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Stable Extending Modules

Mehdi Sadik Abbas and Saad Abdulkadhim Al-Saadi

Department of Mathematics, College of Science, Al- Mustansiriyah University, Iraq
E-mail: amsaj59@yahoo.com E-mail: saadalsaadi08@yahoo.com

Abstract

Let R be a ring and M be an R-module. Recall that M is extends if, every submodule of M is essential in a direct summand of M. Since 1980s, the developments of modules with the extending property have been a major area in ring and module theory. In this paper, we introduce and study a proper generalization of extending modules. We call an R-module M a stable extending if every stable submodule of M is essential in a direct summand of M. Many characterizations and properties of stable extending modules are given. Moreover, it is well known that a direct sum of extending modules need not be extending. Unlike extending modules, we assert that a direct sum of stable extending modules is a stable extending.

Introduction.

Throughout this paper all rings have an identity and modules are unitary. Let R be a ring and M be a left R-module. A submodule N of M is essential if every non-zero submodule of M intersects N nontrivially. Also, a submodule N of M is closed in M, if it has no proper essential extensions in M [3] By Zorn's lemma any submodule of M is contained in a maximal essential extension (a closed submodule) in M.

Recall that an R-module M is extending if, every submodule of M is essential in a direct summand of M. In 2002 Birkenmeier, Muller and Rizvi [2] introduced FI-extending modules as generalization of extending modules. An R-module M is called FI-extending if, every fully invariant submodule of M is essential in a direct summand of M (recall that a submodule N of an R-module M is fully invariant if $f(N) \subseteq N$ for each $f \in End_R(M)$ [19]).

On the other hand, M. S. Abbas introduced and studied stable submodules which are properly stronger than that of fully invariant submodules [1]. A submodule N of an R-module M is called stable if, $f(N) \subseteq N$ for each R-homomorphism $f: N \longrightarrow M$. An R-module M is called fully stable if each submodule of M is stable.

All above motivate us to introduce the following concept which represents a generalization of extending modules (respectively, FI-extending modules).

Definition (1.1): An R-module M is called stable extending (shortly, S-extending) if every stable submodule of M is essential in a direct summand of M.
A ring R is left (right) S-extending if, R is S-extending left (right) R-module.

Recall that an R-module M is uniform if, every submodule of M is essential in M [6, p.85]. As a generalization of uniform modules we introduce the following concept:

Definition (1.2): An R-module M is called stable uniform (shortly, S-uniform) if every stable submodule of M is essential in M.

Remarks and Examples (1.3):

(1) Every extending module (respectively, FI-extending module) is S-extending, while the converse is not true in general.

(2) Every S-uniform module (and hence uniform module) is S-extending. In particular, the ring of integers Z over itself is S-extending.

(3) Consider $M = Z_8 \oplus Z_2$ as Z-module. Since Z_8 and Z_2 are uniform Z-modules, then they are S-extending. By theorem (2.1), $M = Z_8 \oplus Z_2$ is S-extending. But M is not extending since a submodule $N = \{(\bar{0},\bar{0}), (\bar{2},\bar{1}), (\bar{4},\bar{0}), (\bar{6},\bar{1})\}$ is a closed submodule of M which is not direct summand.

(4) We call an R-module M is S-indecomposable if, M and (0) are the only stable direct summands of M. Clearly every indecomposable module is S-indecomposable, but the converse is not true. For example, the vector space $V = F^{(2)}$ over a field F is S-indecomposable F-module which it is not indecomposable since $V = S \oplus S'$ where $S = \{(\alpha, 0) | \alpha \in F\}$ and $S' = \{(0,\beta) | \beta \in F\}$.

(5) It is easy to prove that, an R-module M is S-uniform if and only if M is S-extending and S-indecomposable.

(6) By using the definition (1.1), on can easily see that an R-module M is S-extending if and only if every stable submodule of M lies under a direct summand of M (i.e. for each stable submodule of M, there exists a direct decomposition $M = M_1 \oplus M_2$ with $N \subseteq M_1$ and N is essential in M).

(7) A direct product of S-extending modules need not be an S-extending. For example, consider the Z-module $M = \prod_{p \in P} Z/pZ$ (where P be the set of all primes). Clearly, Z/pZ is S-extending Z-module for each $p \in P$. By [1] the torsion subgroup $\tau(M)$ is a stable submodule of M and it is known that $\tau(M)$ is a closed submodule of M. But, by [12, theorem 10.2], $\tau(M = \prod_{p \in P} Z/pZ)$ is not a direct summand of $M = \prod_{p \in P} Z/pZ$. Hence, M is not S-extending.

(8) It is clear that if M is a fully stable module, then M is extending module if and only if M is S-extending module.

In the following, we obtain a characterization for S-extending modules.

Theorem (1.4): An R-module M is S-extending if and only if for each stable submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ and $A + M_2$ is essential submodule of M.

Proof: ($\Rightarrow$). Suppose that M is S-extending R-module. Let A be a stable submodule of M. Since M is S-extending, then A is essential in a direct summand (say) K_1 of M (i.e.) $M = K_1 \oplus K_2$, where K_2 is a submodule of M. Also, since A is essential in K_1 and K_2 is essential in K_2, thus $A + K_2$ is essential in $K_1 + K_2 = M$. Hence, $A + K_2$ is essential in M.

($\Leftarrow$). Let A be a stable submodule of M. By hypothesis, there is a decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ and $A + M_2$ essential in M. We claim that A is essential submodule of M_1. Let K be a non-zero sumodule of M_1, thus K is a submodule of M. Hence $(A + M_2) \cap K \neq (0)$ (since $A + M_2$ is essential in M). Let $k = a + m \ (\neq 0)$, where $k \in K$, $a \in A$ and $m \in M$, thus $m = k - a$ which implies $m \in M_1 \cap M_2 = (0)$, so we have $k = a \in K \cap A$, then $K \cap A \neq (0)$, and hence A is essential in M_1. Therefore, M is S-extending R-module.□

It is well-known that an R-module M is quasi-injective if and only if $f(M)\subseteq M$ for each $f\in End_R(E(M))$ [10, Theorem (6.73)]. Moreover, an R-module is quasi-continuous (or π-injective) if and only if $f(M)\subseteq M$ for each idempotent $f\in End_R(E(M))$ [4]. Next we provide another characterization of S-extending modules.

Proposition (1.5): An R-module M is S-extending if and only if for each stable submodule K of M, there exists $e=(e^2)\in End_R(E(M))$ such that K is essential in $e(E(M))$ and $e(M)\subseteq M$.

Proof: $(\Rightarrow)$. Assume that M is S-extending. Let K be a stable submodule of M. Then, there is a direct summand D of M such that K is essential in D and so there is a submodule H of M such that $M=D\oplus H$. Thus, we have $E(M)=E(D)\oplus E(H)$ [10]. Let $e:E(M)\longrightarrow E(D)$ be the projection endomorphism of $E(M)$ onto $E(D)$. Clearly, e is an idempotent $(e=e^2)$.Thus, we have $e(M)\subseteq e(D\oplus H)\subseteq D\subseteq M$. Now, since K is essential in D and also D is essential submodule of $E(D)$,then K is essential in $E(D)=e(E(M))$.

$(\Leftarrow)$. Let K be a stable submodule of M. By hypothesis, there is $e=e^2\in End_R(E(M))$ such that K is essential in $e(E(M))$ and $e(M)\subseteq M$. Now, since M is essential in M, thus we have $K=K\cap M$ is essential in $M\cap e(E(M))=e(M)$. But $e(M)$ is a direct summand of M [15, Lemma (8.3)]. Hence, M is S-extending. □

Goldie [5], Johnson and Wong [8], defined for each submodule N of an R-module M, a submodule of M as follows:

$Cl(N)=\{m\in M \mid [N:m]$ is an essential ideal of $R\}$. Equivalently, $Cl(N)=\{m\in M \mid Im\subseteq N$ for some essential ideal I of $R\}$. $Cl(N)$ is a submodule of M. It is called the closure of N (in sense of Goldie), clearly $N\subseteq Cl(N)$.

It is not true that each submodule N is essential in $Cl(N)$. For example, consider Z_6 as Z-module. Let $A=\{\bar{0},\bar{3}\}$. Then, it is an easy to check that $Cl(A)=Z_6$. But A is a direct summand of Z_6 and hence A is not essential in Z_6 (i.e.) A is not essential in $Cl(A)$.

In the next result, we investigate another characterization of S-extending modules by using an extra condition.

Proposition (1.6): Let M be an R-module such that each submodule of M is essential in its closure. Then, M is S-extending if and only if for any submodule N of M with stable closure is essential in a direct summand of M.

Proof :$(\Rightarrow)$. Let N be a submodule of M with stable closure. By S-extending property of M, $Cl(N)$ is essential in a direct summand X of M . But by hypothesis, every submodule of M is essential in its closure (i.e.) N is essential in $Cl(N)$ and hence N is essential in a direct summand X of M.

$(\Leftarrow)$. Let N be a stable submodule of M. By [1,proposition(1.11)], $Cl(N)$ is a stable submodule of M. Thus, by hypothesis, N is essential in a direct summand of M. Therefore, M is S-extending.□

Since every submodule of a non-singular module is essential in its closure [10, p.259], thus we have the next corollary:

Corollary (1.7): Let M be a non-singular R-module. Then, M is S-extending if and only if for any submodule N of M with stable closure is essential in a direct summand of M. □

We observed that every extending module is S-extending module, but the converse is not true in general (Remarks and Examples (1.3) (3)). On the other hand, the closure of an arbitrary submodule need not be stable [1]. In the following result, we give a condition under which the concepts of extending modules and S-extending modules are equivalent.

Proposition (1.8): Let M be an R-module such that the closure of any submodule N of M is a stable and essential extension of N. Then, M is extending if and only if M is S-extending.

Proof: $(\Rightarrow)$. It is obvious.

($\Leftarrow$). Let N be a submodule of M. Since $Cl(N)$ is stable of M, thus by S-extending property of M, $Cl(N)$ is essential in a direct summand D of M. Since, by hypothesis, N is essential in $Cl(N)$, thus N is essential in D. Therefore, M is extending.□

Corollary (1.9): Let M be a non-singular module such that the closure of any submodule of M is stable. Then, M is extending if and only if M isS-extending. □

It is well-known that an R-module M is extending if and only if every closed subomdule of M is a direct summand [3, p.55]. One can raise a question about analogue result for S-extending modules. The following proposition gives a partial answer of this question.

Proposition (1.10): Let M be a non-singular R-module. Then, the following conditions are equivalent:

(1) M is S-extending;

(2) Every closed stable submodule of M is a direct summand;

(3) Every stable submodule of M is essential in a stable direct summand of M.

Proof: **(1) $\Rightarrow$ (2)**. Suppose that M is S-extending R-module. Let N be a closed stable submodule of M. By S-extending property of M, there exists a direct summand D of M such that N is essential in D. But N is closed in M, so $N = D$. Hence, N is a direct summand.

(2) $\Rightarrow$ (3). Let A be a stable submodule of M. Since M is a non-singular, thus there exists a closed submodule $Cl(A)$ such that A is essential in $Cl(A)$ [10, p.259]. But A is a stable submodule of M, then $Cl(A)$ is a stable of M [1,proposition(1.11)]. Thus, by (2), $Cl(A)$ is a direct summand of M (i.e.) A is essential in a stable direct summand ($Cl(A)$) of M. Hence, M is S-extending.

(3) $\Rightarrow$ (1). It is clear. □

Remark (1.11):

(1) In the proof of (3) $\Rightarrow$(1) $\Rightarrow$(2) the condition of non-singularity of M is not necessary.

(2) The condition of non-singularity of M (except (3) $\Rightarrow$(2)$\Rightarrow$(1)) in proposition (3.1.10) can not be removed. For example, let $M = Z\oplus Z_3$ as Z-module. Thus, M is S-extending (by theorem (2.1)), not non-singular [10] and does not satisfy (3).

It is known that, an R-module M is extending if and only if for each direct summand A of $E(M)$ then $A\cap M$ is a direct summand of M [3, p.20]. The following result establishes connection between S-extending modules and their injective hulls.

Proposition (1.12): If M is S-extending R-module, then for each stable direct summand A of $E(M)$, $M\cap A$ is a stable direct summand of M.

Proof: Let M be an S-extending R-module and $K = M\cap A$ where A is a stable direct summand of $E(M)$. Firstly, to show that K is stable of M. Let $f \in Hom_R(K,M)$. By injectivity of $E(M)$, there exists $g\in End_R(E(M))$ which extends f. Let $k\in K$, then $f(k) \in M$,so, since $f(k) = g(k)$ and A is a stable of $E(M)$, then $f(k) \in A$. Hence, we have $f(k) \in K = M\cap A$. Thus, K is a stable of M. Now, we claim that $K = M\cap A$ is a direct summand of M. Since M is S-extending, then there exists a direct summand D of M such that K is essential in D. Since A is a direct summand of $E(M)$, then A is injective R-module. Since $K\subseteq A$, thus $E(K) = A$ (since $K=M\cap A \xrightarrow{e} A\cap E(M)=A$) and $E(D)\cong A$. Also, by stability of A, we have $E(D) = A$. Hence, $D\subseteq M\cap E(D) = M\cap A = K$ .So $K = D$. Therefore, $K = M\cap A$ is a stable direct summand of M. □

We do not know whether the converse of proposition (1.12) is true in general. In the next result, we obtain a condition under which the converse is true. Firstly, we need to introduce the following concept:

Definition (1.13): A submodule N of an R-module M is called hyperstable if, $E(N)$ is stable in $E(M)$. M is called fully hyperstable if each stable submodule of M is hyper stable.

Note that full stability and full hyperstability are different concepts. In fact, Z_6 as Z-module is fully stable [1], while it is not fully hyperstable since $E(\{\bar{0},\bar{3}\}) \cong Z_{2^\infty}$ is not stable submodule of $E(Z_6) \cong Z_{2^\infty} \oplus Z_{3^\infty}$. On the other hand, Z as Z-module is fully hyperstable which is not fully stable since $2Z$ is not stable submodule of Z_Z. Moreover, by [1, Theorem (2.15)], every fully stable uniform module over a Noetherian ring is fully hyperstable.

Now, we are ready to obtain the result which is mentioned.

Proposition (1.14): Let M be a fully hyperstable R-module. Then, the following statements are equivalent:

(1) M is S-extending;

(2) $M \cap X$ is a stable direct summand of M for each stable direct summand X of $E(M)$;

(3) Every stable submodule of M is essential in a stable direct summand of M.

Proof:

(1) $\Rightarrow$ **(2)**. By using proposition (1.12).

(2) $\Rightarrow$ **(3)**.Let A be stable submodule of M. Let B be a relative complement of A in M. Hence $A \oplus B$ is essential in M and M is essential in $E(M)$ [15], so $A \oplus B$ is essential in $E(M)$. Thus $E(A) \oplus E(B) = E(A \oplus B) = E(M)$. Also, by full hyperstability of M, $E(A)$ is a stable of $E(M)$. Hence, by (2), $E(A) \cap M$ is a stable direct summand of M. Since A is essential in $E(A)$ and M is essential in M, thus $A = A \cap M$ is essential in $E(A) \cap M$. Therefore, M is S-extending.

(3) $\Rightarrow$ **(1)**. It is obvious.□

Recall that a ring R is PP-ring if, every principal ideal of R is projective (as an R-module) [10] (recall that an R-module M is projective, if for each R-epimorphism $g: A \longrightarrow B$ (where A, B are R-modules) and for each R-homomorphism $f: M \longrightarrow B$, there exists an R-homomorphism $h: M \longrightarrow A$ such that $g \circ h = f$ [9, p. 117]).

It is known that every left non-singular left extending ring is left PP-ring [3, p.105]. Moreover, left non-singular left S-extending ring need not be extending (see [2, Example (2.6)]).The following result generalizes this result to S-extending rings.

Proposition (1.15): Every left non-singular left S-extending ring is left PP-ring.

Proof: Let R be a left non-singular left S-extending ring and let I be a principle left ideal of R (i.e.) $I = Rx$ where $x \in R$. Now, consider the following exact sequence: $0 \longrightarrow ann_R(x) \xrightarrow{i} R \xrightarrow{f} Rx \longrightarrow 0$ where i is the inclusion homomorphism and $f(r) = rx$ for all $r \in R$ (it is clear that f is well-defined epimorphism). Since $ann_R(x)$ is a stable left ideal of R [1] and R is left S-extending, thus $ann_R(x)$ is essential in a direct summand H of R. We claim that $ann_R(x) = H$. To see this, let $0 \neq r \in H$ and let $L = \{t \in R \mid tr \in ann_R(x)\}$, it is can be easily check that L is essential in R. Since $L \subseteq ann_R(x) \subseteq R$, then $ann_R(x)$ is essential in R. But R is left non-singular, then $rx = 0$ and hence $r \in ann_R(x)$. Thus, $ann_R(x) = H$. So the above mentioned sequence splits and hence Rx is projective. Therefore, R is left PP-ring. □

It is known that, every PP-ring is non-singular [10, Example (7.6) (5)]. Thus, we have the next corollary:

Corollary (1.16): If a ring R is left S-extending, then R is left non-singular if and only if R is left PP-ring.□

In this part, we discuss conditions under which every S-extending module is FI-extending.

Recall that an R-module M is quasi-injective if for each submodule N of M, each R-homomorphism $f: N \longrightarrow M$ can be extended to an R-endomorphism $g: M \longrightarrow M$ [15]. We introduce the following concept as a generalization of quasi-injective modules.

Definition (1.17): An *R*-module *M* is called FI-quasi-injective if, for each fully invariant submodule *N* of *M*, each R-homomorphism $f: N \longrightarrow M$ can be extended to an *R*-endomorphism $g: M \longrightarrow M$.

Remark and Examples (1.18):

(1) Every quasi-injective module is FI-quasi-injective. The converse is not true in general (we have no example yet)

(2) If an *R*-module *M* is duo, then *M* is quasi-injective if and only if FI-quasi-injective.

(3) The *Z*-module *Z* is not FI-quasi-injective (since by (2), *Z* as *Z*-module is not quasi-injective).

(4) A fully invariant submodule of an FI-quasi-injective module is FI-quasi-injective.

Proof: Let *N* be a fully invariant submodule of an FI-quasi-injective *R*-module *M*. Let *K* be a fully invariant submodule of *N* and let $f: K \longrightarrow N$ be any *R*-homomorphism. Now, since *N* is fully invariant submodule of *M*, then *K* is fully invariant of *M*. Also, consider the *R*-homomorphism $(i \circ f): K \longrightarrow M$ where *i* is the inclusion homomorphism from *N* into *M*. By FI-quasi-injectivity of *M*, there exists an *R*-homomorphism $g: M \longrightarrow M$ such that $g_{|K}=f$. But *N* is fully invariant submodule of *M*, then $g(N) \subseteq N$ (i.e.) $g: N \longrightarrow N$ such that $g_{|K}=f$. Thus, *N* is FI-quasi-injective. □

In the next result, we investigate a condition under which fully invariant submodules are stable.

Proposition (1.19): A fully invariant submodule of an FI-quasi-injective module is stable.

Proof: Let *N* be a fully invariant submodule of FI-quasi-injective module *M*. Let $f: N \longrightarrow M$ be any R-homomorphism. Since *M* is FI-quasi-injective, thus there exists an *R*-homomorphism $g: M \longrightarrow M$ such that $g_{|N} = f$. Since *N* is fully invariant of *M*, then $g(N) \subseteq N$ and so $f(N) \subseteq N$ (i.e.) *N* is a stable submodule of *M*.

Corollary (1.20): Every S-extending FI-quasi-injective module is FI-extending. □

Corollary (1.21): A fully invariant submodule of a quasi-injective module is stable. □

Corollary (1.22): Duo FI-quasi-injective modules are fully stable.□

The following result asserts that every non-torsion S-extending module decomposes into torsion and torsion-free submodules.

Proposition (1.23): Let R be an integral domain. Then, every non-torsion S-extending *R*-module decomposes into torsion and torsion-free submodules.

Proof: Assume that *M* is non-torsion S-extending *R*-module. Let $t(M)$ $(\neq 0)$ (the set of all torsion elements of *M*). Since essential extensions of torsion module are torsion [10], then $t(M)$ is closed submodule of M. Moreover, $t(M)$ is a stable submodule of *M* [1]. Hence, by S-extending property of *M*, $t(M)$ is a direct summand of *M* (i.e.) $M = t(M) \oplus F$, where *F* is a non-zero torsion-free submodule of *M*. □

2-Direct sums (and direct summands) of S-extending modules.

It is well-known that a direct sum of extending modules need not be extending. For example, the *Z*-modules Z/Z_p and Z/Z_{p^3} are extending, while the *Z*-module $(Z/Z_p) \oplus (Z/Z_{p^3})$ is not extending because the submodule $K = Z(1+Z_p, p+Z_{p3})$ is closed, but cannot be a direct summand, since it has order p^2 [3, p.56]. Later, many papers appeared which discussed the conditions under which a direct sum of extending modules is extending (for example, see [7]).

Unlike extending modules and similar to FI-extending modules, the next theorem asserts that a direct sum of S-extending modules is S-extending.

Theorem (2.1): A direct sum of S-extending modules is S-extending.

Proof: Suppose that $M = \bigoplus_{i\in I} M_i$ where M_i is S-extending R-module for all $i\in I$. Now, if F be a stable submodule of $M = \bigoplus_{i\in I} M_i$, then $F = \bigoplus_{i\in I} (F\cap M_i)$ [1,proposition (4.5)]. But, we claim that $F\cap M_i$ is a stable submodule of M_i for all $i\in I$, to see that, let $g: F\cap M_i \longrightarrow M_i$ be any R-homomorphism, thus $g(F\cap M_i)\subseteq M_i$. Also, we have the following implications:

$F = \bigoplus_{i\in I} (F\cap M_i) \xrightarrow{\pi} F\cap M_i \xrightarrow{g} M_i \xrightarrow{i} M = \bigoplus_{i\in I} M_i$, then $(i\circ g\circ \pi)(F) \subseteq F$ (since F is a stable submodule of M) and so $g(F\cap M_i) \subseteq F$. Thus, from the above, we have $g(F\cap M_i) \subseteq F\cap M_i$ (i.e.) $F\cap M_i$ is a stable submodule of M_i for all $i\in I$. Now, by S-extending property of M_i for all $i\in I$, then $F\cap M_i$ is essential in a direct summand N_i of M_i. Let $N = \bigoplus_{i\in I} N_i$, then clearly N is a direct summand of $M = \bigoplus_{i\in I} M_i$. Also, since $F\cap M_i$ is essential in N_i for all $i\in I$, then $F = \bigoplus_{i\in I} (F\cap M_i)$ is essential in $N = \bigoplus_{i\in I} N_i$. Therefore, $M = \bigoplus_{i\in I} M_i$ is S-extending.□

One of the most interesting questions concerning extending modules is when a direct sum of extending modules is also extending (see [7]). The next result gives an answer to such a question.

Corollary (2.2): Any direct sum of extending modules is S-extending. □

Corollary (2.3): Let $M = \bigoplus_{i\in I} M_i$ be an R-module such that every closed submodule of M is a stable. If M_i is extending module for all $i\in I$, then $M = \bigoplus_{i\in I} M_i$ is extending. □

Since every finitely generated abelian group is a direct sum of uniform Z-modules. Then, we have the next corollary.

Corollary (2.4): Every finitely generated Z-module is S-extending.□

Example (2.5): By using theorem (2.1), since Z/Z_p and Z/Z_{p^3} are S-extending Z-modules, then the Z-module $(Z/Z_p)\oplus(Z/Z_{p^3})$ is S-extending, while it is not extending [3, p.56].

It is well-known that every direct summand of extending module is extending [11, p.20]. Indeed, we do not know in general whether S-extending property is inherited by direct summands. In the following result, we give a condition under which this inheritance is valid. Firstly, recall that an R-module M has the summand intersection property (SIP) if the intersection of two direct summands of M is a direct summand [14].

Proposition (2.6): Let M be a non-singular R-module with (SIP) property. If M is S-extending, then every direct summand of M is S-extending.

Proof: Let N be a direct summand of M and let A be a closed stable submodule of N. Now, since M is a non-singular then, by [10, p.259], there exists a closed submodule $B = Cl(A)$ in M such that A is essential in B. But N is essential in N, and then $A = A\cap N$ is essential in $B\cap N\subseteq N$. Since A is closed in N, thus $A = B\cap N$. But B is a closed stable submodule of M; hence by S-extending property of M, B is a direct summand of M. So by the (SIP) property of M, we have $A = B\cap N$ is direct summand of M. Thus, [13, lemma (2.4.3)] A is a direct summand of N. Also, since M is non-singular, then N is non-singular. Therefore, (by proposition (1.10)) N is S-extending. □

Corollary (2.7): Every direct summand left ideal of a non-singular S-extending commutative ring is S-extending. □

Corollary (2.8): Every direct summand of non-singular cyclic Z-module is S-extending. □

It is well-known that every fully invariant (and hence stable) submodule of an extending module is extending [2]. In fact, we do not know in general whether S-extending property is inherited by stable submodules. In the following we give an answer to the question: When stable submodules of an S-extending module are S-extending? Firstly, we need to introduce the following concepts:

Definition (2.9): Let X be an R-module. An R-module M is called stable-injective relative to X (simply, S-X-injective) if for each stable submodule A of X, each R-homomorphism $f: A \longrightarrow M$ can be extended to an R-homomorphism $g: X \longrightarrow M$.

Definition (2.10): An R-module M is called S-quasi-injective, if M is S-M-injective.

Remarks and Examples (2.11):

(1) Every X-injective (resp. quasi-injective) module is S-X-injective (resp. S-quasi-injective), but the converse is not true in general. For example, it is easy to check that the Z-module Z is S-Z-injective, while it is not Z-injective.

(2) Every FI-quasi-injective module is S-quasi-injective, while the converse is not true in general (see the same example in (1)).

(3) Every direct summand of S-X-injective module is S-X-injective.

Proof: Let N be a direct summand of an S-X-injective module. Now, let A be a stable submodule of X and $f: A \longrightarrow N$ be any R-homomorphism. Since M is S-X-injective, then there exists an R-homomorphism $g: X \longrightarrow M$ such that $g_{|A} = f$. Define $g': X \longrightarrow N$ such that $g' = \pi \circ g$, where $\pi: M \longrightarrow N$ be the projection mapping. Thus, we have $g'_{|A} = \pi \circ g_{|A} = \pi \circ f = f$. Therefore, N is S-X-injective. □

Now, we are ready to get an answer of the preceding question.

Proposition (2.14): Let M be a stable-injective relative to a stable submodule X. If M is S-extending, then so is X.

Proof: Let A be a stable submodule of X and let $f: A \longrightarrow M$ be any R-homomorphism. Since M is S-X-injective module, then there exists an R-homomorphism $g: X \longrightarrow M$ such that $g_{|A} = f$. Also, since X is a stable submodule of M, then $g(X) \subseteq X$. So $g_{|A}: A \longrightarrow X$, but A is a stable submodule of X, hence $g_{|A}(A) \subseteq A$ (i.e.) A is a stable of M. On other hand, since M is S-extending, thus there exists a direct summand D of M such that A is essential in D. Let $\pi: M \longrightarrow D$ be the projection mapping, thus $A = \pi(A) \subseteq \pi(X) \cap D = \pi(X)$. Therefore, by proposition (1.1.2), A is essential in $\pi(X)$. But $\pi(X)$ is direct summand of X [15, lemma(8.3)]. Therefore, X is S-extending. □

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