A categorical study on the finiteness of specifications

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Abstract

Formal specification practice involve specifications which are finite. We study the concept of finiteness for specifications within the general framework of the so-called π-institutions, which constitute an abstract categorical proof oriented formalization of the informal concept of logical system. In this paper we argue that finite specifications can be captured abstractly by the concept of finitely presented theory in π-institutions.

Key words: Institutions; Specifications; Formal methods

1. Introduction

Although formal specification practice involve only ‘finite’ specifications, i.e. specifications involving a a finite number of symbols and a finite number of sentences, theoretical studies of formal specification have not yet properly studied finiteness concepts for specifications. The aim of the work reported here is to fill this gap.

Formal specification practice has led to an explosion of the population of logical systems used as underlying frameworks for specification systems. In response to this, algebraic specification theory has developed the general framework of the so-called institutions [10] in which much of the specification theory concepts and results can be developed uniformly and independently of the concrete details of particular logical systems (see for example [17,7]). For our work we prefer to use the π-institutions of [9], which replace the model theoretic structure of institutions with abstract co-sequence relations, thus making them even more general than institutions. This choice is motivated by the fact that our study does not involve model theoretic concepts.

In this paper we argue that finitely presented theories constitute the appropriate abstract capture for the informal concept of ‘finite’ specification. The categorical concept of finitely presented object [1] is a rather standard theoretical device for capturing finiteness concepts in the context of various mathematical structures. Our main result provides and ‘if and only if’ characterization of finitely presented theories as having a finitely presented underlying signature and being presented by a finite number of sentences. Our work destills some important causes for this characterization, most notably the compactness of the π-institution and a finiteness property for its sentences. To complete the understanding of finitely presented theories we also show that in the concrete cases finitely presented signatures are those signatures with a finite number of symbols.

We assume the reader is familiar with basic notions and standard notations from category theory; e.g., see [13] for an introduction to this subject. The categorical notions required for this paper are rather basic, the reader is expected to have knowledge only of the concepts of category, functor, and co-limit. With respect to notational conventions, |C| denotes the class of objects of a category C, C(A,B) the set of arrows (morphisms) with domain A and codomain B, and composition is denoted by “;” and in diagrammatic order. While in general arrows f ∈ C(A,B) may be denoted f : A → B, we will denote natural transformations µ between functors F and G by µ : F ⇒ G. The category of sets (as objects) and functions (as arrows) is denoted by Set.

1.1. Finitely presented objects

The concept of finitely presented object plays an important role for our work. Recall [1] that an object A in a category C is finitely presented if and only if the hom-functor C(A,−) : C → Set preserves directed co-limits. This is equivalent to the following conditions.

– for any arrow f : A → C to the vertex of a co-limiting co-cone µ : D ⇒ C of a directed diagram D : (J, ≤) →
Example 1.1 In Set the finitely presented objects are precisely the finite sets.

Example 1.2 In the category of groups Grp the finitely presented groups are exactly the quotients of finitely generated groups by finitely generated congruences.

2. \(\pi\)-institutions

Following the work of Fiadeiro and Sernadas [9], logical systems based on deduction can be formalised as \(\pi\)-institutions, which have for each signature \(\Sigma\) a set of \(\Sigma\)-sentences, but no given models. To compensate for this lack, a consequence relation is given on sentences. We will use the definition of Fiadeiro and Sernadas [9] as modified by Meseguer [15], rather than that of Maibaum and Fiadeiro [14]; Harper, Samella and Tarlecki [11] have given a definition similar to Meseguer’s, but restricted to finite sets of sentences.

**Definition 2.1** A \(\pi\)-institution \(I\) consists of

(i) a category \(\text{Sig}_I^T\) whose objects are called signatures,
(ii) a functor \(\text{Sen}_I^T : \text{Sig}_I^T \to \text{Set}\), giving for each signature a set whose elements are called sentences over that signature, and
(iii) a relation\(^1\) \(\vdash_{\Sigma} \subseteq \mathcal{P}(\text{Sen}_I^T(\Sigma)) \times \text{Sen}_I^T(\Sigma)\) for each \(\Sigma \in \text{Sig}_I^T\), called \(\Sigma\)-consequence, such that the following conditions hold:

A. reflexivity: \(\{\epsilon\} \vdash_{\Sigma} \epsilon\) for each \(\epsilon \in \text{Sen}_I^T(\Sigma)\);
B. monotonicity: if \(E \vdash_{\Sigma} \epsilon\) and \(E \subseteq E'\) then \(E' \vdash_{\Sigma} \epsilon\);
C. transitivity: if \(E \vdash_{\Sigma} \epsilon'\) for each \(\epsilon' \in E'\) and if \((E \cup E') \vdash_{\Sigma} \epsilon\), then \(E \vdash_{\Sigma} \epsilon\);
D. deduction: if \(E \vdash_{\Sigma} \epsilon\) and if \(\varphi : \Sigma \to \Sigma'\) in \(\text{Sig}_I^T\), then \(\text{Sen}_I^T(\varphi)(E) \vdash_{\Sigma'} \text{Sen}_I^T(\varphi)(\epsilon)\).

In general we prefer to write \(\varphi(\epsilon)\) instead of \(\text{Sen}_I^T(\varphi)(\epsilon)\).

Each institution appears canonically as a \(\pi\)-institution by considering the semantic consequence relations \(\models_\Sigma\) in the role of the consequence relations \(\vdash_{\Sigma}\). Conversely, each \(\pi\)-institution can be given a rather artificial model theory by a comma category construction on theories [15]. Recently \(\pi\)-institutions have been adopted by Voutsadakis [19] as the framework for the development of the so-called ‘categorical abstract algebraic logic’. The so-called ‘proof systems’ of [16,5] constitute a refinement of \(\pi\)-institutions which discriminates between different proofs. The work [15] also introduces a similar notion.

**Example 2.2 (First order logic)** Classical first order logic (with equality) in its many sorted version can be captured as a \(\pi\)-institution as follows.

- its signatures are triples \((S, F, P)\) consisting of
  - a set of sort symbols \(S\),
  - a family \(F = \{F_{w,s} | w \in S^*, s \in S\}\) of sets of function symbols indexed by arities (for the arguments) and sorts (for the results), and
  - a family \(P = \{P_{w} | w \in S^*\}\) of sets of relation (predicate) symbols indexed by arities.

**Signature morphisms** map the three components in a compatible way. This means that a signature morphism \(\varphi : (S, F, P) \to (S', F', P')\) consist of

- a function \(\varphi^{\text{st}} : S \to S'\),
- a family of functions \(\varphi^{\text{fp}} = \{\varphi^{\text{fp}}_{w,s} : F_{w,s} \to F'_{\varphi^{\text{st}}(w), \varphi^{\text{st}}(s)} | w \in S^*, s \in S\}\), and
- a family of functions \(\varphi^{\text{rl}} = \{\varphi^{\text{rl}}_{w} : P_{w} \to P'_{\varphi^{\text{st}}(w)} | w \in S^*\}\).

**Sentences** are the usual first order sentences built from equational and relational atoms by iterative application of Boolean connectives and quantifiers. Sentence translations along signature morphisms just rename the sorts, function, and relation symbols according to the respective signature morphisms. They can be formally defined by induction on the structure of the sentences. While the induction step is straightforward for the case of the Boolean connectives it needs a bit of attention for the case of the quantifiers.

For any signature morphism \(\varphi : (S, F, P) \to (S', F', P')\),

\[
\text{Sen}_{\text{FOL}}^\text{FOL}(\varphi)((\forall X)\rho) = (\forall X^{\varphi})\text{Sen}_{\text{FOL}}^\text{FOL}((\varphi')\rho)
\]

for each finite set of variables (i.e. ‘new’ constants) \(X\), each \((S, F \uplus X, P)\)-sentence \(\rho\), and where \(X^{\varphi} = \{\{x: \varphi^{\text{st}}(s)\} | (x: s) \in X\}\), by \(\uplus\) we denote the disjoint union (i.e. \(F \uplus X\) means that the constants \(X\) do not appear in \(F\)), and \(\varphi' : (S, F \uplus X, P) \to (S', F' \uplus X^{\varphi}, P')\) extends \(\varphi\) canonically (here by \(x: \varphi^{\text{st}}(s)\) we denote the fact that \(x\) is a variable of sort \(s\)).

\[
\begin{array}{c}
(S, F, P) \\
\downarrow \varphi \\
(S', F', P')
\end{array}
\]

For each signature \((S, F, P)\), the consequence relation \(\vdash_{(S, F, P)}\) is the ‘provability’ relation, i.e. \(E \vdash_{(S, F, P)} \epsilon\) when there exists a proof of the \((S, F, P)\)-sentence \(\epsilon\) from \(E\) under the proof system generated by the usual first order logic system of proof rules [12]. An equivalent way to define \(\vdash\) is to consider the usual model theory of first order logic and define \(\models\) as \(\models\), the semantic consequence relation. Gödel-Henkin completeness theorem [12] shows that the semantic
consequence relation $\models$ coincides with the provability consequence relation.

Compactness is an important property of logical systems. The definition below is given a straightforward proof-theoretic concept of compactness formulated for $\pi$-institutions.

**Definition 2.3** A $\pi$-institution is compact iff whenever $E \vdash \Sigma \ e$ then there is some finite $E' \subseteq E$ such that $E' \vdash \Sigma \ e$.

In [5] one can find a generic compactness theorem for proof systems, thus applicable also to $\pi$-institutions. A generic model theoretic compactness theorem within the framework of institutions and using the ‘institution-independent’ method of ultraproducts has been produced in [4].

**Definition 2.4** A signature morphism $\varphi : \Sigma \to \Sigma'$ is conservative when $\varphi(E) \vdash_{\Sigma'} \varphi(e)$ implies $E \vdash_{\Sigma} e$ for each $E \subseteq \text{Sen}(\Sigma)$ and each $e \in \text{Sen}(\Sigma)$.

**Example 2.5** In FOL each injective signature morphism\(^2\) is conservative. This is a well known fact which can be established for example by model theoretic arguments. Classical logical studies considers only such signature morphisms, non-injective signature morphisms being used mainly in specification theory.

### 2.1. Theories

Theories represent one of the two major ways to provide denotations for software modules or structured specifications, which has been exploited in works such as [3,7,17]. The other more subtle semantic way, which requires the framework of institutions, is that of [17] in which denotations of specifications are classes of models.

**Definition 2.6** In any $\pi$-institution, a theory $(\Sigma, E)$ consists of a signature $\Sigma$ and a set $E$ of $\Sigma$-sentences closed under consequence, i.e. if $E \vdash_{\Sigma} e$ then $e \in E$. A theory $(\Sigma, E)$ is presented by a set $E_0$ of sentences when $E_0 \subseteq E$ and $E_0 \vdash_{\Sigma} E$ (meaning that $E_0 \vdash_{\Sigma} e$ for each $e \in E$). This is denoted by $E = E_0^*$. A theory morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is just a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'$. Note that theory morphisms form a category under the composition given by the composition of the signature morphisms.

Theory co-limits constitute the main conceptual tool supporting module compositions in specification languages [10]. The following is the $\pi$-institution version of the general and fundamental result on existence of theory co-limits in institutions [10].

**Theorem 2.7** In any $\pi$-institution the forgetful functor from the category of theories to the category of signatures lifts co-limits uniquely (in the sense of [2]). Moreover, if $\mu$ is the co-limiting co-cone of a diagram of theories $(\varphi_i : (\Sigma_i, E_i) \to (\Sigma_j, E_j))_{(i : i \leq j) \in J}$

\[ (\Sigma_i, E_i) \xrightarrow{\varphi_i} (\Sigma_j, E_j) \]

\[ \overset{\mu_i}{\rightarrow} \]

\[ (\Sigma, E) \]

then $E = (\bigcup_{i \in |J|} \mu_i(E_i))^\star$.  

### 3. Finitely presented theories

#### 3.1. Finitely presented signatures

The following result shows that FOL signatures with a finite number of symbols are precisely the finitely presented objects in the category of the signatures.

**Proposition 3.1** A FOL signature $(S, F, P)$ is finitely presented (as an object of $\text{Sig}^{\text{FOL}}$) if and only if $S$, $F$, and $P$ are finite. ($F$ ‘finite’ means that $(\{w, s \mid F_{w \to s} \neq \emptyset\}$ is finite and each non-empty $F_{w \to s}$ is also finite and the same for $P$.)

**Proof** Let us first consider a FOL signature $(S, F, P)$ such that $S$, $F$, and $P$ are finite and let $\theta : (S, F, P) \to (S', F', P')$ be a signature morphism to the vertex of the co-limit of a directed diagram of signature FOL morphisms $(\varphi_{i,j})_{(i < j) \in (J, \leq)}$ of signatures.

\[ (S_i, F_i, P_i) \xrightarrow{\varphi_{i,j}} (S_j, F_j, P_j) \]

\[ (S, F, P) \xrightarrow{\theta} (S', F', P') \]

The first thing we have to show is that there exists $i \in J$ and a signature morphism $h : (S, F, P) \to (S_i, F_i, P_i)$ such that $h_i; \mu_i = \theta$. The sort component $h^n_i$ of $h_i$ is obtained by the fact that $\{\mu^n_i\}_{i \in J}$ is the co-limit of the directed diagram $(\varphi_{i,j})_{(i < j) \in (J, \leq)}$ in $\text{Set}$ and because $S$ is finite (and hence it is a finitely presented set).

In order to define the component of $h_i$ on operation symbols, namely $h^{op}_i$, let us fix an arity $w \in S^n$ and a sort $s \in S$ and define $(h^{op}_i)_{w \to s : F_{w \to s} \to F'_{(\mu^n_i)^{(w)}} \rightarrow (\varphi_{i,j})_{(w \to s)})$. Since the case when $F_{w \to s} = \emptyset$ is trivial, we may assume that $F_{w \to s} \neq \emptyset$. By the construction of co-limits of FOL signatures (see [18,6]) we have that $\{(\mu^n_{ij})_{w \to s} \}_{i \leq j}$ is the co-limiting co-cone for the directed diagram $\{(\varphi_{i,j})_{w \to s} \}_{(i < j) \in (J, \leq)}$.
For each \( k \in J \) we define

\[
\big( T_k \big)_{w-s} = \bigcup (F_k)_{w_s-kw}
\]

where

- for each \( k \in J \) we define \((T_k)_{w-s} = \bigcup (F_k)_{w_s-kw}\)
- \((\mu_k^*)_{w-s}\) is the canonical ‘union’ of all functions \((\mu_k^*)_{w_s-kw}\) for all \( w_k \) \( s_k \) such that \((\mu_k^*)_{w_s-kw} = (\theta^*)_{w_s-kw}\),
- for each \( i, j \), \((\varphi_{i,j}^*)_{w-s}\) is defined as the canonical ‘union’ of all functions \((\varphi_{i,j}^*)_{w_s-kw}\) composed with the canonical injection \((F_{i,j})_{(\varphi_{i,j}^*)} \rightarrow (F_{i,j})_{(\varphi_{i,j}^*)}(s_i) \rightarrow (F_{i,j})_{(\varphi_{i,j}^*)}(s_j)\)

for all \( w_i \) \( s_i \) \( s_j \) \( s_k \) such that \((\mu_k^*)_{w_s-kw} = (\theta^*)_{w_s-kw}\).

Because \( F_{w-s} \) is finite there exists \( k \in J \) and a function \((h^p_k)_{w-s} \rightarrow (h^p_k)_{w_s-kw}\) such that \((h^p_k)_{w_s-kw} = (\theta^*_{w_s-kw})\). Without any loss of generality we may assume that \( i = k \), since otherwise by the directedness of \((J, \leq)\) we may replace \( i \) and \( k \) by an index \( j \) bigger than both. Now we have to show that the image of \((h^p_k)_{w-s}\) is included in \((F_{i,j})_{(\varphi_{i,j}^*)} \rightarrow (F_{i,j})_{(\varphi_{i,j}^*)}(s_j)\).

Proposition 3.4 In any \( \pi \)-institution with finitary sentences, for each finitely presented theory \((\Sigma, E)\)

- \( \Sigma \) is a finitely generated signature, and
- \( E \) is presented by a finite set of sentences.

Proof Consider a co-limit \( \mu \) of a directed diagram \((\varphi_{i,j})_{i, j \in J, i \leq j}\) of signatures

\[
\Sigma_i \xrightarrow{\varphi_{i,j}} \Sigma_j \quad \Sigma \xrightarrow{\varphi_{i,j}} \Sigma_j
\]

and let \( \theta : \Sigma \rightarrow \Sigma' \) be a signature morphism. Let \( E' = \theta(E)^* \) and for each \( i \in J \) let \( E_i = \mu_i^{-1}(E') = \{ e \mid \mu_i(e) \in E' \}. \) We show that these definitions determine a co-limiting co-cone in the category of theories such that \( \theta \) is a theory morphism \((\Sigma, E) \rightarrow (\Sigma', E')\)

By virtue of Theorem 2.7 we have to show that \( E' = \bigcup_{i \in J} E_i \). By the definition of \( E_i \), we have that \( \mu_i(E_i) \subseteq E' \) for each \( i \in J \). Now let \( \rho' \in E' \). Because \( \rho' \) is finitary there exists a finitely presented signature \( \Sigma_0, \rho_0 \in \Sigma_0 \). However this may go against some of the established terminology which calls infinitary rather than finitary such infinite conjunctions involving a finite number of symbols.

3.2. Finitary sentences

The concept below is due to [8] and captures abstractly the concrete situations when sentences are built with a finite number of symbols.

Definition 3.2 A sentence \( \rho \) for a signature \( \Sigma \) of an institution is finitary when there exists a signature morphism \( \varphi : \Sigma_0 \rightarrow \Sigma \) such that \( \Sigma_0 \) is finitely presented and there exists a \( \Sigma_0 \)-sentence \( \rho_0 \) such that \( \rho = \varphi(\rho_0) \).

Example 3.3 Any FOL-sentence is finitary in spite of the fact that FOL admits signatures having an infinite number of symbols. However in the infinitary extension FOL\(_\infty, \omega\) of FOL, which admits infinite conjunctions of sentences, on the one hand there are sentences which are not finitary, and on the other hand there are sentences involving infinite conjunctions but which are finitary.

3.3. Finitely presented theories

The aim of this section is to establish a characterization of finitely presented theories in terms of the corresponding property of the underlying signature and of the set of the sentences of the theory.
Presented theory. Thus there exists a signature morphism \( \theta \) such that \( \theta f_j = f_0 \). Note also that by definition \( \theta E \) is a theory morphism \( (\Sigma, E) \to (\Sigma', E') \), hence \( \theta \) is a theory morphism \( (\Sigma, E) \to (\Sigma', E') \).

Now we can use the fact that \( (\Sigma, E) \) is finitely presented theory. Thus there exists \( i \) and a theory morphism \( h_i : (\Sigma, E) \to (\Sigma_i, E_i) \) such that \( h_i \mu_i = \theta \). This gives an \( i \) and a signature morphism \( \mu_i \) such that \( h_i \mu_i = \theta \).

Now let us assume there exists another index \( k \) and a signature morphism \( h_k : \Sigma \to \Sigma_k \) such that \( h_k \mu_k = \theta \). Because \( \Sigma \) is a finitely presented signature, and the directedness of \( \Sigma \) is based upon the compactness and the finitary of the sentences of \( \Sigma \). We can now put together Propositions 3.4 and 3.5 and obtain the following characterization for finitely presented theories.

**Corollary 3.6** In any compact \( \pi \)-institution with conservative signature morphisms and with finitary sentences, for any theory \( (\Sigma, E) \) the following are equivalent

(i) \( (\Sigma, E) \) is finitely presented, and

(ii) \( \Sigma \) is a finitely presented signature, and \( E \) is presented by a finite set of symbols.

The following instance of the general result above is based upon the compactness property \( \text{FOL} \) and upon the characterization of finitely presented signatures in \( \text{FOL} \) given by Proposition 3.1.

**Corollary 3.7** Let us consider the sub-institution of \( \text{FOL} \) determined by the injective signature morphisms. Then a \( \text{FOL} \) theory \( (\Sigma, E) \) is finitely presented if and only if \( \Sigma \) has a finite number of symbols and \( E \) is presented by a finite number of sentences.

4. Conclusions

In this paper we have argued that finitely presented theories in \( \pi \)-institutions capture abstractly the concept of finitary specifications. We have given a general characterization of finitely presented theories in terms of the underlying signatures being finitely presented and the theory being presented by a finite set of sentences. This characterization is based upon the compactness and the finiteness of the sentences of the \( \pi \)-institution. We have illustrated the general results of our work with the typical instance of classical many sorted logic obtaining there that a theory is finitely presented if and only if the underlying signature has only a finite number of \( (\text{sort}, \text{operation and relation}) \) symbols and can be presented by a finite number of sentences.

References


