Infinitesimal perturbation analysis for optimal production control in a reverse logistic system with different demands

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Abstract—This paper deals with the production control of a manufacturing/ remanufacturing system within a closed loop reverse logistics system with machines subject to random failures and repairs. Three types of inventories are involved in this system. The manufactured and remanufactured products are stored respectively in the manufacturing and remanufacturing inventories. The returned products are collected in the recovery inventory and then remanufactured. The customer demands are constant and known. To describe the system, a stochastic fluid model is adopted and which take into account returned products and remanufacturing products. The objective of this paper is to evaluate the optimal inventory levels of the manufacturing and remanufacturing products that allow minimizing the total cost which is the sum of inventory and lost sales costs. Infinitesimal perturbation analysis is used for optimization of the considered system. The trajectories of the inventories levels are studied and the infinitesimal perturbation analysis estimates are evaluated. These estimates are shown to be unbiased and then they are implemented in an optimization algorithm which determines the optimal inventories levels.

Keywords— reverse logistic system; stochastic fluid model; infinitesimal perturbation analysis; returned products.

I. INTRODUCTION

In recent years, the interest in reverse logistics has increased and the research on the reverse logistic systems has been paying attention on the recovery processes of end of life products for recycling or remanufacturing. Indeed, the reverse logistics has become a matter of strategic importance and which is supposed as an element that companies must consider in decision-making processes concerning the design and development of their supply chains (Flapper et al [1]). We find in the literature many recent studies published on reverse logistic systems and which focus on aspects of production planning and inventory management. Niknejad and Petrovic [2] dealt problem of inventory control and production planning optimization of a generic type of an integrated reverse logistics system which consists of a traditional forward production route, two alternative recovery routes, including repair and remanufacturing and a disposal route. The authors assumed that demand and return quantities are uncertain and they studied the effects of quality of returns and reverse logistic parameters on the network performance and which are: quantity of returned products, unit repair costs, unit production cost, setup costs and unit disposal cost. Jonrinaldi and Zhang [3] proposed a model and solution method for coordinating integrated production and inventory cycles in a whole manufacturing supply chain involving reverse logistics for multiple items with finite horizon period. The considered manufacturing supply chain involving reverse logistic is composed of tier-2 suppliers supplying raw materials to tier-1 suppliers, tier-1 suppliers producing parts, a manufacturer which manufactures and assembles parts from tier-1 suppliers into finished products, distributors distributing finished products to retailers, retailers selling products to end customers and a third party which collects the used finished products from end customers, dissembles collected products into parts, and feed the parts back to the supply chain. Turki et al. [4] studied a manufacturing-remanufacturing system involving reverse logistic with constant demand for evaluating the optimal manufacturing buffer which allows minimizing the total cost. The authors assumed that the remanufactured products meet the same quality level as the new products. However, the most of the works in the literature that study manufacturing-remanufacturing system consider that the remanufactured and manufactured (new) products have the same quality, that means that the manufacturer sells the products that remanufactured and manufactured with same prize. Indeed, in the reality usually this hypothesis is not true and especially when the customer is very demanding and insistent. Therefore, in this paper we consider a manufacturing-remanufacturing system that distinguishes the remanufactured product from the new manufactured product. Indeed, we consider two types of customer demand, one for new products and other for remanufactured products; also we consider two types of inventories for storing separately the remanufactured and the manufactured products. To study the behavior of our considered system we will use stochastic fluid model.

Stochastic fluid model (Cassandras et al [5], Yao and Cassandras [6], Xie, Hennequin, and Mourani [7]) brings an alternative modeling technique to queuing theory with applications including manufacturing and remanufacturing systems. Stochastic fluid models are very simple to study; they allow reducing the simulation time comparing to a classical simulation method. Indeed, this advantage is explained by the fact that the optimization algorithm based on IPA computes at every step the gradient estimates which corresponds to the new value of a parameter of interest. Panayiotou and Cassandras [10]
determined the optimal capacities of the finished goods and work-in-process buffers to minimize the cost function. By using the stochastic fluid model, they estimated the gradient of the cost function with respect to these hedging points for a two stages and single product. Turki et al [11] determined the optimal serviceable inventory level which allows minimizing the sum of inventory and lost sales costs, they adopted a stochastic fluid model to manufacturing/ remanufacturing system composed by two parallel machines, a serviceable inventory, a remanufacturing inventory and customers who demand a stochastic quantity of product. The authors applied the IPA method on the model and determined the gradient estimates, then they proved that these estimates are unbiased. Indeed, the unbiasedness is the principal condition for making the application of IPA useful in practice, since it enables the use of the sample IPA derivative in control and optimization methods that employ stochastic gradient-based techniques. Then, these estimates could be used in stochastic approximation algorithm. In this paper, the IPA estimates will be determined and then they will be used in an optimization algorithm for determining the optimal inventories levels (i.e. manufacturing and remanufacturing inventories).

However, the main contribution of this paper is to apply the IPA method on the stochastic fluid model which describe a manufacturing/remanufacturing system within a closed loop reverse logistics system and take into account of returned products and different demands, then we will derive gradient estimates and we show them their unbiasedness in order to use them in optimization algorithm for determining the optimal inventories levels.

The paper is organized as follows. In section 2, the problem formulation is presented. In section 3, the IPA approach is applied to the stochastic fluid model. In section 4, the IPA estimates are determined, the unbiasedness is proved and numerical results are presented. The last section concludes the paper and gives some perspectives to our work.

II. MODEL AND PROBLEM DEFINITION

In this section, we study a manufacturing/remanufacturing system within a closed loop reverse logistics system (Fig.1) composed by two machines which are subject to random failures and repairs denoted $M_1$ and $M_2$ for manufacturing and remanufacturing, respectively. Production activity in forward direction and reverse logistics is considered in this system (i.e. activity of the remanufacturing of the used products). We assume that remanufactured products have a different quality to the manufactured products (new products). Indeed, in the market we have two different customer demands denoted respectively $d_1$ and $d_2$ and which are satisfied respectively from the manufacturing inventory $S_1$ and the remanufacturing inventory $S_2$. The demands $d_1$ and $d_2$ are known and constant. The inventories $S_1$ and $S_2$ can be filled up respectively by the machines $M_1$ and $M_2$. The recovery inventory $R$ is operated for the stock keeping of the returned products ahead of the remanufacturing process. Returned products will be then remanufactured by the machine $M_2$ and then stoked in $S_2$. We denote $B$ the number of returned products (i.e. the number of the remanufacturable products) and which is constant and proportional to the sum of the demands $d_1$ and $d_2$.

We assume that the machine $M_1$ is never starved means that raw materials are always available. The machine $M_1$ is either down or up. The state of the machine at time $t$, denoted $\alpha(t)$, is given by:

$$
\alpha(t) = \begin{cases} 
1 & \text{machine } M_1 \text{ is up} \\
0 & \text{machine } M_1 \text{ is down} 
\end{cases}
$$

The state of the machine $M_2$ at time $t$, denoted $\beta(t)$, is given by:

$$
\beta(t) = \begin{cases} 
1 & \text{machine } M_2 \text{ is up} \\
0 & \text{machine } M_2 \text{ is down} 
\end{cases}
$$

When the machine is down $u_1(t)=0$ and when the machine is up, the production rate of $M_1$, denoted by $u_1(t)$, could take a value between 0 and its maximum $U_1$ (the machine capacity), i.e., $0 \leq u_1(t) \leq U_1$. The times to repair and times to failure are exponentially distributed with rate $\mu_1$ and $\lambda_1$, respectively. We have the same state for the machine $M_2$, with $u_2(t)$ is the production rate, $U_2$ is the maximal production rate, $\mu_2$ and $\lambda_2$ are times to repair and times to failure of the machine $M_2$.

The following assumptions are considered:

- The maximal production rate of the machine $M_1$ permits to satisfy the demand $d_1$, i.e. $U_1 > d_1$. This assumption allows avoiding having always $S_1$ empty.
- The maximal production rate of the machine $M_2$ permits to satisfy the demand $d_2$, i.e. $U_2 > d_2$. This assumption allows avoiding having always $S_2$ empty.
- The maximal production rate of the machine $M_2$ is upper to the number of returned products, i.e. $U_2 > B$. This assumption allows avoiding having always $R$ very full.
- If the demand $d_1$ is unsatisfied, the demand is lost with a corresponding cost (lost sales cost). The same for the demand $d_2$.
- The failure/repair process is an independent random process. It does not depend on the system parameters.

We assume the case of infinite capacity for $S_1$, $S_2$ and $R$. The levels of the manufacturing, remanufacturing and recovery inventories are denoted, respectively, by $s_1(t)$, $s_2(t)$ and $r(t)$ and are given by the following equations:

$$
\frac{ds_1(t)}{dt} = u_1(t) - d_1 
$$

$$
\frac{ds_2(t)}{dt} = u_2(t) - d_2 
$$

$$
\frac{dr(t)}{dt} = B - u_2(t) 
$$
The control policy is a hedging point policy (Akella and Kumar [12]) and which ensures that the part does not exceed a given number of products, denoted by $h_1$ and $h_2$ for the inventories $s_1(t)$ and $s_2(t)$. The control policy is defined as follows:

For the machine $M_1$:

$$u_1(t) = \begin{cases} U_1 & \text{if } \alpha(t) = 1 \text{ and } s_1(t) < h_1 \\ d_1 & \text{if } \alpha(t) = 1 \text{ and } s_1(t) = h_1 \\ 0 & \text{if } \alpha(t) = 0 \text{ or } s_1(t) > h_1 \end{cases}$$ (6)

For the machine $M_2$:

When $R$ is empty, the machine $M_2$ is supplied directly by the returned products.

$$u_2(t) = \begin{cases} U_2 & \text{if } \beta(t) = 1, \text{ } r(t) > 0 \text{ and } s_2(t) < h_2 \\ d_2 & \text{if } \beta(t) = 1, \text{ } r(t) = 0 \text{ and } s_2(t) = h_2 \\ B & \text{if } \beta(t) = 1, \text{ } r(t) > 0 \text{ and } s_2(t) = h_2 \\ 0 & \text{if } \beta(t) = 0 \text{ or } s_2(t) > h_2 \end{cases}$$ (7)

We assume that the return rate $B$ is proportional to the sum of the demands $d_1$ and $d_2$. We denote by $p$ (0<$p$<1) the percentage of sales which are returned for remanufacturing. Indeed, the customer returns rate may be as high as 15% of sales in the coming years, and in sectors such as catalogue sales and e-commerce it could reach as much as 35% (Rubio and Corominas [13]). Then we have:

$$B = p. (d_1 + d_2)$$ (8)

The numbers of unsatisfied demands (lost) per unit time for the manufactured and remanufactured products are denoted respectively by $D^\alpha_1(t)$ and $D^\alpha_2(t)$ depend on the demands $d_1$ and $d_2$; and the inventories levels $s_1(t)$ and $s_2(t)$. Indeed, for example when the customer orders a demand $d_1$ and $S_1$ is empty; the demand will not be satisfied at all and will be lost. Therefore, $D^\alpha_1(t)$ is null if the inventory level is positive and is equal to the demand $d_1$ if the $S_1$ is empty. The numbers of unsatisfied demands per unit time are defined as follows:

For the manufacturing products:

$$D^\alpha_1(t) = \begin{cases} 0 & \text{if } s_1(t) > 0 \\ d_1 & \text{if } s_1(t) = 0 \end{cases}$$ (9)

For the remanufacturing products:

$$D^\alpha_2(t) = \begin{cases} 0 & \text{if } s_2(t) > 0 \\ d_2 & \text{if } s_2(t) = 0 \end{cases}$$ (10)

The numbers of unsatisfied demands at time $t$ for the manufactured and remanufactured products are denoted respectively by $L_1(t)$ and $L_2(t)$, and are given by:

$$\frac{dl_1(t)}{dt} = D^\alpha_1(t) = d_1 \text{ if } s_1(t) = 0$$

$$L_1(t) = 0 \text{ if } s_1(t) > 0$$ (11)

$$\frac{dl_2(t)}{dt} = D^\alpha_2(t) = d_2 \text{ if } s_2(t) = 0$$

$$L_2(t) = 0 \text{ if } s_2(t) > 0$$ (12)

The cost function $C(t)$, at time $t$, which is composed by the inventory cost and the lost sale cost, is given by:

$$C(t) = cs_1(s_1(t)) + cs_2(s_2(t)) + cr_1 r(t) + cs_r L_1(t) + cs_r L_2(t)$$ (13)

Where:
- $cs_1$, $cs_2$ and $cr$: are the units inventory cost respectively for $S_1$, $S_2$ and $R$;
- $cs_r$: are the units lost sale cost respectively for manufactured and remanufactured products.

The expected average cost, denoted by $J(h_1,h_2)$ depending on $h_1$ and $h_2$ is given by:

$$J(h_1,h_2) = \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C(t) \, dt \right]$$ (14)

$$\forall t \in [0, T] \text{ with } T \text{ the total simulation time.}$$

In the following section, we will investigate the sample path trajectories of $s_1(t)$ and $s_2(t)$. This study will allow us to find the IPA estimates and prove that these estimates are unbiased.

### III. IPA APPROACH

In this section, we apply the IPA method to the stochastic fluid model. Indeed, the IPA is an approach for computing a sample path derivative with respect to an input parameter. This method is intended to estimate gradients of performances metric with respect to some parameters of interest, for example, the optimal inventory level $h_j$ of $S_j$. It consists on observing and analyzing two sample paths, one is the nominal sample path (for example $(s_1(t))$, and the other is the perturbed sample path (for example $(s_1^\epsilon(t))$) (see Fig. 2). We assume that the optimal manufacturing inventory level is increased by a perturbation, denoted by $\eta$. In this work, we consider $\eta > 0$ and we study the resulting changes in the cost function using geometric arguments (similar results could be easily obtained for $\eta < 0$). The optimal inventory level of the perturbed sample path $(s_1^\epsilon(t))$ is $h_1 + \eta$. We assume also that the optimal remanufacturing inventory level $h_2$ is increased by a perturbation, denoted by $\eta$.

The following assumptions are considered:

- The perturbation of $h_1$ is $\eta$.
- The maximal production and unit costs are the same for both sample paths.
- The same distribution of random variables (time to failure, time to repair) is used for both sample paths.

In the following – (Fig. 2), we give an example of simple paths for $s_1(t)$ and $s_1^\epsilon(t)$.

The following notations are used:
- $s_1(t)$: The manufacturing inventory level for the perturbed path;
- $s_1^\epsilon(t)$: The remanufacturing inventory level for the perturbed path;
- $r^\epsilon(t)$: The recovery inventory level for the perturbed path;
- \( u^2(t) \): The production rate of the machine \( M_1 \) at time \( t \) for the perturbed path;
- \( u^3(t) \): The production rate of the machine \( M_2 \) at time \( t \) for the perturbed path;
- \( D^u(t) \): The number of unsatisfied manufacturing demands per unit time for the perturbed trajectory.
- \( D^s(t) \): The number of unsatisfied remanufacturing demands per unit time for the perturbed trajectory.
- \( t_{s1}^1 \): \( j^1 \)th instant for which \( S_1 \) on the nominal path becomes full and causing a lag between the perturbed path and the nominal path;
- \( t_{s2}^1 \): \( j^1 \)th instant for which \( S_1 \) on the disturbed path becomes full.
- \( t_{s1}^2 \): \( j^2 \)th instant for which \( S_1 \) on the perturbed path becomes empty and for which the perturbed and the nominal path merge;
- \( t_{s2}^2 \): \( j^2 \)th instant for which \( S_1 \) on the nominal path becomes full and for which the perturbed and the nominal path merge;
- \( t_{s1}^3 \): is the last instant between \( t_{s1}^1 \) and \( t_{s2}^1 \) for which the manufacturing inventory becomes empty on the nominal path;
- \( t_{s2}^3 \): \( j^3 \)th instant for which \( S_2 \) on the nominal path becomes full and causing a lag between the perturbed path and the nominal path;
- \( t_{s1}^4 \): \( j^4 \)th instant for which \( S_2 \) on the disturbed path becomes full.
- \( t_{s2}^4 \): \( j^4 \)th instant for which \( S_2 \) on the perturbed path becomes empty and for which the perturbed and the nominal path merge;
- \( t_{s1}^5 \): is the last instant between \( t_{s1}^2 \) and \( t_{s2}^2 \) for which the remanufacturing inventory becomes empty on the nominal path;
- \( t_{s2}^5 \): \( j^5 \)th instant which causes a lag between the perturbed path and the nominal path on the recovery inventory \( R \);
- \( t_{s1}^6 \): \( j^6 \)th instant for which \( R \) on the perturbed path becomes empty and for which the perturbed and the nominal path merge.

![Inventory level (S)](image)

**Theorem 1:** If \( \eta > 0 \), \( s^1_1(0) = s_j(0) \) we have:
- If \( t \in \left( t_{s1}^1, t_{s2}^1 \right) \), then \( s^1_2(t) = s_j(t) + \phi(t) \).
- If \( t \in \left( t_{s2}^1, t_{s1}^2 \right) \), then \( s^1_2(t) = s_j(t) \).

The proof of this theorem is similar to the proof of theorem 1 and theorem 2 in [14].

**Theorem 2:** If \( \eta > 0 \), \( s^2_1(0) = s_j(0) \) we have:
- If \( t \in \left( t_{s1}^2, t_{s2}^2 \right) \), then \( s^2_2(t) = s_j(t) + \phi(t) \).
- If \( t \in \left( t_{s2}^2, t_{s1}^3 \right) \), then \( s^2_2(t) = s_j(t) \).

Also the proof of this theorem is similar to the proof of theorem 1 and theorem 2 in [14].

**Theorem 3:** If \( \eta > 0 \), \( r^1(0) = r_0(0) \) we have:
- If \( t \in \left( t_{s1}^1, t_{s2}^1 \right) \), then \( r^2(t) = r_0(t) - \phi(t) \).
- If \( t \in \left( t_{s2}^1, t_{s1}^2 \right) \), then \( r^2(t) = r_0(t) \).

The proof of this theorem is similar to the proof of theorem 1.

**Theorem 4:** If \( \eta > 0 \), \( s^1_2(0) = s_j(0) \) we have:
- If \( t \in \left( t_{s1}^1, t_{s2}^1 \right) \), then \( L_1(t) = L_j(t) - \phi(t) \).
- If \( t \in \left( t_{s2}^1, t_{s1}^2 \right) \), then \( L_1(t) = L_j(t) \).

The proof of this theorem is similar to the proof of theorem 3 in [11].

**Theorem 5:** If \( \eta > 0 \), \( s^2_2(0) = s_j(0) \) we have:
- If \( t \in \left( t_{s1}^2, t_{s2}^2 \right) \), then \( L_1(t) = L_j(t) - \phi(t) \).
- If \( t \in \left( t_{s2}^2, t_{s1}^3 \right) \), then \( L_1(t) = L_j(t) \).

The proof of this theorem is similar to the proof of theorem 4.

In what follows, we will determine the IPA estimates and will prove their unbiasedness.
IV. IPA FOR OPTIMIZATION

In this section, we will use the results of the trajectories study for determining the IPA estimates; these estimates will be implemented in an optimization algorithm, which allows us to determine the value of \( h_1 \) and \( h_2 \). Indeed, the values of IPA estimates allow orienting quickly the algorithm to the optimal values of \( h_1 \) and \( h_2 \). Therefore, the advantage of our optimization algorithm based on IPA compared with an exhaustive optimization algorithm is that it takes smaller computational time (simulation time).

A. IPA estimates

The average cost of the nominal path is given by:

\[
J(h, h) = \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C(t) dt \right]
\]

The average cost of the perturbed path is given by:

\[
J^\eta (h + \eta ; h + \eta) = \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C^\eta (t) dt \right]
\]

With

\[
C^\eta (t) = cs_j, s_j^\eta (t) + cs_j, s_j^\eta (t) + cr, r^\eta (t) + cs_j, L^\eta_j (t) + cs_j, L^\eta_j (t)
\]

The sampled estimation for the expected average cost of the nominal path is given by:

\[
\hat{J}_s (h, h) = \frac{1}{T} E \left[ \int_0^T \hat{C}(t) dt \right]
\]

The sampled estimation for the expected average cost of the perturbed path is given by:

\[
\hat{J}^\eta_s (h + \eta; h + \eta) = \frac{1}{T} E \left[ \int_0^T \hat{C}^\eta (t) dt \right]
\]

We determine the IPA estimates of the cost function by computing the difference between the perturbed average cost and the nominal average cost.

The difference between the perturbed expected average cost and the nominal expected average cost is given by:

\[
J_s^\eta (h + \eta; h + \eta) - J_s (h, h) = \frac{1}{T} E \left[ \int_0^T (C^\eta (t) - C(t)) dt \right]
\]

\[
J_s^\eta (h + \eta; h + \eta) - J_s (h, h) = \frac{1}{T} E \left[ \int_0^T \left( cs_j, (s_j^\eta (t) - s_j(t)) + cs_j, (s_j^\eta (t) - s_j(t)) + cr, (r^\eta (t) - r(t)) + cs_j, (L^\eta_j (t) - L_j(t)) + cs_j, (L^\eta_j (t) - L_j(t)) \right) dt \right]
\]

We assume that in the interval \([0, T]\) we have \( m \) intervals \([t_{ij}, t_{ij+1}]\), \( n \) intervals \([t_{ij}, t_{ij+1}]\), \( g \) intervals \([t_{ij}, t_{ij+1}]\), \( w \) intervals \([t_{ij}, t_{ij+1}]\), \( p \) intervals \([t_{ij}, t_{ij+1}]\), then we have:

- For the trajectories of \( s_j(t) \) and \( s_j^\eta (t) \) the interval \([0, T]\) is divided on two sums of periods: \( T_s^\eta \) is the sum of periods when \( t \in \bigcup_{j=1}^{j\eta} (t_{ij}, t_{ij+1}] \) and \( T_s \) is the sum of periods when

\[
t \in \bigcup_{j=1}^{j\eta} (t_{ij}, t_{ij+1}] \quad (\text{when } s_j^\eta (t) = s_j(t) + \phi(t)) \quad \text{i.e.}
\]

\[
T_s^\eta = \sum_{j=1}^{j\eta} (t_{ij+1} - t_{ij}) \quad \text{and} \quad T_s = \sum_{j=1}^{j\eta} (t_{ij+1} - t_{ij})
\]

\[
T_i = \sum_{j=1}^{j\eta} (t_{ij+1} - t_{ij}) \quad \text{. Thus, according to the theorems 1 we have:}
\]

\[
\int_0^T cs_j, (s_j^\eta (t) - s_j(t)) dt = cs_j, T_s^\eta \phi(t)
\]

In the IPA theory, for simplifying the study we assume that \( \phi(t) = \eta \).

Then we have

\[
\int_0^T cs_j, (s_j^\eta (t) - s_j(t)) dt = cs_j, T_s^\eta \eta
\]

- For the trajectories of \( r(t) \) and \( r^\eta (t) \) the interval \([0, T]\) is divided on two sums of periods: \( T_r^\eta \) is the sum of periods when \( t \in \bigcup_{j=1}^{j\eta} (t_{ij}, t_{ij+1}] \) and \( T_r \) is the sum of periods when \( t \in \bigcup_{j=1}^{j\eta} (t_{ij}, t_{ij+1}] \) (when \( s_j^\eta (t) = s_j(t) + \phi(t) \) i.e.

\[
T_r^\eta = \sum_{j=1}^{j\eta} (t_{ij+1} - t_{ij}) \quad \text{and} \quad T_r = \sum_{j=1}^{j\eta} (t_{ij+1} - t_{ij})
\]

\[
\int_0^T cr, (r^\eta (t) - r(t)) dt = -(cr, T_r^\eta \eta)
\]

- For the trajectories of \( L_j(t) \) and \( L_j^\eta(t) \) the interval \([0, T]\) is divided on two sums of periods: \( T_L^\eta \) is the sum of periods when \( t \in \bigcup_{j=1}^{j\eta} (t_{ij}, t_{ij+1}] \) and \( T_L \) is the sum of periods when \( t \in \bigcup_{j=1}^{j\eta} (t_{ij}, t_{ij+1}] \) (when \( s_j^\eta (t) = s_j(t) \) i.e.

\[
T_L^\eta = \sum_{j=1}^{j\eta} (t_{ij+1} - t_{ij}) \quad \text{and} \quad T_L = \sum_{j=1}^{j\eta} (t_{ij+1} - t_{ij})
\]

\[
\int_0^T cs_j, (L_j^\eta (t) - L_j(t)) dt = -(cs_j, T_L^\eta \eta)
\]

Then we have

\[
J_s^\eta_s (h + \eta; h + \eta) - J_s (h, h) = \frac{1}{T} E \left[ \int_0^T cs_j, T_s^\eta + cs_j, T_s^\eta - cr, T_r^\eta - cs_j, T_r^\eta - cs_j, T_s^\eta \right]
\]

Then, the gradient estimates of the cost function are given by:

\[
\frac{\partial J_s (h, h)}{\partial h} = \frac{1}{T} E \left[ cs_j, T_s^\eta + cs_j, T_s^\eta - cr, T_r^\eta - cs_j, T_r^\eta - cs_j, T_s^\eta \right]
\]
For making these estimates useful in practice, the unbiasedness should be proved.

**Theorem 6:** The gradient estimates of the average cost are unbiased.

The proof of this theorem is similar to the proof of theorem 12 in [14].

In the followings, numerical results are presented to show the impact of the percentage of the returned products on \( h_1 \) and \( h_2 \).

### B. Optimization Using IPA Estimates

In this part, we are interesting to study the impact of the percentage of the returned products \( p \) on the values of \( h_1 \) and \( h_2 \). While the cost function depends on the return rate \( B \), the values of \( h_1 \) and \( h_2 \) which minimize this cost function will certainly also depend on the percentage of the returned products. Thus, we change the values of the percentage of the returned products and we determine the values of \( h_1 \) and \( h_2 \) by using the IPA optimization algorithm.

The following parameters are used for the simulation:

- \( U_1 = 10 \) products /time unit;
- \( U_2 = 6 \) products /time unit;
- \( d_1 = 8 \) products /time unit;
- \( d_2 = 5 \) products /time unit
- The total simulation time is equal to \( T = 10E + 07 \) time units;
- The times to failure or repair are given by exponential distribution, the mean time between failures MTBF is equal to 2.8 and the mean time to repair MTTR is equal to 1.2;
- The unit inventory cost \( cs_1 \) is equal to 1 monetary unit;
- The unit inventory cost \( cs_2 \) is equal to 1 monetary unit;
- The unit inventory cost \( cr \) is equal to 1 monetary unit;
- The unit lost sales cost \( cs_1 \) is equal to 50 monetary units;
- The unit lost sales cost \( cs_2 \) is equal to 50 monetary units;

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<tr>
<th>( p )</th>
<th>( B )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>1.3</td>
<td>14.31</td>
<td>1.58</td>
</tr>
<tr>
<td>20%</td>
<td>2.6</td>
<td>11.09</td>
<td>2.68</td>
</tr>
<tr>
<td>30%</td>
<td>3.9</td>
<td>7.86</td>
<td>4.09</td>
</tr>
<tr>
<td>40%</td>
<td>5.2</td>
<td>3.07</td>
<td>7.11</td>
</tr>
<tr>
<td>50%</td>
<td>6.5</td>
<td>1.19</td>
<td>8.24</td>
</tr>
</tbody>
</table>

As we see, the optimal manufacturing inventory level \( h_1 \) decreases when the percentage of the returned products increases (i.e. the number of remanufacturable products). Besides, the optimal remanufacturing inventory level \( h_2 \) increases when the percentage of the returned products increases. Indeed, when the number of returned products increases, the recovery inventory \( R \) fills up and then the machine \( M_2 \) can fill more remanufacturing inventory \( S_2 \). Therefore, the demand \( d_2 \) is more satisfied and the unsatisfied demand decreases and normally the lost sales cost decreases; consequently, the optimal remanufacturing inventory level which minimizes the total cost increases.

### V. CONCLUSION

In this paper, we studied a manufacturing/remanufacturing system within a closed loop reverse logistics system with machines subject to random failures and repairs. Two different demands are considered: demand for manufacturing products and the other for remanufacturing products. These demands are constant and known. Stochastic fluid model is adopted to describe the system and which take into account returned products and remanufacturing products. The trajectories of the inventories levels are studied and the infinitesimal perturbation analysis estimates are evaluated. These estimates are shown to be unbiased and then they are implemented in an optimization algorithm which determines the optimal inventories levels. These optimal inventories levels allow minimizing the total cost which is the sum of inventory and lost sales costs. The impact of the percentage of the returned products on the values of \( h_1 \) and \( h_2 \) is studied. Indeed, the optimal manufacturing inventory level \( h_1 \) decreases when the percentage of the returned products increases. However, the optimal remanufacturing inventory level \( h_2 \) increases when the percentage of the returned products increases.

For future research, we will consider a more complex model with random customer demands and random number of returned products (i.e. return rate).

### REFERENCES


