shows, that such a structure or the approach towards attractors is in many cases relatively robust to the choice of initial configurations or microscopic details and, hence, suggests an ensemble picture.

Furthermore, since the early days of statistical mechanics, the ensemble point of view is, at least partly, corroborated by the philosophy that time averages transform (under favorable conditions) into ensemble averages. In our context this means the following. Denoting the typical length/time scale of ordinary quantum theory by \([t_{qm}], [t_{qm}]\), we have

\[
[t_{qm}] \gg [t_{pl}], \quad [t_{qm}] \gg [t_{pl}]
\] (5.1)

where the latter symbols denote the Planck scale. Under renormalization, the mesoscopic scales comprise a huge number of microscopic clock time intervals and degrees of freedom of the network under discussion.

A fortiori, the networks we are interested in correspond to graphs having a large vertex degree, i.e. channels entering a given typical vertex of the graph. That is, we expect large local fluctuations in microscopic grains of space or time. Put differently, the network locally traverses a large number of different microscopic states in a typical mesoscopic time interval, \([t_{qm}]\). This observation suggests that on a mesoscopic or macroscopic scale, microscopic patterns will be washed out or averaged over.

### B. The Random Graph Framework

One kind of probability space is constructed as follows. Take all possible labeled graphs over \(n\) vertices as probability space \(\mathcal{G}\) (i.e. each graph represents an elementary event). The maximal possible number of edges is \(N := \binom{n}{2}\), which corresponds to the unique simplex graph (denoted usually by \(K_n\)). Give each edge the independent probability \(0 \leq p \leq 1\), (more precisely, \(p\) is the probability that there is an edge between the two vertices under discussion). Let \(G_m\) be a graph over the above vertex set, \(V\), having \(m\) edges. Its probability is then

\[
pr(G_m) = p^m \cdot q^{N-m},
\] (5.2)

where \(q := 1-p\). There exist \(\binom{N}{m}\) different labeled graphs \(G_m\), having \(m\) edges, and the above probability is correctly normalized, i.e.

\[
pr(\mathcal{G}) = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m} = (p + q)^N = 1.
\] (5.3)

This probability space is sometimes called the space of binomially random graphs and denoted by \(\mathcal{G}(n,p)\). Note that the number of edges is binomially distributed, i.e.

\[
pr(m) = \binom{N}{m} p^m q^{N-m}
\] (5.4)

and

\[
\langle m \rangle = \sum m \cdot pr(m) = N \cdot p.
\] (5.5)

The really fundamental observation made already by Erdös and Rényi (a rigorous proof of this deep result can e.g. be found in [46]) is that there are what physicists would call phase transitions in these random graphs. To go a little bit more into the details we have to introduce some more graph concepts.
Definition V.1 (Graph Properties) Graph properties are certain particular random variables (indicator functions of so-called events) on the above probability space \( \mathcal{G} \). i.e., a graph property, \( Q \), is represented by the subset of graphs of the sample space having the property under discussion.

To give some examples: i) connectedness of the graph, ii) existence and number of certain particular subgraphs (such as subsimplices etc.), iii) other geometric or topological graph properties etc.

In this context Erdös and Rényi made the following important observation:

Observation V.2 (Threshold Function) A large class of graph properties (e.g. the monotone increasing ones, cf. [46] or [47]) have a so-called threshold function, \( p^*(n) \) or \( m^*(n) \), with \( m^*(n) := N \cdot p^*(n) \), \( p^*(n) \) a \( n \)-dependent edge probability, so that for \( n \to \infty \) the graphs under discussion have property \( Q \) almost surely for \( p(n) > p^*(n) \) and almost surely not for \( p(n) < m^*(n) \) or vice versa (for the details see the above cited literature). That is, by turning on the probability \( p \), one can drive the graph one is interested in beyond the phase transition threshold belonging to the graph property under study. Note that by definition, threshold functions are only unique up to “factorization”, i.e. \( m^*_2(n) = O(m^*_1(n)) \) is also a threshold function.

Calculating these graph properties is both a fascinating and quite intricate enterprise. In our context we are mainly interested in properties of \emph{cliques} (defined below), their distribution (with respect to their order, \( r \), i.e. number of vertices), frequency of occurrence of cliques of order \( r \), degree of mutual overlap etc. (cf. also [45] and [11]). These cliques shall be the building blocks of our geometric renormalization process being described in the following section. We shall relate these properties to the various assumed stages and phases of our space-time manifold.

Definition V.3 A subsimplex is a subgraph with all its vertices being connected to each other. A clique is a maximal subsimplex, i.e., adding another vertex to the subsimplex will destroy the property of being a subsimplex.

We can introduce various random function on the above probability space. For each subset \( V_i \subset V \) of order \( r \), we define the following random variable:

\[
X_i(G) := \begin{cases} 
1 & \text{if } G_i \text{ is an } r \text{-simplex}, \\
0 & \text{else} 
\end{cases}
\]  
\[ (5.6) \]

where \( G_i \) is the corresponding induced subgraph over \( V_i \) in \( G \in \mathcal{G} \) (the probability space). Another random variable is then the number of \( r \)-simplices in \( G \), denoted by \( Y_r(G) \) and we have:

\[
Y_r = \sum_{i=1}^{\binom{n}{r}} X_i 
\]  
\[ (5.7) \]

with \( \binom{n}{r} \) the number of \( r \)-subsets \( V_i \subset V \). With respect to the probability measure introduced above we have for the expectation values:

\[
\langle Y_r \rangle = \sum_i \langle X_i \rangle 
\]  
\[ (5.8) \]

and

\[
\langle X_i \rangle = \sum_{G \in \mathcal{G}} X_i(G) \cdot pr(G_i = r \text{-simplex in } G). 
\]  
\[ (5.9) \]

These expectation values were calculated in [45]. We have for example

\[
\langle X_i \rangle = p^{(2)}. 
\]  
\[ (5.10) \]

The probability that such a subsimplex is maximal, i.e. is a clique, is then (cf. [45])

\[
pr(G_r \text{ is a clique}) = (1 - p^r)^{n-r} \cdot p^r. 
\]  
\[ (5.11) \]

As there exist exactly \( \binom{n}{r} \) possible different \( r \)-subsets in the vertex set \( V \), we arrive at the following conclusion:
Conclusion V.4 (Distribution of Subsimplices and Clique}s) The expectation value of the random variable “number of r-subsimplices” is

$$\langle Y_r \rangle = \binom{n}{r} \cdot p^r (\bar{z})$$.

(5.12)

For $Z_r$, the number of r-cliques (i.e. maximal r-simplices) in the random graph, we have then the following relation

$$\langle Z_r \rangle = \binom{n}{r} \cdot (1 - p^r)^{n-r} \cdot p^r (\bar{z})$$.

(5.13)

These quantities, as functions of $r$ (the order of the subsimplices) have quite a peculiar numerical behavior. We are interested in the typical order of cliques occurring in a generic random graph (where typical is understood in a probabilistic sense).

Definition V.5 (Clique Number) The maximal order of occurring cliques contained in $G$ is called its clique number, $cl(G)$. It is another random variable on the probability space $\mathcal{G}(n,p)$.

It is remarkable that this value is very sharply defined in a typical random graph. Using the above formula for $\langle Z_r \rangle$, we can give an approximate value, $r_0$, for this expectation value and get

$$r_0 \approx \frac{2 \log(n)}{\log(p^{-1})} + O(\log(\log(n)))$$

(5.14)

(cf. chapt. XI.1 of [40]). It holds that practically all the occurring cliques fall in the interval $(r_0/2, r_0)$. We illustrate this with the following tables. Our choice for $n$, the number of vertices, is $10^{100}$. The reason for this seemingly very large number is that we want to deal with systems ultimately simulating our whole universe or continuous space-time manifolds (see the more detailed discussion in [45]). We first calculate $r_0$.

$$
\begin{array}{l|llllllll}
 p & 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\
 r_0 & 4370 & 2063 & 1291 & 901 & 664 & 502 & 382 & 286 & 200 \\
\end{array}
$$

(5.15)

(for reasons we do not understand, we made some numerical errors in the original table 1 in [45], p.2043. The correct numerical calculations can be found in [12]). It is more complicated to give numerical estimates of the distribution of cliques, that is $(Z_r)$. After some manipulations and approximations we arrived (in [45], p.2051f) at the following approximate formula and numerical table (the numerical values are given for $p = 0.7$; note that for this parameters the maximal order of occurring cliques, $r_0$, was approximately 1291)

$$\log(\langle Z_r \rangle) \approx r \cdot \log(n) + n \cdot \log(1 - p^r) + r^2/2 \cdot \log(p)$$

(5.16)

(with $r^2/2$ an approximation of $r(r - 1)/2$ for $r$ sufficiently large).

$$
\begin{array}{l|c|c|c|c|c|c|c|c|c}
 r & 600 & 650 & 800 & 1000 & 1200 & 1300 & 1400 \\
\log(\langle Z_r \rangle) & -5.7 \cdot 10^5 & 3.2 \cdot 10^4 & 3.2 \cdot 10^4 & 2.5 \cdot 10^4 & 8.4 \cdot 10^3 & -0.75 \cdot 10^2 & -1.1 \cdot 10^2 \\
\end{array}
$$

(5.17)

(In the original table 2 of [45] the numerical values for small and large $r$’s, lying outside the interval $(r_0/2, r_0)$, were wrong as we neglected numerical contributions which are only vanishingly small in the above interval. The correct calculations are taken from [12]). The above table nicely illustrates how fast the frequency of cliques of order $r$ drops to zero outside the above interval.

As to the interpretation of these findings, one should remind the reader that the above results apply to the generic situation, that is, do hold for typical graphs (in very much the same sense as in corresponding discussions in the foundations of statistical mechanics). An evaluation of the combinatorial expressions in this and the following sections shows that frequently the same kind of extreme probabilistic concentration around, for example, most probable values occurs as in ordinary statistical mechanics.

What is not entirely clear is, how far the random graph approach can be applied to our complex dynamical networks. Our working philosophy is that these results serve to show what we hope to be the qualitative behavior of such systems. As our systems follow deterministic dynamical laws, starting from certain initial conditions, the behavior cannot be entirely random in the strict sense. This holds the more so since we expect the systems to evolve towards attracting sets in phase space and/or generate some large scale patterns. On the other hand, due to the constant reorientation
of the edges, being incident to an arbitrary but fixed vertex and the generically large vertex degrees of the vertices, one may assume that the system is sufficiently random on small scales so that the random graph picture reproduces at least the qualitative behavior of such extremely complex systems.

To make this picture more quantitative, the general strategy is the following. We count the typical number of active edges in our evolving network at a given clock time $t$, calculate from this the corresponding edge probability, $p(t)$, and relate this snapshot of our network to a random graph with the same edge probability. This should yield at least some qualitative clues. That is, we expect that qualitative characteristics of our evolving network can, at each given clock time, be related to the characteristics of a corresponding random graph. In this specific sense, one may regard the edge probability, $p(t)$, as the crucial dynamical parameter of our network, regarded as a statistical system.

VI. A GEOMETRIC RENORMALIZATION GROUP AND THE CONTINUUM LIMIT OF DISCRETE GEOMETRIES

In the preceding section we introduced the notion of cliques as maximal complete subgraphs of a given (random) graph. We are interested in them because physically they represent lumps of vertices which are maximally entangled among each other. That means, viewing the graph as the geometric substrate on which our dynamical network lives, the vertices or the respective DoF of a clique are directly interacting with each other. Invoking the Wilson/Kadanoff picture of the renormalization group in statistical mechanics these cliques are assumed to act as dynamical entities of their own in the larger network.

In this context we want to mention an older approach developed by Menger and coworkers some time ago and which we discussed in greater detail in [48]. Ideas in this direction (a geometry of lumps in which points are not primary entities) were also briefly mentioned in Mengers contribution to the anthology: Albert Einstein:Philosopher Scientist, volume II ([49]); note also the comments of Einstein in the same volume.

Remark VI.1 The cliques in our dynamical networks may change their shape under the imposed dynamics which may create and/or delete edges. This was one of the reasons why we developed the concept of fuzzy lumps or fuzzy cliques in [48].

Similar to the block spin approach in the theory of critical phenomena we promote the cliques of the initial (random) graph, $G$ (level zero), to the vertices of the graph of the next level (level one). We draw an edge between these vertices of level one if the corresponding cliques have an overlap of a certain order of vertices of level zero (see below). In this sense we get a new graph, the so-called clique graph, $Cl(G)$ or $G_{cl}$. We can repeat this process, i.e. we can form the graphs $Cl(G)$, $Cl^2(G) = Cl(Cl(G))$ … and study the change of geometric characteristics and/or emergence of new collective patterns on the various levels of this process.

So what is the physical picture underlying this process? The idea of the ordinary (infrared) renormalization process in e.g. condensed matter physics is to integrate out the microscopic details of the model and its dynamics and, via coarse graining and rescaling, make visible the collective and large scale properties of the model. This is accomplished by integrating out in each step the DoF in the blocks or lumps of the preceding level and make the new averaged DoF into the constituents of the next level, thus establishing a new coarse grained and rescaled model together with a new coarse grained Hamiltonian.

Slightly deviating from this philosophy, we want to concentrate in our geometric renormalization process, for the time being, primarily on the underlying geometric substratum. While we presume that one can treat e.g. gravitation as being emergent from some dynamical network model as we have introduced it in section [], (including the existence of a metric field $g_{ij}(x)$), we want to deal at the moment only with the geometric large scale properties of our dynamic graphs.

Remark VI.2 The following point is important. In the ordinary renormalization process in for example some spin models, the spins in a certain block are regarded as behaving in a coherent way, i.e., provided the correlation length is larger than the diameter of the blocks, the spins in some block are expected to be more or less aligned. Therefore it makes sense to build an average spin as a new block variable. In our network we want, on the contrary, to implement not only the wild vacuum fluctuations which are expected to be very large on small scales but also create some smooth classical space-time continuum on large scales.

As a consequence of the dynamical laws we introduced above, the individual vertex states in a certain clique fluctuate considerably but, due to the close entanglement induced by the edges, we expect them to cooperate strongly so that some global coherent pattern may emerge. The picture we have in mind is that of an array of coupled phase oscillators as will be described in the following section about wormhole spaces and small world networks.

Observation VI.3 As a result of the process described in the following, we arrive at a picture where lumps or cliques are contained in larger cliques (of the next level) which are again contained in larger cliques of the following level etc. So we get a hierarchical picture of the concept of physical points which have a rich internal structure.
Definition VI.4 We want to denote this double structure of an underlying erratic network coexisting with a conjectured coherently behaving smooth classical macroscopic surface structure by QX/ST (quantum space plus classical space time).

We talked in in subsection V B about the existence of threshold functions which resemble phase transitions and phase transition lines in statistical mechanics. Furthermore, the different phases are frequently characterized by non-vanishing order parameters. In this spirit we make the following definition:

Definition VI.5 If such a super structure ST emerges in our dynamical network in a certain parameter regime we call it an order parameter manifold.

Remark VI.6 This implies that the existence of such a ST is a particular property prevailing only in a certain region of the phase space of the model, that is, we may also have such a microscopic substratum without an overlying classical space time.

This is, to some extent, similar to the situation in some other mainstream theories about the structure of spacetime like loop quantum gravity. There, a certain state of a spatial region (in the canonical formulation) is described by a certain vector in a Hilbert space. This vector in general corresponds to a certain graph (which is a superposition of basis states or graphs called spin networks) which is considered to underly that region. The state (or the corresponding graph) might or might not correspond to any “classical” space. There are very strong indications that for example singular regions like the ones inside a black hole correspond to states that do not have any classical counterpart. In other words those states are highly quantum and there is no classical notion of space that can be attached to them. Our model also seems to have the capability of incorporating this idea in a rather natural way.

It is interesting to consider what takes the place of the so called critical systems of the ordinary renormalization group. Critical systems converge to fixed points which, on their part, describe the large scale properties of the critical system under discussion.

Conjecture VI.7 In our geometric case, where cliques and their entanglements are expected to be the crucial building blocks, geometric critical systems are conjectured to display some geometric long range order and a certain selfsimilarity. This will be worked out in more detail in the following.

It turns out that it is advisable to split the investigation into two subsections. In the first one which deals with the clique structure, we develop the geometric coarse graining process. In the second section we discuss the rescaling process which leads to a continuum limit. Both parts of the renormalization process have problems of their own and lead to quite deep mathematics. The material of the following two subsections consists mainly of a review of the content of the papers [45], [12], parts of [14] and [13]. That is due to length limitations we have to refer the reader to these papers as to the partly quite complicated and intricate combinatorial and numerical calculations.

A. The Geometric Coarse-Graining

As indicated above, we assume that presently we live in a network scenario where in the underlying microscopic network, called QX, there exists a superstructure or order parameter manifold, ST (classical space-time). The emergence of ST signals the transition from a chaotic initial phase, QX0, to a phase developing a near/far-order, i.e., a causal structure and relatively stable physical points or (fuzzy) lumps (Menger).

Our physical picture concerning the initial scenario (big bang era) is the following. The network QX started from a presumably densely entangled initial phase QX0, in which on average every pair of vertices, x1, xk is connected by an active edge with high probability p ≈ 1 or a Jik ̸= 0 (in our examples of dynamical laws).

We then envisage two main epochs of our evolving network, a so-called embryonic epoch with a still large edge probability near p = 1 and an unfolded epoch with a much smaller value of p and a large diameter (see below for the definition) at least on some higher clique level.

Observation VI.8 One should note that pure random graphs have a very small diameter for a large range of p-values (cf. e.g. [74]) so that p must be quite small if we want to have an unfolded epoch.

Definition VI.9 The diameter of a graph is defined by maxx1,xk d(x1,xk).

In the subsection V B we calculated the typical order and number of cliques and provided two tables for the order of the largest expected cliques (the clique number of the random graph), r0, and number of r-cliques. The numerical values were p = 0.7, n = 10^100. We showed that almost all cliques have an order between r0 and 2p.

In a first step we want to clarify the mathematical and physical process of constructing the clique graph Cl(G) of a graph G.
Definition VI.10 We employ two methods of constructing the clique graph:

1. The mathematical clique graph operator $\text{Cl}_m$ is defined as follows. The vertices of the (mathematical) clique graph are the cliques of $G$, an edge is drawn if two cliques have a non-empty overlap.

2. The physical clique graph operator $\text{Cl}_p$ is constructed in the following way. We delete too small (marginal) cliques which do not lie in the above described interval $(r_0, r_0/2)$. An edge between cliques $C_i, C_k$ is drawn if the overlap is non-marginal (we discuss the physical implications of this notion below). That is, we say, the overlap of the respective cliques is non-marginal if it is larger than a certain value $l_0(r)$ which depends on $r$ and the underlying physics.

Observation VI.11 In this way the original graph and the mathematical clique graph are purified, put differently, the iterated mathematical clique graph is coarse-grained, that is, on each level some marginal structure is deleted.

Remark VI.12 To give examples for $l_0$; in [12] section V.B we took $n = 10^{100}$, $p = 0.7$ which yielded $r_0 = 1291$. Our choice for $l_0$ was $l_0 = 50$. We however convinced ourselves that the physical picture does not critically depend on this choice. As an example, for $l_0 = 30$ we got similar results.

The physical motivation behind this procedure is the following. Our aim is it to construct a space-time substratum in which we have, on the one hand, a near order structure concerning interaction and the flow of information among neighboring physical points of $ST$ (i.e., the classical space-time concept under a certain magnification, so that the internal structure of the points of the manifold becomes visible). On the other hand, we have weak, translocal ties between lumps, which are with respect to the ordinary space-time metric some distance apart. These weak ties result from edges which have been present in the original graph $G$ or on some lower clique level, say $\text{Cl}_m^r(G)$, relative to some clique level $\text{Cl}_m^n(G)$ with $n > m$ or a level which is already near our presumed classical space-time $ST$ and have been neglected in the construction of the consecutive levels of the physical, purified clique graph. This dual picture will become clearer in the next section on wormhole spaces and small world networks.

Conjecture VI.13 We conjecture that these weak translocal ties are responsible for some important aspects of quantum theory as for example entanglement.

In this context we regard the cliques of some given level as approximately autonomous subunits which are coupled both with their neighbors via strong ties and with more distant cliques via some weak ties. We conjecture that this dual structure will lead to a sort of global coherence which results in the emergence of the order parameter manifold $ST$.

We now want to briefly discuss some analytic and numerical results of our construction of the (purified) clique graph. For the details of the mostly quite intricate calculations we refer to the above mentioned papers. We begin with the notion of the embryonic epoch (cf. subsection 4.1 of [12]). In this epoch most of the edges are still active, i.e. $p \approx 1$. In subsection 4.1 of [12] we dealt with the question that under what conditions all the cliques of the graph do have a common nonempty overlap. On the other hand, for sufficiently small $p$ we could show that this overlap is empty with high probability. This last regime describes presumably the so called unfolded epoch. The main work consists of providing combinatorial/probabilistic expressions for these graph properties.

In a next step, we want to calculate the order of the so called local group of a fixed clique. This local group comprises of the cliques which have non-marginal overlap with an arbitrary but fixed clique, $C_0$. This is an interesting graph property in the unfolded epoch because it defines the infinitesimal (first) neighborhood of a vertex in the clique graph. The necessary combinatorial/probabilistic analysis is made in subsection 4.1 of [12] and section V.B of [12].

The number of cliques overlapping with some given clique $C_0$ is, as all these properties, a random variable in our random graph picture. We denote it by $N(C_0; r', l)$ and its expectation value by $< N(C_0; r', l) >$. In this expression $C_0$ is a clique of fixed order $r$. Its overlap is considered with respect to cliques of some order $r'$, both lying in the interval $(r_0, r_0/2)$ i.e. $r_0 < r < r' < r_0/2$. The overlap is denoted by $l \geq l_0$ with $l_0$ some minimal value (denoting the non-marginal overlap). In the end this expression is summed over all $r'$ lying in the interval $(r_0, r_0/2)$.

In the same way we analyze the other defining parameters of the clique graph (cf. section V.B of [12]).

Remark VI.14 One should note that one can infer from our second table in subsection V.B that, generically, there do exist much more cliques than vertices in a typical random graph, i.e., $\text{Cl}(G)$ has typically much more vertices than $G$.

Observation VI.15 Our numerical analysis shows after some intricate calculations and estimates that the clique graph of a random graph $G$ with

$$n = 10^{100}, \ p = 0.7, \ \text{clique overlap} \ l \geq l_0 = 50$$

(6.1)
has typically $10^{10^4}$ vertices, the edge probability is approximately $10^{-7\cdot10^3}$ and the vertex degree is of the order of $10^{0.3\cdot10^4}$.

**Remark VI.16** Note that the small edge probability is compensated by the huge vertex degree.

Proceeding in the same way, while however readjusting the non-marginal overlap $p_0(r)$ on each level, we get the *iterated purified clique graph*. We convinced ourselves that the gross parameters of the clique graphs on the consecutive levels seem to reach stable values after only a few renormalization steps.

**Observation VI.17** As the whole scenario is much more irregular in our case compared to the situation in e.g. statistical mechanics, the existence of a fixed limit network under the geometric renormalization group (more properly, modulo graph isomorphisms) should presumably be refined a little bit. We introduce the important concept of quasi-isometry or coarse isometry in the following subsection. We hence expect that the limit of an iterated clique graph is only invariant under the physical clique operator $Cl_p$ modulo quasi-isometries.

**Definition VI.18** An ordinary graph isomorphism $I$ is a bijective map between graphs so that the graphs are form-invariant, i.e., an edge exists between two image vertices iff there exists an edge between the pre-images.

**Remark VI.19**

1. In section VI of [12] we gave simple examples which show that our expectations are apparently not completely far fetched. These examples should however not be regarded as examples of possible space-times. We see that both limit points and limit cycles are possible.

2. In pure mathematics the study of the iterated clique graph is indeed an interesting field in graph theory. cf. for example [50], [51], [52].

We want to close this subsection with giving some important results for the unpurified mathematical clique graph. We begin with the graph property connectedness.

**Lemma VI.20** If $G$ is connected (i.e., every two vertices are connected by a path), the same holds for $Cl_m(G)$.

**Proof:** Let $C, C'$ be two cliques and $x, x'$ two vertices with $x \in C, x' \in C'$. There exists a path connecting $x$ and $x'$. We denote the corresponding vertices by $x = x_0, x_1, \ldots, x_k = x'$. There exist cliques, $C_i$ containing the pairs $(x_i, x_{i+1})$ with $i = 0, \ldots, k - 1$. Note that the cliques $C_i, C_{i+1}$ have non-zero overlap. This proves the statement.

**Remark VI.21** The property of connectedness need of course no longer hold for the coarse-grained clique graph, $Cl_p(G)$. In the random graph framework this property becomes a random variable and its probability can be calculated (cf. e.g. Conclusion 5.10 in [12]).

Another interesting property is the possible change of distance under the clique operator. The following lemma shows that distances do not change significantly.

**Lemma VI.22** Let $G$ be a connected graph and $C, C'$ two cliques in $Cl_m(G)$, having the distance $d_{Cl(G)}(C, C')$. Then we have

$$d_{Cl(G)}(C, C') = \min_{x, x'} d(x, x') + 1, \ x \in C, \ x' \in C'$$  \hspace{1cm} (6.2)

**Proof:** The proof is essentially already contained in the proof of the following theorem (cf. theorem 7.4 in [12] or see [52]). Choose a minimal-length path in $Cl_m(G)$ connecting $C$ and $C'$ and consisting of the cliques

$$C = C_0, C_1, \ldots, C_k = C'$$ \hspace{1cm} (6.3)

implying that $C_i, C_{i+1}$ have non-empty overlap. Choose a path in $G$ starting at some $x_1 \in C_0 \cap C_1$, with $x_2 \in C_1 \cap C_2, \ldots$. This is a path with initial point in $C$, endpoint in $C'$, having the length $k - 1$. It is easy to show that such a path is minimal under this condition, i.e. it holds $k = (k - 1) + 1$, which proves the statement.

The following observation is non-trivial and important.

**Theorem VI.23** We assume that $G$, having a globally bounded vertex degree, has scaling dimension $D$ (cf. section IV). It follows that $Cl_m(G)$ has also a dimension with

$$D_{Cl_m(G)} = D.$$  \hspace{1cm} (6.4)
The proof is more intricate and can be found in section VII of [12].

Remark VI.24 While there exists the restriction of a globally finite vertex degree, the result shows that when building the iterated unpurified clique graph, the microscopic dimension remains fixed in the transitions to higher levels. In other words, if we surmise that our physical space-time is smoother than the underlying microscopic substratum, implying among other things that also the graph dimension on the different levels of the iterated clique graph is expected to vary, we have to employ the concept of the physical clique graph operator $C_{1p}$ with its purification and coarse-graining properties. That it is not easy to change the graph dimension at all is shown in section VIII of [13]. We will briefly come back to this point in the following section.

B. The Rescaling Process and the Continuum Limit

In the preceding subsection we remained within the class of discrete networks or graphs. I.e., at all levels of our constructions, the models under discussion were discrete. We now will develop the framework which allows to construct continuum limit models of our discrete networks. While such a process is perhaps transparent in the context of models living on a periodic Bravais lattice, it becomes very intricate in the case of general irregular networks with a relatively deep amount of mathematics being involved. The general context is the theory of general metric spaces. We shall make heavy use of material being developed for example in [50,51] and [39]. This subsection reviews the content of [12] where more references can be found.

An important conceptual tool is the notion of quasi-isometry. This is the appropriate generalization of the notion of isometry to disordered and irregular spaces where marginal details and variations are partly ignored.

Definition VI.25 Let $F$ be a map from a metric space, $X$, to a metric space, $Y$, with metrics $d_X, d_Y$ respectively. It is called a quasi-isometric embedding if the following holds: There exist constants, $\lambda \geq 1, \epsilon \geq 0$, such that

$$\lambda^{-1} \cdot d_X(x, y) - \epsilon \leq d_Y(F(x), F(y)) \leq \lambda \cdot d_X(x, y) + \epsilon$$

(6.5)

If, furthermore, there exists a constant $\epsilon'$ such that for all $y \in Y$ we have $d_Y(y, F(X)) \leq \epsilon'$, that is, $Y \subset U_{\epsilon'}(F(X))$, (for the definition of $U_{\epsilon}(A)$ see below) $F$ is called a quasi-isometry; the spaces are then called quasi-isometric. There is an equivalent definition which shows that the preceding definition is in fact symmetric between $X$ and $Y$ (see for example [39]). That is, there exists a quasi-isometric map $G$ from $Y$ to $X$ with corresponding constants and $d_X(G \circ F(x), x) \leq \rho$ and $d_Y(F \circ G(y), y) \leq \rho$ for some $\rho$. If $\lambda = 1$ it is called a rough isometry.

It is an important observation that in our framework of networks and graphs many properties are stable under quasi-isometries. We have, for example, the following results:

Observation VI.26 The “growth type” of graphs with globally bounded vertex degree is stable under quasi-isometry (cf. section [11] and [20]). We have in particular that quasi-isometric graphs have the same dimension, i.e. (with $G_1, G_2$ quasi-isometric graphs)

$$\bar{D}_1 = \bar{D}_2 \quad , \quad D_1 = D_2$$

(6.6)


A further interesting observation is the following. We introduced in section [14] the concept of local edge insertions/deletions and showed that these procedures do not alter the dimension in the case of graphs with globally bounded vertex degree. We have

Observation VI.27 Local edge insertions/deletions lead to quasi-isometries. By the same token we see that via this method we get a rich class of examples of quasi-isometric graphs.

It is of great interest to derive criteria under which a network has a finite growth degree (in contrast to e.g. an exponential growth) and, a fortiori, an integer dimension. This is discussed in greater detail in section 3 of [13]. We want furthermore to emphasize that perhaps rather surprisingly our physically motivated interests turn out to be closely related to a field of pure mathematics, i.e. geometric group theory via the concept of Cayley graph (cf. [39]).

We now come to the construction of the continuum limit of an infinite graph (for finite graphs the construction is not particularly interesting as it leads to a single point). To this end we have in a first step to construct a metric on a set of graphs or set of subsets of a certain space (i.e., a metric on a set of metric spaces). We begin with the definition of the so called Hausdorff-metric on a space of subsets of a metric space.
Definition VI.28 Let $X$ be a metric space, $U_\epsilon(A)$ the $\epsilon$-neighborhood of a subset $A \subset X$ ($U_\epsilon(A) := \{ y : d(x,y) < \epsilon \text{ for some } x \in A \}$). The Hausdorff-distance between $A,B \subset X$ is then given by
\[
d_H(A,B) := \inf \{ \epsilon : A \subset U_\epsilon(B), B \subset U_\epsilon(A) \}. \quad (6.7)
\]

We have the following lemma

Lemma VI.29 With $X$ a compact metric space, the closed subsets of $X$ form a compact (i.e. complete) metric space with respect to $d_H$ (see e.g. [53] or the book by Edgar, [37]).

In the following it is sometimes useful to make a slight generalization to pseudo metric spaces as we will encounter situations where spaces or sets have zero Gromov-Hausdorff-distance (for example, one being a dense subset of the other) while they are not strictly the same. Everything we will state for metric spaces in the following will also hold for pseudo metric spaces.

Definition VI.30 A pseudo metric fulfills the same axioms as a metric with the exception that $d(a,b) = 0 \rightarrow a = b$ does not necessarily hold.

The above distance concept is too narrow to be useful in a more general context. It was considerably generalized by M.Gromov in an important way (see [60] and later slightly modified by himself and other authors ([53], [54], [55]). What is really beautiful in our view is that while it seems to be more abstract, it encodes the really important and crucial aspects of similarity or “nearness” of spaces in a more satisfying way. That is, it measures their structural similarity and not simply the nearness of two structureless sets of points in a space. In general it is a pseudo metric which may even take the value infinity. For compact spaces it is always finite. If one forms equivalence classes of compact spaces under isometries, it becomes a true metric.

The Gromov-Hausdorff distance, $d_{GH}$, can be formulated in two equivalent ways.

Definition VI.31 $d_{GH}(X,Y)$ between two metric spaces, $X,Y$, is defined as the infimum of $d_H^2(f(X),g(Y))$ over all metric spaces $Z$ and isometric embeddings, f.g. of $X,Y$ into $Z$.

Equivalently, one can define $d_{GH}$ by the infimum over $d_H(X,Y)$ in $X \sqcup Y$ (disjoint union) equipped with the metrics $d_{X,Y}$ which extend the respective metrics $d_X,d_Y$ in $X,Y$.

The crucial part of the distance concept is always the triangle-inequality. Furthermore we have to show that the above infimum is again a metric. This is proved in section 4 of [13] where quite a few more interesting results are discussed. Due to the lack of space we directly embark on the deep results of Gromov concerning the formulation of convergence of spaces towards each other.

We now present the fundamental Gromov-compactness theorem, first for compact spaces, then for more general cases.

Definition VI.32 We call a family of compact spaces, $X_\lambda$, uniformly compact if their diameters are uniformly bounded and if for each $\epsilon > 0$, $X_\lambda$ is coverable by $N_\epsilon < \infty$ balls of radius $\epsilon$ independent of the index $\lambda$.

Theorem VI.33 (Gromov) A sequence $\{X_\lambda\}$ contains a convergent subsequence in $d_{GH}$ iff $\{X_\lambda\}$ is uniformly compact.

Proof: see [50, 55] or [54]. Typically an Arzela-Ascoli-Cantor-diagonal-sequence-like argument is used in the proof.

In our framework we are mainly interested in infinite graphs, i.e. non-compact metric spaces being however frequently proper.

Definition VI.34 A metric space, $X$, is called proper if all its closed balls, $B(x,r)$, are compact.

We can then extend the above result in the following way. Ordinary GH-convergence works well in the category of compact metric spaces. If the spaces are non-compact, a slightly modified approach is more satisfactory. One problem which may arise is that things in unbounded spaces can “wander away” to infinity. So it is reasonable to pin down the members of the sequence of spaces at certain points (base points), so that they can be better compared. More precisely, we work in the category of pointed metric spaces, $(X,x)$, which is, a fortiori, pretty normal from the physical point of view as it is like introducing a reference point or a coordinate system.

Definition VI.35 The sequence of pointed metric spaces, $(X_i,x_i)$, is said to converge to $(X,x)$ in pointed GH-sense if for every $r > 0$ the sequence of closed balls, $B(x_i,r)$, converges to $B(x,r)$ in $d_{GH}$.

The Gromov-uniform-compactness theorem now reads:

Theorem VI.36 If for all $r$ and $\epsilon > 0$ the balls $B(x_i,r)$ of a given sequence $(X_i,x_i)$ are uniformly compact, then a subsequence of spaces converge in pointed GH-sense.
Remark VI.37 There exist various slightly different notions of pointed convergence in the literature. One can for example define pointed GH-distance by admitting only isometries which map the base points onto each other (53). Another possibility is to include the distance of the images of the base points in the definition (57). The above definition is used in (54).

We now apply these techniques to the following sequence of graphs. We start with a graph, $G$, of globally bounded vertex degree, $v$, and take $G$ with the original graph metric, $d$, as the initial metric space. Then we generate a sequence, or more generally a directed system of metric spaces, $\lambda G$, by taking the same graph, $G$, but now with the scaled metric, $\lambda d$, defined as

$$\lambda d(x, y) := \lambda \cdot d(x, y)$$

(6.8)

and (usually) taking $\lambda \to 0$. One may, in particular, take subsequences of the kind

$$G_n, d_n := n^{-1} \cdot d, n \to \infty$$

(6.9)

or replace $n$ by $2^{-k}$.

In a next step we have to show that all the above criteria are fulfilled in this case which is a non-trivial task (see section 5 of (13)). Among other things several new notions and concepts have to be introduced like e.g. doubling measures etc. Finally we can show that our sequence of rescaled graphs has a continuum limit! It is now very important to learn something about the structure of this limit space. Some steps are done in section 5 of (13). It is of particular importance to understand under what conditions this limit space is a smooth manifold or, on the other hand, a chaotic space of rather fractal type. We are very interested in the possibility of a limit space having a superficially smooth structure together with an internal infinitesimal more erratic structure “around” the “classical” points of the base manifold, being kind of a generalized fiber space.

VII. WORMHOLE SPACES, HOLOGRAPHY AND THE TRANSLOCAL STRUCTURE OF QUANTUM THEORY

In this last short section, which is also kind of a conclusion, we want to discuss in a very sketchy way various fundamental (open) questions in modern physics and show how our above framework can be applied to them. As we shall treat these important topics in a quite cursory way we refer the interested reader to the following papers (16), (14), (58) for a more thorough discussion.

In section VI we came already to the conclusion that presumably the cellular network substratum which has the propensity to lead to a continuum limit, resembling our physical (quantum) space-time, has to be in a peculiar critical state which resembles a scale free small world network as we described it in (14). This means that apart from a certain nearorder leading to a lumpy local structure in the network on the various scales of coarse graining and rescaling, there remains on all scales a certain sparse network of the so-called translocal connections (i.e., edges in the graph language relating distant regions with respect to the ordinary local metric). Rather surprisingly such a structure was analyzed roughly at the same time in a quite different area of science, dubbed small world networks (for a detailed treatment see e.g. (59). A brief discussion can also be found in (61).

In (61) we described such models in the following way. We start with a regular graph having a nearorder structure like e.g. $\mathbb{Z}^n$ with edges to nearest neighbors in the vertical and horizontal direction. In a scaling limit this would lead to a space like $\mathbb{R}^n$ with the Euclidean distance metric. In a next step we superimpose $\mathbb{Z}^n$ or a similar regular space with a sparse random graph on the same vertex set and a small edge probability $p$. These random edges now lead to an additional translocal structure on $\mathbb{Z}^n$. We conjecture that this serves as a toy model for the kind of substratum underlying our physical (quantum) space-time which we dubbed QX/ST (i.e. an underlying discrete substratum having both a near and a far order together with a continuous surface structure on a low resolution of space-time).

Observation VII.1 A characteristic of small world networks is their surprisingly small diameter or mean distance (cf. e.g. (14), (61) or (59)), given the sparseness of additional translocal edges.

In (16) we developed and studied this phenomenon in quite some detail and coined the notion wormhole spaces for such structures like our QX/ST. We showed that the BH entropy area law and the holographic principle follow quite naturally from our framework.

Finally, in (58) we undertook to develop a framework which describes the quantum phenomena as emergent properties on a mesoscopic scale. More precisely, at various places in our paper we indicated that our dynamic network of local lumps or cliques can be associated to a network of coupled phase oscillators (cf. remark II.6 or remark VI.2).
Remark VII.2 Many non-linear systems approach limit cycles in their evolution on which they then evolve according to a law like $\dot{\theta} = \omega$, $\omega$ a certain specific natural frequency, $\theta$ the phase of the system. As to more details see \cite{54,60} or \cite{58}.

The famous Kuramoto model describes a large population of coupled limit cycle or phase oscillators whose natural frequencies are drawn from some prescribed distribution (see for example \cite{62}). The hallmark of such models is that there may occur a particular kind of phase transition in which all the initially different natural frequencies, $\omega_i$, become dynamically synchronized.

Conjecture VII.3 We expect that our dynamical network models show a similar behavior with the cliques or lumps representing the limit cycle or phase oscillators.

It is an important observation \cite{63} that this emergent property of synchronization is strongly enhanced and stabilized by a certain sparse non-local network of random couplings superimposed on the prevailing network of local couplings in the array of oscillators. This is exactly what we found in our network models.

Conjecture VII.4 This possibility of phase locking may be a hint how a global time function emerges from the array of initially different local times with the phase oscillators viewed as local clocks.

It is remarkable that Bohm, starting from a different direction, also speculates about the existence of a hierarchy of coupled oscillators on consecutive scales of (quantum) space-time (see \cite{64}). He came to a conclusions similar to the ones uttered in \cite{58} concerning the consequences for quantum theory as being emergent from such a deep structure. We discussed this in greater detail in \cite{58}.

VIII. THE (QUANTUM) GRAPHITY APPROACH

In this section, we provide a brief review of another bottom-up approach to quantum gravity, called quantum gravity \cite{2,3}. We divide the discussion into kinematics and dynamics. In the next section, we try to give a brief comparison of the two approaches.

A. Kinematics

1. States and the Hilbert space

Quantum gravity like some other bottom-up approaches starts from a family of graphs living on a fixed set of vertices as underlying the notion of spacetime. This family comprises the complete graph $K_N$ (i.e., all vertices are linked to each other) and all the other possible graphs on the $N$ vertices. The vertices are distinguishable but are undirected, so for the edge $e_{ij}$ between two vertices $x_i$ and $x_j$, we have $e_{ij} = e_{ji}$. A Hilbert space $H_{\text{edge}}$ is associated to each edge such that

$$H_{\text{edge}} = \text{span} \{ |0\rangle, |1\rangle \}$$

where an edge being in a state $|0\rangle$ is considered “off” and one with a state $|1\rangle$ is considered an “on” edge. This Hilbert space is the same as one that belongs to a fermionic oscillator with $|0\rangle$ and $|1\rangle$ corresponding to no particles or 1-particle states respectively. Another assumption is that the degrees of freedom are only associated to edges and vertices do not contribute to them. This means that the total Hilbert space of a typical graph is the tensor product of the edge Hilbert spaces $H_{\text{edge}}$ such that

$$H_{\text{total}} = \bigotimes_{e \in E} H_{\text{edge}}$$

where $\frac{N(N-1)}{2}$ is the total number of edges in a complete graph $K_N$ with $N$ vertices. As mentioned above, this total Hilbert space is a space with members being the complete graph $K_N$ with $N$ vertices, and all of its subgraphs. Thus one can decompose $H_{\text{total}}$ as a tensor sum

$$H_{\text{total}} = \bigoplus_G H_G$$

over all subgraphs $G$ of the complete graph $K_N$, including $K_N$. 

2. Operator representation

In the next step, one defines the creation and annihilation operators $a^\dagger$ and $a$ acting on the Hilbert space of each edge $\mathcal{H}_{\text{edge}}$ (and by extension, on $\mathcal{H}_{\text{total}}$) such that they anticommute
\[
\{a, a^\dagger\} = 1
\]
and
\[
a|0\rangle = 0, \quad a|1\rangle = |0\rangle, \quad a^\dagger|0\rangle = 1.
\]
One can also define the “number operator” for the edge $e_{ij}$ between the vertices $x_i$ and $x_j$
\[
N_{ij} = a_{ij}^\dagger a_{ij}
\]
such that $N_{ij}$ acting on its eigenstates, i.e. “number states” that are the basis of $\mathcal{H}_{\text{edge}}$, yields
\[
N_{ij}|n_{ij}\rangle = n_{ij}|n_{ij}\rangle, \quad n_{ij} = 0, 1
\]
which means that it returns an eigenvalue 0 if the edge is off and 1 if it is on. Also note that since the graphs considered are undirected, it follows that
\[
a_{ij} = a_{ji},
a_{ij}^\dagger = a_{ji}^\dagger,
N_{ij} = N_{ji}.
\]

Then the authors make an observation that the matrix corresponding to the eigenvalue of the number operator $N_{ij}$ is analogous to the adjacency matrix $A_{ij}$ of an undirected graph in which an element $A_{kl}$ is 1 if there is an edge between the vertices $x_k$ and $x_l$ and 0 otherwise.

On the other hand, the powers of the adjacency matrix such as
\[
A^n_{i_1i_{n+1}} = \sum_{i_2} \cdots \sum_{i_n} A_{i_1i_2}A_{i_2i_3} \cdots A_{i_{n+1}i_{n+1}}
\]
contain information about the open (if $i_1 \neq i_{n+1}$) and closed (if $i_1 = i_{n+1}$) paths between the vertices $x_{i_1}$ and $x_{i_{n+1}}$ in the graph under discussion. So for example $A^n_{i_1i_{n+1}}$ denotes the number of possible paths between the vertices $x_{i_1}$ and $x_{i_{n+1}}$ by $n$ jumps or steps. But this does not necessarily mean a non-overlapping path (or non-retracing path) of non-overlapping length $n$, since one can jump back and forth over some of the edges more than once. As an example they mention an element of the fourth power $A^4$ of the adjacency matrix of a certain graph such that $i_1 = 1$, $i_2 = 2$, $i_3 = i_1 = 1$, $i_4 = i_2 = 2$, $i_5 = 3$ and thus the path is between $x_{i_1} = x_1$ to $x_{i_5} = x_3$ and is represented by the sequence of edges $\{e_{12}, e_{21}, e_{12}, e_{23}\}$. This is an overlapping path with 4 jumps or steps but with a non-overlapping length of 2, i.e. the minimum number of edges connecting $x_{i_1} = x_1$ to $x_{i_5} = x_3$ is only 2 represented by $\{e_{12}, e_{23}\}$.

In order to avoid this and only count the number of non-retracing paths between two vertices, they make the following observation. In a first step, one uses the number operator $N_{ij}$ instead of $A_{ij}$ to extract the information about the number of paths between two vertices. Then, like in quantum field theory, a normal ordering, using the sign $N^n_{ij}$ : is used in the powers of $N_{ij}$. As usual it is such that all the annihilation operators are brought to the right of creation operators. For example for the second power we have
\[
N^2_{ik} =: N_{ij}N_{jk} := a_{ij}^\dagger a_{ij} a_{jk}^\dagger a_{jk} = a_{ij}^\dagger a_{jk}^\dagger a_{ik} a_{jk}.
\]
Thus in the product appearing in $N^n$, $n \geq 2$, if any of the edges appear more than once in the path, its number operator $N_{kl}$ will also appears more than once in the form $N_{kl}N_{ik}$ (which is equal to $N_{kl}N_{ik}$ due to symmetry of $N$ in Section 10) in the product, and then we will have at least two of the (same) corresponding annihilation and creation operators in the product as
\[
N^n = a_{kl}^\dagger a_{ik} a_{kl} a_{ik} \cdots = a_{kl}^\dagger a_{kl} \cdots = 0
\]
and it will vanish due to the anticommutation [8,4]. This way the eigenvalues of the operator $N^n_{ik}$ return the number of non-overlapping paths between the vertices $x_i$ and $x_k$.  

B. Dynamics

In bottom-up models, one starts from a discrete underlying structure like a graph, and aims to derive structures like spacetime as an emergent phenomenon through basic evolution laws on the discrete system, i.e. on the graph. Then the evolution laws, or the Hamiltonian in the case of quantum graphity, should be defined to make this possible in one way or another. The guidelines for defining the evolution laws or the Hamiltonian are generally based on educated guesses and having the final result of evolution, i.e. emergence of some structure, in mind. In quantum graphity, there are several guidelines to define a Hamiltonian, including the followings:

1. There are two types of Hamiltonians: the first type are the ones that just measure the amount of “energy” of a graph and do not change the linking structure and topology of it, and are called the free Hamiltonians. The second type are those that do change the linking structure and topology of the graph and are called the interaction Hamiltonians. The free Hamiltonians preserve the graph automorphism symmetry, i.e. permutation symmetry: a map of graph $G$ onto itself, $\sigma : G \rightarrow G$, such that the edge $(x_i, x_j)$ is in the domain of $\sigma$, iff the edge $(\sigma(x_i), \sigma(x_j))$ is in the image of $\sigma$.

2. All the Hamiltonians are written in terms of the adjacency matrix or equivalently the number operator and its powers. They also may include the creation and annihilation operators.

3. The evolution should have a preference for global vertex degree $d$ such that after a long “time”, it settles to a state that is of this nature.

The full Hamiltonian of the system then would be the sum of all the free and interacting Hamiltonians. In what follows, we describe in some details the explicit form of the Hamiltonians, suggested by the authors.

1. The free Hamiltonians

In their model, the authors introduce two free Hamiltonians, one related to the valence or degree of vertices (although vertices themselves do not carry any degree of freedom) and another one related to the number of closed non-overlapping paths.

For the first term associated to the vertex degree, there are many possibilities that conform to the criteria 1 and 2 mentioned above, such as the trace of the number operator, or the sum of the off-diagonal elements of it. For item 3 above, first it is desirable to find an expression, based on the number operator (or adjacency matrix) that counts the number of edges attached to a vertex $x_i$, i.e. its vertex degree. This expression is

$$v_i = \text{deg}(x_i) = \sum_j N_{ij}. \quad (8.14)$$

It basically sums over the the elements of the $i$’th row in the matrix of $N_{ij}$, i.e. counts the number of 1’s in row $i$. Using this, they seek to find a Hamiltonian in the form

$$H'_{V} = g_V \sum_i f_i \left( \sum_j N_{ij}, d \right) \quad (8.15)$$

where $g_V$ is a positive coupling constant, and to make the “energy” of a graph with all the vertex degrees equal to $d$ a minimum state (or in other words, the “equilibrium point” of the evolution), the function $f_i$ is defined such that its minimum occurs when vertex $x_i$ has a vertex degree $v_i = d$. This way if all the vertices of the graph have a vertex degree exactly equal to $d$, the function $H'_{V}$ will have its minimum. Based on these observations, the authors then introduce a specific form of this class of Hamiltonians

$$H_V = g_V \sum_i e^{p(d - \sum_j N_{ij})^2} \quad (8.16)$$

where $p$ is a real constant. The exponential is defined by its series expansion in $p$ and they claim that for each deviation of a vertex degree from $d$ such as $v_i \neq d$, there is a “penalty” in energy, i.e. a deviation from the minimum energy which is roughly of the form

$$\delta E_v \sim e^{p(v_i - d)^2}. \quad (8.17)$$
The second free Hamiltonians, $H_B$, measures the contributions to the energy associated to a graph due to closed non-overlapping paths in it. It is a sum over all vertices

$$H_B = \sum_i H_{B,i}$$ (8.18)

such that the term $H_{B,i}$ is centered on vertex $x_i$ and written in the form

$$H_{B,i} = -g_B \sum_j \delta_{ij} e^{r N_{ij}} = -g_B \sum_j \delta_{ij} \sum_{L=0}^{\infty} \frac{r^L}{L!} N_{ij}^L$$ (8.19)

Here $g_B$ is another positive coupling constant, $r \in \mathbb{R}$ and $L$ is the length of a non-overlapping path. The term $N_{ij}^L$ returns the number of paths of length $L$ that connect vertices $x_i$ and $x_j$. Note that as mentioned before, $N_{ij}^L$ does not count overlapping paths and also because of presence of the $\delta_{ij}$, only closed path are counted that start and end on the same vertex. The energy it assigns to a graph state $|\psi_G\rangle$

$$\langle \psi_G | H_B | \psi_G \rangle = E_B(G)$$ (8.20) is

$$E_B(G) = -\sum_i \sum_{L=0}^{\infty} g_B(L) P(x_i, L)$$ (8.21)

where $P(i, L)$ is the number of closed paths of length $L$ at each vertex $x_i$ and $g_B(L)$ is an effective coupling of the form

$$g_B(L) = \frac{r^L}{L!} g_B.$$ (8.22)

Note that $L = 1$ and $L = 2$ terms in (8.19) will vanish since there is no closed path with length 1, and also any closed path of length 2 is an overlapping one which is prohibited by the properties of $N_{ij}$ due to (8.4) as discussed before. Terms with $L = 0$ are also simple constants. So the interesting contributions to $H_B$ start from $L = 3$. Looking at (8.21) we notice that the more closed paths a graph has, the lower energy it possesses. But since the graph is finite, the energy of the system is bounded from below. Also due to the observation that the effective coupling $g_B(L)$ in many cases falls faster than $P(x_i, L)$, very large closed paths do not contribute significantly to the energy of the system. So $H_B$ is a “quasilocal” operator and also the energy per vertex remains finite.

The system exhibits two scales $L^*$ and $L_i^*$ where $L^*$ is the number of closed paths that maximize the effective coupling, i.e.

$$g_B(L^*) > g_B(L), \quad \forall L \neq L^*$$ (8.23)

and $L_i^*$ defined for each vertex $x_i$ is the number that maximizes $g_B(L) P(x_i, L)$, i.e.

$$g_B(L_i^*) P(x_i, L_i^*) > g_B(L) P(x_i, L), \quad \forall L \neq L_i^*$$ (8.24)

which is a graph dependent scale.

So this free Hamiltonian operator has information about the energy of the system due to the number of closed paths with the minimum energy favored by the scale $L_i^*$.

2. Interaction Hamiltonian

The interaction Hamiltonians are the ones that provide the evolution of the states of the graph, and this evolution is what leads to the emergence of specific structures or lead to stable states that can mimic spacetime. One physical guideline for the authors of the model is the locality of these interactions, meaning that given an edge or a vertex, only a small neighborhood of them can be affected by the evolution. As an example, the famous Game of Life has a local evolution where only cells in the first neighborhood of a certain cell are affected by the state of that cell. In the case of quantum graphity, the following paragraphs will make this locality notion more clear.

The authors divide the interaction Hamiltonians into two cases: $H_{(ad)}$ which describes the addition or subtraction of edges between three vertices in which the vertex degree of at least one of the involved vertices changes, and $H_{(exch)}$
which describes the exchange of already existing edges between four neighboring vertices, such that the vertex degrees of the involved vertices remain invariant.

The exchange Hamiltonian is defined as

\[ H_{\text{exch}} = g_{\text{exch}} \sum_{ijkl} N_{ij} \left( a^\dagger_{il} a^\dagger_{jk} a_{jl} a_{ik} \right) \]  

(8.25)

which as can be seen, already possesses the normal ordering mentioned before. Here the prime on the summation indicates that the vertices \( x_i \) to \( x_l \) involved in the Hamiltonian are all different. It is seen easily that the evolution only involves four vertices that are in a very close neighborhood of each other. This evolution deletes the edges between the vertices \( x_i \) and \( x_k \) and between \( x_j \) and \( x_l \), but due to the presence of \( N_{ij} \), all that will happen only if the link between \( x_i \) and \( x_j \) is already on.

The addition/subtraction Hamiltonian is introduced in the following form

\[ H_{\text{add}} = g_{\text{add}} \sum_{ijk} N_{ij} N_{ik} \left( a_{jk} + a^\dagger_{jk} \right) \]  

(8.26)

Here again, due to the presence of \( N_{ij} \) and \( N_{ik} \), given that there is already an on link between \( x_i \) and \( x_j \), and between \( x_i \) and \( x_k \), then a link between \( x_j \) and \( x_k \) is created if there was no link between them before, or it will be deleted if it existed before.

In the following subsection, we discuss an extended version of the model, similar to the older version of quantum graphity, where edges have internal degrees of freedom and will review the idea that this can be used to incorporate matter degrees of freedom.

3. Inclusion of more degrees of freedom; matter

The internal states of edges can be extended by letting them have several “on” states \( |1_s\rangle \) for \( s \in \mathbb{N} \) instead of just one, i.e \( |1\rangle \). Of course the off state is still only \( |0\rangle \). Then the creation and annihilation operators acting on the Hilbert space of each edge will also have an index \( s \) such that

\[ \left\{ a_s, a^\dagger_{s'} \right\} = \delta_{ss'} \]  

(8.27)

This is similar to the case of several types of fermions in a fermionic system. One can then define several Hamiltonians corresponding to these new degrees of freedom.

If one chooses \( s \in \{1, 2, 3\} \), then the system can be cast into a form such that it resembles a spin-1 particle and the states can be represented by

\[ \mathcal{H}_G = \text{span} \{ |j, m\rangle : j \in \{0, 1\}, -j \leq m \leq j \} \]  

(8.28)

Then one can define operators acting on the Hilbert space of each edge, similar to the \( J_z \) and the ladder operators \( J_+ \) and \( J_- \) of the spin-1 particle.

Note that now the edges have internal states and these can be used to describe matter degrees of freedom. This is done by associating \( m = 0 \) to ground states and \( m = \pm 1 \) to the excited states interpreted as matter.

IX. A BRIEF COMPARISON OF THE TWO APPROACHES

In this last section we undertake to describe the characteristics of the two frameworks. It serves at the same time as a conclusion.

- Quantum graphity (QGph) starts with a Hilbert space description and (anti)commutator operator representation on it, which implies that quantum theory (in a general sense; a description based on a Hilbert space etc.) is assumed to be valid at the bottom-most layer of the universe, i.e. the universe is fundamentally quantum. Furthermore the existence of a Hamiltonian is assumed and most of the calculations are performed by employing something like the canonical ensemble in a given heat-bath. In SDCN, on the other hand, the description on the most fundamental level is instead inspired by the discrete dynamics of large disordered networks and cellular automata (CA). The quantum-like behavior of physics is expected to emerge on a higher level. This holds also for most of the other continuum concepts.
In QGph, the degrees of freedom are all associated to edges, and vertices do not play an important role, while in the SDCN the dynamics and the degrees of freedom correspond to both edges and vertices. Exclusion of vertices from having degrees of freedom looks a bit unnatural. It also limits the richness of the model and its ability to produce more interesting physics and complex behavior. It is important that in SDCN the elementary vertex and edge states coevolve under the dynamics, following the philosophy of general relativity, that is, matter degrees of freedom (DoF) interacting with the geometric DoF.

An important difference between the two models is the existence of several coarse-grained layers in the SDCN compared to only one layer in the QGph. In other words, in the SDCN, we not only have an evolution in “time”, but also at each step, there exist several layers in the network (graph) that represent the effective physics at different scales of resolution of the universe. In our view, this makes more sense due to the fact that we already know that at different scales, physical descriptions and variables used may be different. A simple example is the description and the relevant variables in thermodynamics of a gas compared to it statistical mechanics where the variables of the former can be seen as coarse-grained variables of the latter and the dynamical laws also look different.

The cornerstone of the SDCN-framework is the formulation of a geometric renormalization group, with the various steps consisting of coarse graining and rescaling. In this context advanced concepts in graph theory are employed like cliques and clique graphs. We argue that classical space-time and its microstructure (the vacuum fluctuations) occur as a fixed point of such a renormalization process.

We provide arguments that only particular network states can lead to such a classical limit. We call them critical network states. They are characterized by both a local nearorder structure plus a superimposed sparse translocal farorder structure. This picture resembles the famous small world networks. It leads to a finestructure of classical space-time which we dub wormhole spaces. In contrast to this picture QGph assumes that the deep structure of space-time is built by quite regular latticelike graphs. This view is perhaps inspired by the spinnetwork picture in which only relatively simple graphs occur.

The existence of several layers in SDCN also allows us to explain, some bizarre properties of the quantum realm such as superpositions, entanglements and the “spooky action-at-a-distance” as referred to by Einstein. This follows from an analysis of the critical states on the primordial network level which are capable to lead to a macroscopic (quantum) space-time on the surface level with a built-in translocal entanglement structure. Furthermore the emergence of holography and the BH entropy-area law is explained with the help of this translocal wormhole structure. In QGph, since the fundamental description is quantum, these issues are taken for granted or as-is and there is no deeper explanation as to why these phenomena appear in the universe.

Another issue is the notion of dimension. In the SDCN we have an explicit definition of the dimension which can in general adopt arbitrary real values. It can remain invariant or vary after each coarse-graining step (in e.g. the critical network states). We also explore the condition that this dimension is for example integer (these properties are connected with deep mathematical problems). In QGph there is no discussion of this kind and in fact the notion is not explored.

Although the dynamics of both models are guided by educated guesses and desired end results, it seems that the dynamics in QGph is more fine tuned in the sense that the criteria for the equilibrium state look more hand picked. The criteria for a “minimum energy state” there, corresponds to a graph with certain vertex degree and number of closed paths. The final or equilibrium state in the SDCN happen in a more natural and less fine tuned way and it is related to the more or less fixed patterns that appear at the highest coarse-graining level (although they might happen in the lower levels too).

In the face of these observations we think that the two approaches are markedly different but may complement each other.

ACKNOWLEDGMENTS

S.R. would like to acknowledge the support of the PROMEP postdoctoral fellowship (through UAM-I) and the grant from Sistema Nacional de Investigadores of the CONACyT.