A note on “New fundamental relation of hyperrings”

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**Abstract**

In the theory of hyperrings, fundamental relations make a connection between hyperrings and ordinary rings. Commutative fundamental rings and the fundamental relation \(\alpha^*\) which is the smallest strongly regular relation in hyperrings were introduced by Davvaz and Vougiouklis (2007). Recently, another strongly regular relation named \(\theta^*\) on hyperrings has been studied by Ameri and Norouzi (2013). Ameri and Norouzi proved that \(\theta^*\) is the smallest strongly regular relation such that \(R/\theta^*\) is a commutative ring. In this paper, we show that \(\theta^* \neq \alpha^*\) and \(\theta^*\) is not the smallest strongly regular relation. Moreover, we show that some results of Ameri and Norouzi do not hold.

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**1. Hyperrings and fundamental relations**

\((R, +, \cdot)\) is a hyperring if + and \(\cdot\) are two hyperoperations such that \((R, +)\) is a hypergroup, \((R, \cdot)\) is a semihypergroup and the hyperoperation “ \(\cdot\) ” is distributive over the hyperoperation “\(+\)”, which means that for all \(x, y, z\) of \(R\) we have: \(x \cdot (y + z) = x \cdot y + x \cdot z\) and \((x + y) \cdot z = x \cdot z + y \cdot z\). We call \((R, +, \cdot)\) a hyperfield if \((R, +, \cdot)\) is a hyperring and \((R, \cdot)\) is a hypergroup. There are different types of hyperrings. If only the addition \(+\) is a hyperoperation and the multiplication \(\cdot\) is a usual operation, then we say that \(R\) is an additive hyperring. A special case of this type is the Krasner hyperring. We
recall the following definition from [3]. A \textit{Krasner hyperring} is an algebraic structure \((R, +, \cdot)\) which satisfies the following axioms: (1) \((R, +)\) is a canonical hypergroup, i.e., \(x + (y + z) = (x + y) + z\) for all \(x, y, z \in R\); \(x + y = y + x\) for all \(x, y \in R\); there exists \(0 \in R\) such that \(0 + x = x\) for all \(x \in R\); for every \(x \in R\) there exists a unique element \(x' \in R\) such that \(0 \in x + x'\) (we shall write \(-x\) for \(x'\) and we call it the opposite of \(x\)); \(z \in x + y\) implies that \(y \in -x + z\) and \(x \in z - y\); (2) Relating to the multiplication, \((R, \cdot)\) is a semigroup having zero as a bilaterally absorbing element; (3) The multiplication is distributive with respect to the hyperoperation \(+\).

An equivalence relation \(\rho\) is called strongly regular over a hyperring \(R\), if the quotient \(R/\rho\) is a ring.

For a hyperring \(R\), we denote \(\delta_R = \{(x, x) | x \in R\}\) and \(\Delta_R = R \times R\).

At the fourth AHA congress [8] which took place in 1990, Vougiouklis introduced the concept of a fundamental relation on hyperrings, analyzed afterwards by himself and many other authors, for example see [4–6].

\textbf{Remark 1.} A relation \(\rho^*\) is the transitive closure of a binary relation \(\rho\) if

(1) \(\rho^*\) is transitive,
(2) \(\rho \subseteq \rho^*\),
(3) for any relation \(\rho'\), if \(\rho \subseteq \rho'\) and \(\rho'\) is transitive, then \(\rho^* \subseteq \rho'\), that is, \(\rho^*\) is the smallest relation that satisfies (1) and (2).

\textbf{Definition 1.1} ([8]). Let \(R\) be a hyperring. We define the relation \(\Gamma^*\) as follows:

\[
x \Gamma^* y \iff \exists n \in \mathbb{N}, \exists k_i \in \mathbb{N}, \exists (x_{i_1}, \ldots, x_{i_k}) \in R^k, 1 \leq i \leq n \text{ such that } \{x, y\} \subseteq \left( \sum_{i=1}^{n} \left( \prod_{j=1}^{k_i} x_{ij} \right) \right).
\]

\textbf{Theorem 1.2} ([8]). Let \(R\) be a hyperring and \(\Gamma^*\) be the transitive closure of \(\Gamma\). Then, we have

(1) \(\Gamma^*\) is a strongly regular relation both on \((R, +)\) and \((R, \cdot)\).
(2) The quotient \(R/\Gamma^*\) is a ring.
(3) The relation \(\Gamma^*\) is the smallest equivalence relation such that the quotient \(R/\Gamma^*\) is a ring.

Let \(S_n\) be the symmetric group. In 2007, Davvaz and Vougiouklis defined the following notion [4].

\textbf{Definition 1.3.} Let \(R\) be a hyperring and let \(\alpha_0 = \{(x, x) | x \in R\}\). For every integer \(n \geq 1\), \(\alpha_n\) is the relation defined as follows:

\[
x \alpha_0 y \iff \exists (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n, \exists \sigma \in S_n \text{ and } \exists (x_{i_1}, \ldots, x_{i_k}) \in R^k, \exists \sigma_i \in S_{k_i}, (i = 1, \ldots, n) \text{ such that } x \in \sum_{i=1}^{n} \left( \prod_{j=1}^{k_i} x_{ij} \right) \text{ and } y \in \sum_{i=1}^{n} A_{\sigma(i)},
\]

where \(A_i = \prod_{j=1}^{k_i} x_{i\sigma(j)}\).

Obviously, for every \(n \geq 0\), the relation \(\alpha_n\) is symmetric, and \(\alpha = \bigcup_{n \geq 0} \alpha_n\) is reflexive and symmetric.

\textbf{Theorem 1.4} ([4]). Let \(R\) be a hyperring and \(\alpha^*\) be the transitive closure of \(\alpha\). Then, we have

(1) \(\alpha^*\) is a strongly regular relation both on \((R, +)\) and \((R, \cdot)\).
(2) The quotient \(R/\alpha^*\) is a commutative ring.
(3) The relation \(\alpha^*\) is the smallest equivalence relation such that the quotient \(R/\alpha^*\) is a commutative ring.

Ameri and Nozari [2] defined a new strongly regular relation \(\theta\) as follows:
**Definition 1.5.** Let $R$ be a hyperring and $\theta_1 = \{(x, x) | x \in R\}$. For every integer $n \geq 2$, $\theta_n$ is the relation defined as follows:

$$x \theta_n y \iff \exists f \in \mathbb{N}[z_1, \ldots, z_n], \quad \exists \sigma \in \mathbb{S}_n : x \in f \text{ and } y \in \sigma f,$$

where $\mathbb{N}[z_1, \ldots, z_n]$ is the set of all polynomials in the $n$-variables $z_1, \ldots, z_n$ with coefficient in $\mathbb{N}$.

In fact, $f = \sum n_{k_1} \ldots z_{\alpha(1)}^{k_1} \ldots z_{\alpha(n)}^{k_n}$ and $\sigma f = \sum n_{k_1} \ldots z_{\alpha(1)}^{k_1} \ldots z_{\alpha(n)}^{k_n}$. Since for every $n \geq 1$, $\theta_n$ is symmetric, then the relation $\theta = \bigcup_{n \geq 1} \theta_n$ is reflexive and symmetric.

Let $\theta^*$ be the transitive closure of $\theta$. In the following, Theorem 1.6 is an adapted version of results of [2].

**Theorem 1.6.** In any hyperring $R$, we have

1. The relation $\theta^*$ is a strongly regular equivalence (cf. [2], Theorem 3.2).
2. The quotient $R/\theta^*$ is a commutative ring (cf. [2], Corollary 3.4).

Later, Ameri and Norouzi in Theorem 3.1 of [1] proved the next theorem:

**Theorem 1.7.** Let $(R, +, \cdot)$ be a hyperring. Then, the relation $\theta^*$ is the smallest strongly regular equivalence.

Based on other results from [2], the authors conclude that $\theta^*$ is the smallest equivalence relation such that $R/\theta^*$ is a commutative ring.

Now, the following question arises:

**Question:** Are two relations $\alpha^*$ and $\theta^*$ equal?

By Theorems 1.4 and 1.7, we conclude that under same assumptions the relations $\alpha^*$ and $\theta^*$ are the smallest strongly regular relations such that quotients $R/\alpha^*$ and $R/\theta^*$ are commutative rings. Therefore, we should obtain $\alpha^* = \theta^*$.

But, this equality is not true. Indeed, we have $\alpha^* \subset \theta^*$ but $\theta^* \not\subset \alpha^*$. So, the relations $\theta^*$ is not the smallest strongly regular relation such that the quotient $R/\theta^*$ is a commutative ring. The authors proved Theorem 1.7 by induction, but their proof (cf. [1], p. 886, proof of Theorem 3.1) is not correct, since for $n = 2$, $\theta_0 \subseteq \rho$ is not true.

Also, the following examples show that $\alpha^* \neq \theta^*$.

**Example 1.** Let $R = \mathbb{Z}_{64}$. Then, $\alpha^* = \{(x, x) | x \in \mathbb{R}\}$ but $\theta^* \neq \alpha^*$. In $\mathbb{Z}_{64}$ we have $32 = 2^5 \cdot 1$ and $0 = 2^1 \cdot 2^2$ and so $0 \theta^* 32$. It is easy to see that $\{2n|0 \leq n \leq 31\} \subset \theta^*(1)$ and $\{2n+1|0 \leq n \leq 31\} \subset \theta^*(1)$ and $|R/\theta^*| \leq 2$ and $|R/\alpha^*| = 64$.

**Example 2.** Let $(\mathbb{F}, +, \cdot)$ be a finite field and $|F^*| = n$. Then for every $x \in F^*$ we have $x = 1^n \cdot x$ and $1 = x \cdot x$ and so $F^* \subset \theta^*(1)$. But $\alpha^*(1) = \{1\}$ and so $\alpha^* \neq \theta^*$. Therefore $|F/\theta^*| \leq 2$ and $|F/\alpha^*| = n + 1$.

If $\text{char}(F) = p > 4$ then we have $p - 2 = 2^1 \cdot 1^2 + (p - 4)^1 \cdot 1^1$ and $0 = 2^2 \cdot 1^1 + (p - 4)^1 \cdot 1^1$, ($\sigma = (12)$), so $\theta^*(1) = F$ and $\theta^* = F \times F = \Delta_F$ and $\alpha^* = \{(x, x) | x \in F\} = \delta_F$.

Thus, there exist many examples such that $\theta^*$ is not the smallest strongly regular relation such that $R/\theta^*$ is a commutative ring. Therefore, Theorems 3.1, 3.4, 3.9 and Corollary 3.13 in [1] are not true.

Also, Corollary 3.11 and Theorem 3.12 in [1] are not true in a general case. Corollary 3.11 and Theorem 3.12 are true (we refer to Theorems 3.2, 3.3, 3.4 and 3.5 of [7]) only in particular cases, such as for every hyperring $(R, +, \cdot)$ with an identity element for hyperoperation · or for a hyperfield $(\langle R, \cdot \rangle)$ be a hypergroup).
References