Garside structure for the braid group of $G(e, e, r)$

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Abstract

We give a new presentation of the braid group B of the complex reflection group $G(e, e, r)$ which is positive and homogeneous, and for which the generators map to reflections in the corresponding complex reflection group. We show that this presentation gives rise to a Garside structure for B with Garside element a kind of generalised Coxeter element, and hence obtain solutions to the word and conjugacy problems for B.

Introduction

The work in this paper arose from some questions posed in [BMR]. In that paper, the authors associate a braid group to each complex reflection group. For all but six exceptional irreducible complex reflection groups, they exhibit particular presentations for these braid groups. These presentations have certain remarkable properties, including that the relations are positive and homogeneous, and that the generators are "generators of the monodromy" (natural analogues of reflection in braid groups). However, many questions about the braid groups remain open. For example, solutions to the word and conjugacy problems are not known in all cases.

We consider the family of imprimitive reflection groups $G(e, e, r)$, where e and r are positive integral parameters. This family contains two infinite families of real reflection groups: $G(2, 2, r)$ is the reflection group of type D_r, whereas $G(e, e, 2)$ is the dihedral group $I_2(e)$. We denote by $B(e, e, r)$ the associated braid group.

Except when $e = 2$ or $r = 2$, Broué-Malle-Rouquier's presentation for $B(e, e, r)$ does not have good algorithmic properties: the main obstruction is that the monoid defined by their positive homogeneous presentation does not embed in the group. This negative answer to Question 2.28 in [BMR] was first observed in the second author's PhD thesis ([C1]) – in the last section of the present article, we prove that an even stronger obstruction holds.

This obstruction led us to search for a new presentation for $B(e, e, r)$.

Our new presentation has more generators than the original presentation; our generators are conjugates of the original generators, thus are "generators of the monodromy". It is positive and homogeneous. For $e = 2$ or $r = 2$, it coincides with the "dual" presentation introduced in [B]. Our main result, Theorem 21, states that the monoid defined by our presentation is a Garside monoid.

Garside monoids were introduced in different ways in [BDM] and [DP]. We use the approach of [DP] in this article. A monoid M defined by a homogeneous presentation is *Garside* if it is cancellative, the partial orders on M defined by left and right division are lattices, and M possesses a special *Garside element*, for which the sets of left divisors and right divisors are equal and generate

M. The group of fractions of a Garside monoid is a *Garside group*, which is the group defined by the same presentation (considered now as a group presentation) as the original monoid. Garside monoids embed in their groups of fractions; Garside groups have explicitly solvable word [DP] and conjugacy [P] problems. Thus our Theorem 21 provides the first known solutions to the word and conjugacy problems in $B(e,e,r)$.

This article is in four sections. We begin by exhibiting a positive homogeneous presentation, defined in terms of two parameters e and r. In the second section, we show that the monoid defined by this presentation has a *complete presentation* (see [D]), and thus that it is a Garside monoid. Hence the group defined by the presentation is a Garside group. The third section is devoted to showing that this group is indeed the braid group of the complex reflection group $G(e,e,r)$. In the last section, we show that, with the original generating set of [BMR], one cannot write a finite positive homogeneous presentation satisfying the embedding property.

Note. The proofs presented in this article rely on complex computations, including some case-by-case verifications performed using the computer algebra system `GAP`. After completing this first version, we realised that computer calculations could be avoided with a more geometric approach. While we will present this new approach in a forthcoming sequel, we decided to post this article with our first proof, since the generality of the techniques used may be applicable to other contexts. We also postpone explanation of the origin of our new presentation, since this is not required for the algebraic approach used here, and is better understood in the geometric setting of the sequel.

1 The presentation

1.1 Statement of the presentation

Here we give a presentation which we will show in Section 2 to be a presentation of the braid group of the reflection group $G(e,e,r)$. For notational reasons, we will use $r = n+1$ throughout. The generators are $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ where

$$\mathcal{A}_0 = \{a_{pq} \mid 0 \le p,q < n\} \quad \text{and } \mathcal{A}_1 = \{a_p^{(i)} \mid 0 \le p < n,\ 0 \le i < e\}.$$

We show in Theorem 21 that these generators map to reflections in the reflection group $G(e,e,r)$, which can be thought of as the group of monomial $r \times r$ matrices whose non-zero entries firstly must come from $\{\exp\left(\frac{2\pi i}{e}\right) \mid 0 \le i < e\}$, and secondly, must multiply to give 1. To give a little meaning to the generators $\mathcal{A}$ above, and why they are so denoted, we describe their images in $G(e,e,r)$. We will label the entries in matrices over the indices $\{0,1,\ldots,n\}$. Let E_{ij} denote the $r \times r$ matrix with 0's everywhere but 1 in the (i,j) place. Define a matrix $M_{ij}(a,b) := I_r - E_{ii} - E_{jj} + aE_{ij} + bE_{ji}$ (which is like a permutation matrix corresponding to the transposition $(i\ j)$, but with the off-diagonal entries replaced by a and b). Let $\zeta = \exp\left(\frac{2\pi}{e}\right)$; then the matrices corresponding to our generators are: for $0 \le p < q < n$,

$$a_{pq} \stackrel{\nu}{\longmapsto} M_{pq}(1,1) \quad \text{and} \quad a_{qp} \stackrel{\nu}{\longmapsto} M_{pq}(\zeta,\zeta^{-1});$$

and for $0 \le p < n$ and $0 \le i < e$,

$$a_p^{(i)} \stackrel{\nu}{\longmapsto} M_{pn}(\zeta^i,\zeta^{-i}).$$

There are six types of relations, they are given modulo e where appropriate. The sequence $(a_1,\ldots,a_k)$ of distinct integers $\texttt{mod}\, n$ is said to be *n-cyclically ordered* if the sequence of roots of unity $(\mu^{a_1},\ldots,\mu^{a_k})$ has strictly increasing arguments, where μ is the nth root of unity $\exp\left(\frac{2\pi}{n}\right)$. For example, if $0 \le p < q < r < n$ then certainly (p,q,r) is n-cyclically ordered; so are (q,r,p) and (r,p,q); and these three are the only n-cyclically ordered sequences on these numbers.

For every n-cyclically ordered sequence (p,q,r,s) there are relations:

$$\begin{array}{ll}(\mathcal{R}_1) & a_{pq}a_{rs} = a_{rs}a_{pq} \quad \text{and} \\ (\mathcal{R}_2) & a_{ps}a_{qr} = a_{qr}a_{ps}.\end{array}$$

For every n-cyclically ordered sequence (p,q,r) and for every i $(0 \le i < e)$ there are relations:

$$\begin{array}{ll}(\mathcal{R}_3) & a_{pq}a_{qr} = a_{qr}a_{pr} = a_{pr}a_{pq} \\ (\mathcal{R}_4) & a_{pq}a_r^{(i)} = a_r^{(i)}a_{pq}.\end{array}$$

For every pair $0 \le p, q < n$ and for every i $(0 \le i < e)$ there are relations:

$$(\mathcal{R}_5) \quad a_{pq}a_q^{(j)} = a_q^{(j)}a_p^{(i)} = a_p^{(i)}a_{pq}$$

where

$$j = \begin{cases} i & \text{if } p < q, \text{ and} \\ i+1 \ \texttt{mod}\, e & \text{otherwise.}\end{cases}$$

Finally for every p $(0 \le p < n)$ there are relations:

$$(\mathcal{R}_6) \qquad a_p^{(0)}a_p^{(e-1)} = \cdots = a_p^{(i)}a_p^{(i-1)} = \cdots = a_p^{(2)}a_p^{(1)} = a_p^{(1)}a_p^{(0)}.$$

We will write B^+ for the monoid defined by this presentation.

B^+ is isomorphic to its reverse

Let $\texttt{rev}$ be the involution on $\mathcal{A}^*$ defined by $\texttt{rev}(a_1\cdots a_f) = a_f\cdots a_1$. Recall that the monoid B^+ is defined by the presentation $\langle \mathcal{A} \mid \mathcal{R}\rangle$ where $\mathcal{A}$ and $\mathcal{R}$ are the generators and relations respectively defined at the beginning of Section 1.1. Define $B^+_{\texttt{rev}}$ to be the monoid defined by the presentation $\langle \mathcal{A} \mid \texttt{rev}(\mathcal{R})\rangle$. We call this monoid the reverse of B^+. Clearly $\texttt{rev}$ defines an anti-isomorphism $B^+ \longrightarrow B^+_{\texttt{rev}}$. We will show that they are in fact isomorphic.

Let χ be any map: $\mathcal{A} \longrightarrow \mathcal{B}$. Then χ extends to a homomorphism between the free monoids $\mathcal{A}^*$ and $\mathcal{B}^*$. For any set of relations $\mathcal{R}$ over $\mathcal{A}$, write $\chi(\mathcal{R})$ for the set $\{\chi(u) = \chi(v) \mid (u = v) \in \mathcal{R}\}$ of relations over $\mathcal{B}$. Then χ defines a homomorphism between the monoids $M_1 = \langle \mathcal{A} \mid \mathcal{R}\rangle$ and $M_2 = \langle \mathcal{B} \mid \chi(\mathcal{R})\rangle$. In particular, $\chi(M_1) = M_2$ precisely when $\chi(\mathcal{A}) = \mathcal{B}$, and $\chi(M_1) \cong M_1$ precisely when $\chi : \mathcal{A} \to \mathcal{B}$ is one-one. Thus $\chi : M_1 \to M_2$ is an isomorphism whenever $\chi : \mathcal{A} \to \mathcal{B}$ is a bijection.

Define an operator χ on our set of generators $\mathcal{A}$ by

$$\chi(a_{pq}) := a_{q'p'} \qquad \text{and} \qquad \chi(a_p^{(i)}) := \begin{cases} a_{p'}^{(i')} & \text{if } p \ne 0, \text{ and} \\ a_0^{(i'-1)} & \text{otherwise,}\end{cases}$$

where $p' := n - p$, $q' = n - q$ and $i' = e - i$. Clearly χ is a permutation of $\mathcal{A}$, so by the previous paragraph, we have an isomorphism

$$\chi : \mathcal{B}^+ \xrightarrow{\sim} \langle \mathcal{A} \mid \chi(\mathcal{R})\rangle. \tag{1}$$

Moreover,

Lemma 1 $\chi(\mathcal{R}) = \texttt{rev}(\mathcal{R})$

Proof Since χ and `rev` are bijections on words, it suffices to show that $\texttt{rev}(\chi(\mathcal{R})) \subseteq \mathcal{R}$, as since `rev` is an involution this implies $\chi(\mathcal{R}) \subseteq \texttt{rev}(\mathcal{R})$, which in turn by bijectivity of χ, and finiteness of $\mathcal{R}$, ensures that $\chi(\mathcal{R}) = \texttt{rev}(\mathcal{R})$.

We will use p' and i' to denote $n-p$ and $e-i$ respectively, with context (subscript *vs* exponent) making it clear which is intended. Observe that any sequence $(p_1, \ldots, p_f)$ is cyclically ordered precisely when $(p'_f, \ldots, p'_1)$ is cyclically ordered, which is precisely when $(p'_1, \ldots, p'_f)$ is reverse cyclically ordered. In the rest of this proof, $x < y < z$ will mean cyclically ordered, and $0 < x < y < n$ will mean linearly (usually) ordered.

Consider the relations of type $\mathcal{R}_1$ and $\mathcal{R}_2$; for $p < q < r < s$ we have

$$\begin{array}{l} \texttt{rev}(\chi(a_{pq}a_{rs} = a_{rs}a_{pq})) = \texttt{rev}(a_{q'p'}a_{s'r'} = a_{s'r'}a_{q'p'}) = (a_{s'r'}a_{q'p'} = a_{q'p'}a_{s'r'}) \quad \text{and} \\ \texttt{rev}(\chi(a_{ps}a_{qr} = a_{qr}a_{ps})) = \texttt{rev}(a_{s'p'}a_{r'q'} = a_{r'q'}a_{s'p'}) = (a_{r'q'}a_{s'p'} = a_{s'p'}a_{r'q'}) \end{array}$$

which are relations of type $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively also, with $s' < r' < q' < p'$.

Next consider the typical relation of type $\mathcal{R}_3$ for $p < q < r$:

$$\begin{array}{rcl} \texttt{rev}(\chi(a_{pq}a_{qr} = a_{qr}a_{pr} = a_{pr}a_{pq})) & = & \texttt{rev}(a_{q'p'}a_{r'q'} = a_{r'q'}a_{r'p'} = a_{r'p'}a_{q'p'}) \\ & = & (a_{q'p'}a_{r'p'} = a_{r'p'}a_{r'q'} = a_{r'q'}a_{q'p'}), \end{array}$$

which is a relation of type $\mathcal{R}_3$ with $r' < q' < p'$. Looking at a relation of type $\mathcal{R}_4$:

$$\texttt{rev}(\chi(a_{pq}a_r^{(i)} = a_r^{(i)}a_{pq})) = \texttt{rev}(a_{q'p'}a_{r'}^{(j)} = a_{r'}^{(j)}a_{q'p'}) = (a_{q'p'}a_{r'}^{(j)} = a_{r'}^{(j)}a_{q'p'}),$$

where $j = i' - 1$ if $r = 0$ and $j = i'$ otherwise; this is a relation of type $\mathcal{R}_4$ also, with $r' < q' < p'$.

We will consider the image of relations of type $\mathcal{R}_5$ in four cases. Suppose $0 < p < q < n$. Then we have $0 < q' < p' < n$, and:

$$\begin{array}{rcl} \texttt{rev}(\chi(a_{0q}a_q^{(i)} = a_q^{(i)}a_0^{(i)} = a_0^{(i)}a_{0q})) & = & \texttt{rev}(a_{q'0}a_{q'}^{(i')} = a_{q'}^{(i')}a_0^{(i'+1)} = a_0^{(i'+1)}a_{q'0}) \\ & = & (a_{q'0}a_0^{(i'+1)} = a_0^{(i'+1)}a_{q'}^{(i')} = a_{q'}^{(i')}a_{q'0}), \\ \\ \texttt{rev}(\chi(a_{pq}a_q^{(i)} = a_q^{(i)}a_p^{(i)} = a_p^{(i)}a_{pq})) & = & \texttt{rev}(a_{q'p'}a_{q'}^{(i')} = a_{q'}^{(i')}a_{p'}^{(i')} = a_{p'}^{(i')}a_{q'p'}) \\ & = & (a_{q'p'}a_{p'}^{(i')} = a_{p'}^{(i')}a_{q'}^{(i')} = a_{q'}^{(i')}a_{q'p'}), \\ \\ \texttt{rev}(\chi(a_{q0}a_0^{(i+1)} = a_0^{(i+1)}a_q^{(i)} = a_q^{(i)}a_{q0})) & = & \texttt{rev}(a_{0q'}a_0^{(i')} = a_0^{(i')}a_{q'}^{(i')} = a_{q'}^{(i')}a_{0q'}) \\ & = & (a_{0q'}a_{q'}^{(i')} = a_{q'}^{(i')}a_0^{(i')} = a_0^{(i')}a_{0q'}), \\ \\ \texttt{rev}(\chi(a_{qp}a_p^{(i+1)} = a_p^{(i+1)}a_q^{(i)} = a_q^{(i)}a_{qp})) & = & \texttt{rev}(a_{p'q'}a_{p'}^{(i'-1)} = a_{p'}^{(i'-1)}a_{q'}^{(i')} = a_{q'}^{(i')}a_{p'q'}) \\ & = & (a_{p'q'}a_{q'}^{(i')} = a_{q'}^{(i')}a_p^{(i'-1)} = a_{p'}^{(i'-1)}a_{p'q'}), \end{array}$$

where in the third and fourth calculations we use the fact that $(i+1)' = e - i - 1 = i' - 1$, and so $\chi(a_0^{(i+1)}) = a_0^{(i')}$. Each of these calculations results in a relation of type $\mathcal{R}_5$ also, although permuting the types.

Finally we consider the relation of type $\mathcal{R}_6$: for every $0 < p < n$ we have

$$\begin{aligned}
\texttt{rev}(\chi(a_p^{(0)} a_p^{(e-1)} &= \cdots = a_p^{(i)} a_p^{(i-1)} = \cdots = a_p^{(2)} a_p^{(1)} = a_p^{(1)} a_p^{(0)})) \\
&= \texttt{rev}(a_{p'}^{(0)} a_{p'}^{(1)} = \cdots = a_{p'}^{(i')} a_{p'}^{(i'+1)} = \cdots = a_{p'}^{(e-2)} a_{p'}^{(e-1)} = a_{p'}^{(e-1)} a_{p'}^{(0)}) \\
&= (a_{p'}^{(0)} a_{p'}^{(e-1)} = a_{p'}^{(e-1)} a_{p'}^{(e-2)} = \cdots = a_{p'}^{(i'+1)} a_{p'}^{(i')} = \cdots = a_{p'}^{(1)} a_{p'}^{(0)}),
\end{aligned}$$

which is clearly a relation of type $\mathcal{R}_6$ also. The case where the subscript is 0 is similar, with the exponents shifted by 1; also resulting in a relation of type $\mathcal{R}_6$. Thus we have that $\texttt{rev}(\chi(\mathcal{R})) \subseteq \mathcal{R}$, and hence $\chi(\mathcal{R}) = \texttt{rev}(\mathcal{R})$, as desired. □

Substituting the equality thus obtained into the isomorphism (1) above, we obtain

Theorem 2 *The monoid B^+ is isomorphic to its reverse.*

Some consequences of the relations

In this subsection we observe a number of easily derived equations which are consequences of the relations. For every n-cyclically ordered sequence (p, q, r, s) we have

$$\begin{array}{ll}
(\mathcal{C}_1) & a_{pr} a_{pq} a_{rs} = a_{qs} a_{ps} a_{qr} \\
(\mathcal{C}_2) & a_{ps} a_{pq} a_{qr} a_s^{(1)} a_s^{(0)} = a_{rq} a_{rp} a_{rs} a_q^{(1)} a_q^{(0)}
\end{array}$$

For every n-cyclically ordered sequence (p, q, r) and for every i $(0 \le i < e)$ we have

$$\begin{array}{ll}
(\mathcal{C}_3) & a_{pr} a_{pq} a_r^{(1)} a_r^{(0)} = a_{qp} a_{qr} a_p^{(1)} a_p^{(0)} \\
(\mathcal{C}_4) & a_{pr} a_{pq} a_r^{(k)} = a_q^{(i)} a_{qr} a_p^{(j)}
\end{array}$$

where

$$j = \begin{cases} i & \text{if } p < q, \text{ and} \\ i - 1 \ \texttt{mod}\, e & \text{otherwise;} \end{cases} \qquad k = \begin{cases} i & \text{if } q < r, \text{ and} \\ i + 1 \ \texttt{mod}\, e & \text{otherwise.} \end{cases}$$

And lastly, for each pair p, q and for each pair i, j we have

$$\begin{array}{ll}
(\mathcal{C}_5) & a_{pq} a_q^{(1)} a_q^{(0)} = a_{qp} a_p^{(1)} a_p^{(0)} \\
(\mathcal{C}_6) & a_p^{(i)} a_p^{(i-1)} a_{pq} = a_q^{(j)} a_q^{(j-1)} a_{qp}
\end{array}$$

In fact, if we denote by t_p the word $a_p^{(1)} a_p^{(0)}$, we have that $a_{pq} t_q = t_q a_{qp} = a_{qp} t_p = t_p a_{pq}$, from which $(\mathcal{C}_5)$ and $(\mathcal{C}_6)$ follow directly.

Call the presentation consisting of the original generating set, along with relations $(\mathcal{R}_1), \ldots, (\mathcal{R}_6)$ together with their consequences $(\mathcal{C}_1), \ldots, (C_6)$ the *augmented presentation*. Since the added relations are consequences of the original relations, the monoid and group presented by it are the same as for the original presentation. We observe that for every pair of generators x and y there exists exactly one relation of the form $xu = yv$ in the augmented presentation.

2 Garside Structure

We want to show that B^+ is a Garside monoid with Garside element Δ, the least common multiple of the generators. To do so, we will show that it is cancellative, that the partial orders on M defined by left and right division are lattices, and that the sets of left divisors and right divisors of Δ are equal and generate M.

2.1 Completeness

A positive presentation $\mathcal{P}$ is said to be *right complemented* if firstly, for every generator x, $\mathcal{P}$ contains no relations of the form $xu = xv$ for any words u and v, and secondly, for all generators $x \neq y$, there is exactly one relation of the form $xu = yv$. Thus, by inspection, the augmented presentation for B^+ defined above can be seen to be right complemented.

Completeness is introduced in [D]. If a monoid possesses a presentation which is both right complete and right complemented then in particular the monoid is left cancellative (6.2 of [D]) and the poset defined by left division is a lattice (6.10 of [D]). We will show in Theorem 8 that it the augmented presentation for B^+ right complete. Invoking Theorem 2 we will then have that B^+ is both left and right cancellative, and that the posets defined by left and right division are lattices, as desired.

Word reversing

Completeness is defined in [D] via a technique called *word reversing.* We introduce this in the context of a right complemented (positive) presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$. We begin by defining word reversing on generators: for any generators x and y, write

$$x^{-1}y \curvearrowright^{1}_{\mathcal{R}} \begin{cases} \varepsilon & \text{if } x = y, \text{ and} \\ uv^{-1} & \text{if } xu = yv \text{ is a relation from } \mathcal{R}. \end{cases}$$

We extend in the obvious way to arbitrary words over $X = \mathcal{A} \cup \mathcal{A}^{-1}$: for any w, w' from X^*, write $wx^{-1}yw' \curvearrowright^{1}_{\mathcal{R}} wzw'$ where $x^{-1}y \curvearrowright^{1}_{\mathcal{R}} z$. We say that w *reverses to* w' *in one step.* Finally, for any w, w' from X^*, we write $w \curvearrowright_{\mathcal{R}} w'$ if there exists a sequence of words $w \equiv w_0, w_1, \ldots, w_f \equiv w'$ from X^* such that $w_{i-1} \curvearrowright^{1}_{\mathcal{R}} w_i$ for each $i \in \{1, \ldots f\}$. In this case we say that w *reverses to* w' (modulo $\mathcal{R}$). (The notation $w \equiv w'$ indicates that the words w and w' are letterwise identical.)

While reversing a word of the form $s^{-1}t$ where s and t are generators gives a unique result (due to the presentation being complemented), in general a word may be reversed in a single step to any of a number of other words: for example $s^{-1}st^{-1}t$ reverses in one step to $t^{-1}t$ and to $s^{-1}s$. However there is confluence (again due to being complemented): if $w \curvearrowright_{\mathcal{R}} w'$ and $w \curvearrowright_{\mathcal{R}} w''$ where w' and w'' are both fully word reversed (that is, no further reversings are possible), then $w' \equiv w''$. Thus the particular order chosen to fully reverse a word is not important: the end result is unique.

Clearly, a word which is fully word reversed is one in which there is no occurrence of a subword of the form $s^{-1}t$ for generators s and t. Thus these are precisely the words of the form uv^{-1} where u and v are words over $\mathcal{A}$. (For $v \equiv a_{i_1} \cdots a_{i_f}$ a word over $\mathcal{A}$, we write v^{-1} to denote the word $a_{i_f}^{-1} \cdots a_{i_f}^{-1}$.)

Definition of completeness

We say that a triple of generators (x, y, z) over $\mathcal{A}$ satisfies the *completion condition modulo* $\mathcal{R}$ if

whenever $x^{-1}zz^{-1}y \curvearrowright_{\mathcal{R}} uv^{-1}$ for u and v positive words, we have $(xu)^{-1}yv \curvearrowright_{\mathcal{R}} \varepsilon$

where ε represents the empty word. It is shown in [D] that a presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is right complete if the completion condition holds modulo $\mathcal{R}$ for every triple of generators over $\mathcal{A}$. Until the end of this section we use the notation $\mathcal{A}_{e,r}$ and $\mathcal{R}_{e,r}$ to refer to the generators and augmented relations respectively of the braid group of type $G(e,e,r)$. (Thus $r = n+1$ in the presentation given in Section 1.1.) For ease on the eye, we write $\curvearrowright_{e,r}$ for $\curvearrowright_{\mathcal{R}_{e,r}}$.

We programmed `GAP` [GAP] to take as input a set (or subset) of generators G and a set of relations R, to calculate for each triple (x,y,z) over G whether the above condition holds. The procedure outputs to a file the calculations at each one-reversing step. The code to do this, as well as the output calculations we will refer to below can be found at `http//www.math.jussieu.fr/~corran`.

For example, this can be used to check, for fixed values of e and r, that the presentation $\langle A_{e,r} \mid \mathcal{R}_{e,r} \rangle$ is complete. In order to prove the presentation is complete for general e and r we will show that checking the completion condition modulo $\mathcal{R}_{e,r}$ for a triple (x,y,z) over $\mathcal{A}_{e,r}$ can be reduced to checking the completion condition for another triple, this time modulo small values of e and r, which then is done with the `GAP` code (see Propositions 6 and 7).

In Table 1 (found on page 8) we list all the one-step reversings of words of the form $x^{-1}y$ where x and y are generators; that is, for a pair of generators x and y, we show the reversings $x^{-1}y \curvearrowright_{e,r} uv^{-1}$. Since the set of augmented relations has exactly one relation of the form $xu = yv$ for every pair $\{x, y\}$ of generators, $x^{-1}y$ will always reverse, in one step, and uniquely, to a word of the form uv^{-1} where u and v are positive words over the generators. Furthermore, $x^{-1}y \curvearrowright uv^{-1}$ precisely when $y^{-1}x \curvearrowright vu^{-1}$, so in the table we only include one of each of these.

Observe that the set of subscripts appearing on the right hand side of any of the one-step reversings in Table 1 is the same as the set of subscripts for the left hand side. Furthermore, every "exponent" (the i appearing in $a^{(i)}$) on the right hand side of a one-step reversing differs by at most one from an exponent appearing on the left hand side; except for the exponents 0 and 1 which may appear from nowhere.

The *reversing calculation of* w, denoted ρ_w, where w is a word over $\mathcal{A}_{e,r} \cup \mathcal{A}_{e,r}{}^{-1}$, is defined to be the sequence of words obtained from one-step reversings, where at each step the left-most occurrence of an (inverse generator, positive generator) pair is reversed, (choosing this convention because it coincides with how we programmed the reversing procedure in `GAP`), and continuing until no more reversings are possible, or otherwise continuing indefinitely. The last word in the sequence ρ_w, if it exists, is called the result of the reversing calculation. All the reversing calculations we undertook for the results we needed in this article terminated. As remarked above, whenever the sequence ρ_w is finite (in other words when the reversing procedure terminates) the result of ρ_w is a word of the form uv^{-1} where u and v are positive words.

Block mappings

The *exponent set* I_w of a word is defined to be the set of exponents appearing in the word; more precisely, if $w = a_{i_1} \cdots a_{i_f}$ where $a_{i_j} \in \mathcal{A}_{e,r} \cup \mathcal{A}_{e,r}{}^{-1}$, then

$$I_w = \bigcup_{j=1}^{f} I_{a_{i_j}}, \quad \text{where} \quad I_{a_{pq}} := \varnothing, \quad I_{a_p^{(i)}} := \{i\} \quad \text{and} \quad I_{a^{-1}} := I_a \text{ for all } a \in \mathcal{A}.$$

The *exponent set of a reversing calculation* is then the union of the exponent sets for each word in the reversing calculation. By definition, the exponent set will always be a subset of $\{0, 1, \ldots, e-1\}$.

1. For any p, and any distinct i and j, we have $\left(a_p^{(i)}\right)^{-1} a_p^{(j)} \curvearrowright_1 a_p^{(i-1)} \left(a_p^{(j-1)}\right)^{-1}$;
2. For any $p < q$, and any i and j, we have
   (a) $a_{pq}^{-1} a_{qp} \curvearrowright_1 a_q^{(1)} a_q^{(0)} \left(a_p^{(1)} a_p^{(0)}\right)^{-1}$,
   (b) $a_{pq}^{-1} a_q^{(i)} \curvearrowright_1 a_q^{(i)} \left(a_p^{(i)}\right)^{-1}$ and $a_{qp}^{-1} a_p^{(i)} \curvearrowright_1 a_p^{(i)} \left(a_q^{(i-1)}\right)^{-1}$,
   (c) $a_{pq}^{-1} a_p^{(i)} \curvearrowright_1 a_q^{(i)} a_{pq}^{-1}$ and $a_{qp}^{-1} a_q^{(i)} \curvearrowright_1 a_p^{(i+1)} a_{qp}^{-1}$, and
   (d)
$$\left(a_p^{(i)}\right)^{-1} a_q^{(j)} \curvearrowright_1 \begin{cases} a_{pq} \left(a_p^{(i)}\right)^{-1} & \text{if } i = j, \\ a_p^{(i)} a_{qp}^{-1} & \text{if } i = j+1, \\ a_p^{(i)} a_{pq} \left(a_q^{(j-1)} a_{qp}\right)^{-1} & \text{if } i \neq j, j+1 \ ; \end{cases}$$
3. For any n-cyclically ordered sequence (p, q, r) and any i, we have
   (a) $a_{pq}^{-1} a_{qr} \curvearrowright_1 a_{qr} a_{pr}^{-1}$, $a_{pq}^{-1} a_{pr} \curvearrowright_1 a_{qr} a_{pq}^{-1}$, and $a_{qr}^{-1} a_{pr} \curvearrowright_1 a_{pr} a_{pq}^{-1}$,
   (b) $a_{pq}^{-1} a_r^{(i)} \curvearrowright_1 a_r^{(i)} a_{pq}^{-1}$,
   (c) $a_{pr}^{-1} a_{qr} \curvearrowright_1 a_{pq} a_r^{(1)} a_r^{(0)} \left(a_{qr} a_p^{(1)} a_p^{(0)}\right)^{-1}$,
   (d)
$$(a_{pr})^{-1} a_q^{(i)} \curvearrowright_1 \begin{cases} a_{pq} a_r^{(i)} \left(a_{qr} a_p^{(i)}\right)^{-1} & \text{if } 0 \leq p < q < r < n, \\ a_{pq} a_r^{(i)} \left(a_{qr} a_p^{(i-1)}\right)^{-1} & \text{if } 0 \leq q < r < p < n, \\ a_{pq} a_r^{(i+1)} \left(a_{qr} a_p^{(i)}\right)^{-1} & \text{if } 0 \leq r < p < q < n \ ; \end{cases}$$
4. For any n-cyclically ordered sequence (p, q, r, s), we have
   (a) $a_{pq}^{-1} a_{rs} \curvearrowright_1 a_{rs} a_{pq}^{-1}$,
   (b) $a_{ps}^{-1} a_{qr} \curvearrowright_1 a_{qr} a_{ps}^{-1}$,
   (c) $a_{pr}^{-1} a_{qs} \curvearrowright_1 a_{pq} a_{qr} (a_{ps} a_{qr})^{-1}$, and
   (d) $a_{ps}^{-1} a_{rq} \curvearrowright_1 a_{pq} a_{qr} a_s^{(1)} a_s^{(0)} (a_{rp} a_{rs} a_q^{(1)} a_q^{(0)})^{-1}$.

Table 1: All the possible one-step reversings with respect to $\mathcal{R}_{e,n+1}$ for pairs of generators from $\mathcal{A}_{e,n+1}$. Exponents are always modulo e.

The *exponent blocks* of an exponent set are the equivalence classes generated by the relation "$i \sim j$ if $|i-j| = 1$ (modulo e)". For example, given an exponent set $I_{\rho_w} = \{0, 1, 2, 3, 6, 7, 11, 12, 13, 17\}$ where $e = 18$, the exponent blocks are $\{17, 0, 1, 2, 3\}$, $\{6, 7\}$ and $\{11, 12, 13\}$. We will write this $I_{\rho_w} = [17, 3]_{18} \cup [6, 7]_{18} \cup [11, 13]_{18}$, where $[i, k]_e$ denotes the e-cyclically ordered interval from i to k. We will write $[e]$ to denote $\{0, \ldots, e-1\}$ with e-cyclic ordering.

Let w be a word over $\mathcal{A}_{e,r} \cup \mathcal{A}_{e,r}{}^{-1}$, with reversing calculation ρ_w, and corresponding exponents $I_{\rho_w} = I_1 \cup \cdots \cup I_f$, in exponent block decomposition. We have $I_l = [i_l, i_l + p_l - 1]_e$ where p_l is the size of I_l, and i_l its cyclically ordered least element. A *block mapping* of I_{ρ_w} to $[d]$ maps blocks to blocks, acting identically on 0 and 1 if they appear in I_{ρ_w}. More precisely, fix $\mathbf{j} = (j_1, \ldots, j_l)$ where each $j_t \in [d]$. Define $\alpha_\mathbf{j} : I_{\rho_w} \to [d]$ as follows: every $x \in I_{\rho_w}$ is of the form $x \equiv i_l + p$ modulo e for some l where $0 \le p < p_l$; then we set

$$\alpha_\mathbf{j}(x) :\equiv j_l + p \bmod d \quad \text{for} \quad x \equiv i_l + p \bmod e \quad \text{and } 0 \le p < p_l.$$

Then $\alpha_\mathbf{j}$ is a *block mapping* if whenever $\{0, 1\} \subseteq I_{\rho_w}$, we have $\alpha_\mathbf{j}(0) \equiv 0 \bmod d$ and $\alpha_\mathbf{j}(1) \equiv 1 \bmod d$. Thus in particular:

Lemma 3 *For α a block mapping: $I_{\rho_w} \to [d]$ and k a positive integer, whenever $[i, i+k] \subseteq I_l$ for some block I_l of I_{ρ_w}, we have*

$$\alpha(i+k) \equiv \alpha(i) + k \bmod d.$$

For example, given an exponent set $I_{\rho_w} = [17, 3]_{18} \cup [6, 7]_{18} \cup [11, 13]_{18}$, ordering the exponents from left to right I_1, I_2, I_3, we have that any block map $\alpha_\mathbf{j} : I_{\rho_w} \to [d]$ must have $j_1 \equiv d - 1 \bmod d$, but that j_2 and j_3 can be chosen arbitrarily. Thus $d = 3$ and $\mathbf{j} = (2, 1, 1)$ gives rise to the block mapping

$$\begin{array}{rlllll} \text{on } I_1: & 17 \overset{\alpha_\mathbf{j}}{\mapsto} 2 & 0 \overset{\alpha_\mathbf{j}}{\mapsto} 0 & 1 \overset{\alpha_\mathbf{j}}{\mapsto} 1 & 2 \overset{\alpha_\mathbf{j}}{\mapsto} 2 & 3 \overset{\alpha_\mathbf{j}}{\mapsto} 0, \\ \text{on } I_2: & 6 \overset{\alpha_\mathbf{j}}{\mapsto} 1 & 7 \overset{\alpha_\mathbf{j}}{\mapsto} 2, & & & \\ \text{and on } I_3: & 11 \overset{\alpha_\mathbf{j}}{\mapsto} 1 & 12 \overset{\alpha_\mathbf{j}}{\mapsto} 2 & 13 \overset{\alpha_\mathbf{j}}{\mapsto} 0. & & \end{array}$$

We extend a block mapping α to the generators appearing in the reversing calculation for w, and consequently homomorphically to words over these generators, and so to the entire reversing calculation ρ_w, as follows:

$$\alpha(a_{pq}) := a_{pq}, \quad \alpha(a_p^{(i)}) := a_p^{(\alpha(i))} \quad \text{and} \quad \alpha(a^{-1}) := \alpha(a)^{-1} \quad \text{for every } a \text{ in } \mathcal{A}_{e,r}.$$

Write $\alpha(\rho_w)$ for the sequence of words obtained by applying α to the sequence of words in the reversing calculation ρ_w for w.

Proposition 4 *Let ρ_w be the reversing calculation with respect to $\mathcal{R}_{e,r}$ of the word w. Then a block map $\alpha : I_{\rho_w} \to [d]$ sends ρ_w to the reversing calculation of $\alpha(w)$ with respect to $\mathcal{R}_{d,r}$ — that is,*

$$\alpha(\rho_w) = \rho_{\alpha(w)}.$$

Proof It suffices to show that for every one-step reversing $a^{-1}b \curvearrowright^1_{(e,r)} uv^{-1}$ appearing in ρ_w, we have that $\alpha(a^{-1}b) \curvearrowright^1_{(d,r)} \alpha(uv^{-1})$. We remark that a and b appear in ρ_w only if all the exponents appearing in a, b, u and v lie in I_w. For the numbering, we refer to Table 1 where all the one-step reversings are enumerated. For the one-step reversings 3(a) and 4(a), (b) and (c) there is nothing to prove, as α acts trivially on the generators appearing, and the relations are the same in $\mathcal{R}_{d,r}$. For the reversing 3(b), which is $a_{pq}^{-1}a_s^{(i)} \curvearrowright_1 a_s^{(i)}a_{pq}^{-1}$, we have

$$\alpha\left(a_{pq}^{-1}a_s^{(i)}\right) \equiv a_{pq}^{-1}a_s^{(\alpha(i))} \curvearrowright_{1(d,r)} a_s^{(\alpha(i))}a_{pq}^{-1} \equiv \alpha\left(a_s^{(i)}a_{pq}^{-1}\right).$$

Let's now consider the one-step reversing numbered (1) in Table 1: for any p, and any distinct i and j, we have $\left(a_p^{(i)}\right)^{-1}a_p^{(j)} \curvearrowright_1 a_p^{(i-1)}\left(a_p^{(j-1)}\right)^{-1}$. For this to occur in ρ_w means that $i-1$ and i (resp. $j-1$ and j) lie in some block I_l (resp. I_k) of I_w. Then we have

$$\begin{aligned}
\alpha\left(\left(a_p^{(i)}\right)^{-1}a_p^{(j)}\right) &\equiv \left(a_p^{(\alpha(i))}\right)^{-1}a_p^{(\alpha(j))} \\
&\curvearrowright_{1(d,r)} a_p^{(\alpha(i)-1)}\left(a_p^{(\alpha(j)-1)}\right)^{-1} && \text{(where exponents are } \texttt{mod}\, d\text{)} \\
&\equiv a_p^{(\alpha(i-1))}\left(a_p^{(\alpha(j-1))}\right)^{-1} && \text{(Lemma 3)} \\
&\equiv \alpha\left(a_p^{(i-1)}\left(a_p^{(j-1)}\right)^{-1}\right).
\end{aligned}$$

(The blocks I_l and I_k need not be distinct.) Similar arguments go through for all the one-step reversings 2(b), (c) and (d), and 3(d).

The reversings remaining are 2(a), 3(c) and 4(d) which involve explicitly the exponents 0 and 1, and only these. By the definition of the block mapping, since both 0 and 1 lie in I_w, we have that α acts as the identity on these two exponents; hence its extension to words acts identically on the entirety of these relations. □

2.1.1 Results using GAP, applied to block mappings.

In the following result, we collect together some results obtained using the GAP code. Recall that since all pairs of generators have a corresponding one-step reversing, then a terminating reversing calculation ρ_w for any word w results in a word of the form uv^{-1} where u and v are positive (but possibly empty) words.

Proposition 5 *1. Let p, q, r be any elements of $\{0,1,2\}$ (repeats allowed), and denote $x = a_p^{(2)}$, $y = a_q^{(8)}$, $z = a_r^{(5)}$ and $w = x^{-1}zz^{-1}y$. Then the reversing calculation ρ_w terminates, and writing uv^{-1} for the result of ρ_w, where u and v are positive words, we have*

(a) $w' :\equiv (xu)^{-1}yv \curvearrowright_{9,4} \varepsilon$, the empty word, and

(b) $I_{\rho_w}, I_{\rho_{w'}} \subseteq \{1,2\} \cup \{4,5\} \cup \{7,8\}$.

2. Let p, q, r, s be any elements of $\{0,1,2,3\}$ (repeats allowed), let $\{x,y,z\} = \{a_{pq}, a_r^{(2)}, a_s^{(6)}\}$ (any ordering, but not repeating), and let $w = x^{-1}zz^{-1}y$. Then ρ_w terminates, and writing uv^{-1} for the result of ρ_w, where u and v are positive words, we have

(a) $w' :\equiv (xu)^{-1}yv \curvearrowright_{8,5} \varepsilon$, and

(b) $I_{\rho_w}, I_{\rho_{w'}} \subseteq \{1,2,3\} \cup \{5,6,7\}$.

3. Let p, q, r, s, t be any elements of $\{0,1,2,3,4\}$ (repeats allowed), let $\{x,y,z\} = \{a_{pq}, a_{rs}, a_t^{(4)}\}$ (any ordering, but not repeating), and let $w = x^{-1}zz^{-1}y$. Then ρ_w terminates, and writing uv^{-1} for the result of ρ_w, where u and v are positive words, we have

(a) $w' :\equiv (xu)^{-1}yv \curvearrowright_{7,6} \varepsilon$, and

(b) $I_{\rho_w}, I_{\rho_{w'}} \subseteq \{0,1\} \cup \{3,4,5\}$.

4. Let p,q,r,s,t,f be any elements of $\{0,1,2,3,4,5\}$ (repeats allowed), let $x = a_{pq}$, $y = a_{rs}$ and $z = a_{tf}$, and let $w = x^{-1}zz^{-1}y$. Then ρ_w terminates, and writing uv^{-1} for the result of ρ_w, where u and v are positive words, we have

   (a) $w' :\equiv (xu)^{-1}yv \curvearrowright_{4,7} \varepsilon$, and

   (b) $I_{\rho_w}, I_{\rho_{w'}} \subseteq \{0,1,2\}$.

Recall that the completion condition with respect to $\mathcal{R}$ holds for a triple x,y,z if whenever the word $x^{-1}zz^{-1}y$ reverses modulo $\mathcal{R}$ to a word of the form uv^{-1} where u and v are positive words, then $(xu)^{-1}yv \curvearrowright_{\mathcal{R}} \varepsilon$. Via well-chosen block mappings, we use the previous result to show that the completion condition holds for certain r with no restrictions on e.

Proposition 6 *1. For any d, the completion condition with respect to $\mathcal{R}_{d,4}$ holds for every triple $(a_p^{(i)}, a_q^{(j)}, a_r^{(k)})$ where $p,q,r \in \{0,1,2\}$ and $i,j,k \in [d]$.*

2. For any d, the completion condition with respect to $\mathcal{R}_{d,5}$ holds for every triple consisting of a_{pq}, $a_r^{(i)}$ and $a_s^{(j)}$ (any ordering) where $p,q,r,s \in \{0,1,2,3\}$ and $i,j,k \in [d]$.

3. For any d, the completion condition with respect to $\mathcal{R}_{d,6}$ holds for every triple consisting of a_{pq}, a_{rs} and $a_t^{(i)}$ (any ordering) where $p,q,r,s,t \in \{0,1,2,3,4\}$ and $i,j,k \in [d]$.

4. For any d, the completion condition with respect to $\mathcal{R}_{d,7}$ holds for every triple $(a_{pq},\ a_{rs},\ a_{tf})$ where $p,q,r,s,t,f \in \{0,1,2,3,4,5\}$ and $i,j,k \in [d]$.

Proof 1. We continue to use the notation w, u and v for the words in Proposition 5, and define

$$\overline{w} :\equiv \left(a_p^{(i)}\right)^{-1} a_r^{(k)} \left(a_r^{(k)}\right)^{-1} a_q^{(j)}.$$

Define a block map $\alpha_{\mathbf{t}} : I_{\rho_w} \to [d]$ by $\mathbf{t} := (i-1, j-1, k-1)$; that is

$$\begin{array}{lll} 1 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i-1, & & 2 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i, \\ 4 \overset{\alpha_{\mathbf{t}}}{\longmapsto} j-1, & & 5 \overset{\alpha_{\mathbf{t}}}{\longmapsto} j, \text{ and} \\ 7 \overset{\alpha_{\mathbf{t}}}{\longmapsto} k-1 & \text{and} & 8 \overset{\alpha_{\mathbf{t}}}{\longmapsto} k. \end{array}$$

Thus $\overline{w} \equiv \alpha(w)$. Since $w \curvearrowright_{9,4} uv^{-1}$, we have $\alpha(w) \curvearrowright_{d,4} \alpha(uv^{-1}) \equiv \alpha(u)\alpha(v)^{-1}$. Furthermore,

$$\alpha\left((xu)^{-1} yv\right) \equiv \left(a_p^{(i)}\alpha(u)\right)^{-1} a_q^{(j)}\alpha(v),$$

so $(xu)^{-1} yv \curvearrowright_{9,4} \varepsilon$ implies

$$\left(a_p^{(i)}\alpha(u)\right)^{-1} a_q^{(j)}\alpha(v) \curvearrowright_{d,4} \alpha(\varepsilon) \equiv \varepsilon,$$

as required. The arguments for the other three cases are the same as for 1., but with respect to different block mappings, being, respectively:

2. $\alpha_{\mathbf{t}} : \{1,2,3\} \cup \{5,6,7\} \to [d]$ where $\mathbf{t} = (i-1, k-1)$; that is

$$\begin{array}{llll} 1 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i-1, & 2 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i, & & 3 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i+1, \\ 5 \overset{\alpha_{\mathbf{t}}}{\longmapsto} k-1, & 6 \overset{\alpha_{\mathbf{t}}}{\longmapsto} k, & \text{and} & 7 \overset{\alpha_{\mathbf{t}}}{\longmapsto} k+1. \end{array}$$

3. $\alpha_{\mathbf{t}} : \{0,1\} \cup \{3,4,5\} \to [d]$ where $\mathbf{t} = (0, i-1)$; that is

$$0 \overset{\alpha_{\mathbf{t}}}{\longmapsto} 0, \qquad 1 \overset{\alpha_{\mathbf{t}}}{\longmapsto} 1,$$
$$3 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i-1, \qquad 4 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i, \qquad \text{and} \qquad 5 \overset{\alpha_{\mathbf{t}}}{\longmapsto} i+1.$$

4. $\alpha_4 : \{0,1,2\} \to [d]$ where $\mathbf{t} = (0)$; that is $\quad 0 \overset{\alpha_{\mathbf{t}}}{\longmapsto} 0, \quad 1 \overset{\alpha_{\mathbf{t}}}{\longmapsto} 1 \quad$ and $\quad 2 \overset{\alpha_{\mathbf{t}}}{\longmapsto} 2.$ ☐

It remains to extend these results to the case of general r; having done which we will have shown that each presentation $\langle \mathcal{A}_{e,r} \mid \mathcal{R}_{e,r} \rangle$ is right complete.

Principal subscript maps

Let S_w denote the set of subscripts appearing in the word w; that is

$$S_{a_{pq}} = \{p, q\}, \quad S_{a_p^{(i)}} = \{p\} \quad \text{and} \quad S_{a^{-1}} = S_a \quad \text{for all } a \in \mathcal{A},$$

extending to words in the obvious way: $S_{wx} = S_w \cup S_x$. By inspection of Table 1, if $x^{-1}y \curvearrowright_1 uv^{-1}$ then $S_{x^{-1}y} = S_{uv^{-1}}$. Thus the subscript set of a reversing calculation is exactly the subscript set of its first word.

Define the *principal subscript map* σ_w of a word w as follows: let $S_w = \{p_0, \ldots, p_f\}$ where $0 \leq p_0 < \cdots < p_f$, and define $\sigma_w(p_i) = i$. Thus any ordering $(p_{i_1}, \ldots, p_{i_k})$ of a subset of S_w is cyclically ordered precisely when $(i_1, \ldots, i_k)$ is cyclically ordered. As an example, the word $w = a_{1,7}^{-1} a_{6,9} a_{6,9}^{-1} a_4^{(15)}$ has $S_w = \{0, 4, 6, 7, 9\}$ and corresponding principal subscript map

$$\sigma_w : \quad 1 \mapsto 0, \quad 4 \mapsto 1, \quad 6 \mapsto 2, \quad 7 \mapsto 3 \quad \text{and} \quad 9 \mapsto 4.$$

We then extend this in the obvious way to arbitrary words whose subscript set is a subset of S_w:

$$\sigma_w(a_{p_s p_t}) := a_{st}, \quad \sigma_w(a_{p_s}^{(i)}) := a_s^{(i)} \quad \text{and} \quad \sigma_w(a^{-1}) := \sigma(a) \quad \text{for every } a \in \mathcal{A}_{e,r}.$$

For the example above, this gives $\sigma_w(w) = a_{0,3}^{-1} a_{2,4} a_{2,4}^{-1} a_1^{(15)}$. In general, for w' a word whose subscripts lie inside S_w, $\sigma_w(w')$ is a word whose subscripts are all strictly less than $|S_w|$.

Because the relations $\mathcal{R}_{e,r}$ are defined in terms of cyclic order on the subscripts, inspection of Table 1 will convince the reader that for principal subscript map σ_w whose domain contains $S_{x^{-1}y}$,

$$x^{-1}y \curvearrowright_{1\;e,r} uv^{-1} \;\Leftrightarrow\; \sigma(x^{-1}y) \curvearrowright_{1\;e,r'} \sigma(uv^{-1})$$

where $r' = |S_w|$.

Proposition 7 *Let $r' = |S_{xyz}|$ for $x, y, z \in \mathcal{A}_{e,r}$. Then the completion condition holds for (x, y, z) modulo $\mathcal{R}_{e,r}$ if and only if the completion condition holds for $(\sigma(x), \sigma(y), \sigma(z))$, which is a triple over $\mathcal{A}_{e,r'}$, modulo $\mathcal{R}_{e,r'}$, where σ is the principal subscript map on S_{xyz}.*

Observe that in Proposition 6 we proved that completion holds for each different type of triple with respect to the corresponding principal subscript set. Combining Propositions 6 and 7, we obtain:

Theorem 8 *The augmented presentation $\langle \mathcal{A}_{e,r} \mid \mathcal{R}_{e,r} \rangle$ is right complete for any choice of e and r. Thus the B^+ is left cancellative, and the partial order defined by left division is a lattice.* □

Invoking Theorem 2 we observe that $B^+_{\texttt{rev}}$ is also left cancellative and its left division poset is a lattice. Applying the anti-isomorphism `rev`, we deduce that B^+ must be right cancellative, and that the partial order defined by right division must be a lattice. Hence we have:

Theorem 9 *The monoid B^+ is cancellative and the posets defined by left and right division are lattices.* □

Left completeness

A presentation is left complete if its reverse is right complete, where the reverse presentation is the presentation with the same generators and all relations of the form $\texttt{rev}(\rho_1) = \texttt{rev}(\rho_2)$ whenever $\rho_1 = \rho_2$ is a relation from the original presentation.

We observe that the presentation which we showed to be left complemented and left complete is in general neither right complemented (since it contains, for example, relations of type $\mathcal{C}_2$ which are of the form $ux = vx$) nor right complete (since it contains, for example, no relation of the form $ua_{pr} = va_q^{(i)}$, for $p < q < r$, while by Theorem 9, a_{pr} and $a_q^{(i)}$ do have a left common multiple, excluding completeness). We suspect that there is no presentation for B^+ on our generating set which is simultaneously left and right complete. Our point here is to show that $\langle \mathcal{A} \mid \texttt{rev}(\chi(\mathcal{R}^+)) \rangle$ is a presentation for B^+ which is left complete. This result is not necessary in the sequel, but is included for reasons of completeness.

Recall that we defined a permutation χ of $\mathcal{A}$ on page 3. Since the augmenting relations of types $\mathcal{C}_1, \ldots, \mathcal{C}_6$ are consequences of the relations $\mathcal{R}$, the relations of types $\chi(\mathcal{C}_1), \ldots, \chi(\mathcal{C}_6)$ are consequences of the relations $\chi(\mathcal{R}) = \texttt{rev}(\mathcal{R})$. Thus

$$\langle \mathcal{A} \mid \chi(\mathcal{R}^+) \rangle = \langle \mathcal{A} \mid \chi(\mathcal{R}) \rangle = B^+_{\texttt{rev}}.$$

This presentation is left complemented, because $\langle \mathcal{A} \mid \chi(\mathcal{R}^+) \rangle$ is left complemented. Furthermore, since

$$\begin{aligned} a^{-1}b \curvearrowright_{1\,\mathcal{R}} uv^{-1} &\iff (au = bv) \in \mathcal{R} \\ &\iff (\chi(au) = \chi(bv)) \in \chi(\mathcal{R}) \iff \chi(a^{-1}b) \curvearrowright_{1\,\chi(\mathcal{R})} \chi(uv^{-1}) \end{aligned}$$

we have that

Lemma 10 *The operator χ on $\mathcal{A}^*$ preserves word reversing: that is, for any word w over $\mathcal{A}$,*

$$\chi(\rho_w) = \rho_{\chi(w)}.$$

Hence application of χ maps the verification of the completeness condition for $\langle \mathcal{A} \mid \mathcal{R}^+ \rangle$ to the verification of the completeness condition for $\langle \mathcal{A} \mid \chi(\mathcal{R}^+) \rangle$. Thus $\langle \mathcal{A} \mid \chi(\mathcal{R}^+) \rangle$ is right complete and by applying `rev` we have that $\langle \mathcal{A} \mid \texttt{rev}(\chi(\mathcal{R}^+)) \rangle$ is left complete. Thus in summary,

Theorem 11 *Let $\mathcal{Q}^+ := \texttt{rev}(\chi(\mathcal{R}^+))$. Then the presentation $\langle \mathcal{A} \mid \mathcal{Q}^+ \rangle$ is a left complemented and left complete presentation for B^+.* □

2.2 Garside element

To show that the monoid B^+ is Garside, it remains to show that it has a Garside element – an element with the property that its set of left divisors is equal to its set of right divisors. We will use the following general result.

Lemma 12 *Let M be a cancellative monoid and suppose that $\overline{} : M \longrightarrow M$ is an automorphism of M for which there is a fixed element Δ of M such that $m\Delta = \Delta\overline{m}$ for all m in M. Then the set of left and right divisors of Δ are equal.*

Proof Suppose that $\Delta = mn$. Then $mn\overline{m} = \Delta\overline{m} = m\Delta$. Left cancelling m we have that $n\overline{m} = \Delta$, showing that every right divisor of Δ is also a left divisor. Furthermore, since $\overline{\Delta} = \Delta$, applying the inverse of the automorphism to $n\overline{m} = \Delta$ shows that every left divisor is also a right divisor. □

We will use the notation $\binom{q}{p}$ to represent the product $a_{p,p+1}a_{p+1,p+2}\cdots a_{q-1,q}$, where the subscripts are taken modulo n; if $p = q$ then this is the empty word, representing the identity; $\binom{p+1}{p} = a_{p,p+1}$; and if $p > q$ then $\binom{q}{p} = a_{p,p+1}\cdots a_{n-1,0}a_{0,1}\cdots a_{q-1,q}$. Clearly if (p,q,r) is cyclically ordered then $\binom{q}{p}\binom{r}{q} = \binom{r}{p}$. The following relations will be useful in the sequel.

Lemma 13 *1. For any $p \leq q$ and any i, $a_p^{(i)}\binom{q}{p} = \binom{q}{p}a_q^{(i)}$,*

2. For any (p,q,r) cyclically ordered, $a_{p,r}\binom{q}{p} = \binom{q}{p}a_{q,r}$ and $a_{r,p}\binom{q}{p} = \binom{q}{p}a_{r,q}$.

Proof Recall that by definition, $\binom{p+1}{p} = a_{p,p+1}$. Translating relations $(\mathcal{R}_5)$ and $(\mathcal{R}_3)$ respectively into this language, we have

1. for all p, $a_p^{(i)}\binom{p+1}{p} = \binom{p+1}{p}a_{p+1}^{(i)}$, and
2. for $(p,p+1,r)$ cyclically ordered, $a_{p,r}\binom{p+1}{p} = \binom{p+1}{p}a_{p+1,r}$ and $a_{r,p}\binom{p+1}{p} = \binom{p+1}{p}a_{r,p+1}$.

Since each (non-empty) $\binom{q}{p}$ is of the form $\prod_{t=p}^{q-1}\binom{t+1}{t}$, the result follows immediately. □

Let $\alpha := a_0^{(1)}a_0^{(0)}\binom{n-1}{0}$ (this is exactly the image of the element β of [BMR] under our isomorphism of the next section). In this section, we will show that this is the lcm of the generators $\mathcal{A}$, and is a Garside element for the braid monoid.

We begin by observing that the permutation ρ of $\mathcal{A}$ given by

$$a_{pq} \overset{\rho}{\longmapsto} a_{p+1,q+1} \quad \text{and} \quad a_p^{(i)} \overset{\rho}{\longmapsto} \begin{cases} a_0^{(i+1)} & \text{if } p = n-1, \text{ and} \\ a_{p+1}^{(i)} & \text{otherwise,} \end{cases}$$

(subscripts considered modulo n, exponents modulo e) extends to an automorphism of the braid monoid. It is enough to check that $\rho(\mathcal{R}) = \mathcal{R}$; by finiteness of $\mathcal{R}$ it suffices to show $\rho(\mathcal{R}) \subseteq \mathcal{R}$, which is clear by inspection. The action of ρ on $\mathcal{A}$ has n orbits, with representatives $a_{0,q}$ for each q in $0 < q < n$, together with $a_0^{(0)}$, where

$$\rho^p(a_{0,q}) = a_{p,p+q}, \quad \text{and for } 0 \leq p < n, \quad \rho^{ni+p}(a_0^{(0)}) = a_p^{(i)}.$$

Clearly, for any p, q, we have $\binom{q}{p} \overset{\rho}{\mapsto} \binom{q+1}{p+1}$. Also, writing t_q for $a_q^{(1)} a_q^{(0)}$, observe that $\rho(t_q) = t_{q+1}$. Using these facts, along with Lemma 13.2 and $\mathcal{C}_6$, we see that the element α is preserved by ρ:

$$\rho(\alpha) = \rho\left(t_0\binom{n-1}{0}\right) = t_{n-1}\binom{n-1}{n-2} = t_{n-1}a_{n-1,0}\binom{n-2}{0} = t_0 a_{0,n-1}\binom{n-2}{n-1} = t_0\binom{n-2}{0}a_{n-2,n-1} = \alpha.$$

Proposition 14 *There is a permutation $\bar{\cdot}$ of $\mathcal{A}$, such that for all a in $\mathcal{A}$,*

$$\alpha = a\alpha_a = \alpha_a\overline{a}$$

for some element α_a.

Proof Define $\gamma_0 = a_0^{(e-1)}\binom{n-1}{0}$ and $\gamma_q = t_q\binom{q-1}{0}\binom{n-1}{q}$ for $0 < q < n$. Then we have

$$\begin{aligned}
\alpha &= a_0^{(0)}\gamma_0 \\
&= a_0^{(e-1)}a_0^{(e-2)}\binom{n-1}{0} = a_0^{(e-1)}\binom{n-1}{0}a_{n-1}^{(e-2)} = \gamma_0 a_{n-1}^{(e-2)}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\alpha &= t_0\binom{q-1}{0}a_{q-1,q}\binom{n-1}{q} \\
&= t_0 a_{0q}\binom{q-1}{0}\binom{n-1}{q} = a_{0q}t_q\binom{q-1}{0}\binom{n-1}{q} = a_{0q}\gamma_q \\
&= t_q a_{q0}\binom{q-1}{0}\binom{n-1}{q} = t_q\binom{q-1}{0}a_{q,q-1}\binom{n-1}{q} = t_q\binom{q-1}{0}\binom{n-1}{q}a_{n-1,q-1} = \gamma_q a_{n-1,q-1}.
\end{aligned}$$

That is, for each of the ρ-orbit representatives x, $\alpha = x\alpha_x = \alpha_x\rho^k(x)$ where $k = -(n+1)$. Applying ρ an appropriate number of times, and recalling that $\rho(\alpha) = \alpha$, gives the equation

$$\alpha = x\alpha_x = \alpha_x\rho^{-(n+1)}(x)$$

for each generator x, for some α_x. Defining $\overline{x} := \rho^{-(n+1)}(x)$, we obtain the stated result. □

This proposition also shows that α is a common multiple of $\mathcal{A}$. Since the braid monoid has a complete presentation, every set with a common multiple has a least common multiple, so in particular $\mathcal{A}$has a least common multiple.

Proposition 15 *The element α is the least common multiple of the generators.*

Proof We use the fact that if a generator a does not divide x but a divides xb for some generator b then xb is the least common multiple of x and a. This follows directly from the fact that the relations are homogeneous.

We show firstly that $\binom{r-1}{0}$ is the least common multiple of $\{a_{pq} | 0 \le p < q < r\}$: this proceeds via induction on r. The base case $r = 2$ holds since a_{01} is the lcm of $\{a_{01}\}$. For any r, it is easily seen that $\binom{r}{0}$ is a common multiple of $\{a_{pq} | 0 \le p < q < r+1\}$. Furthermore, $\binom{r}{0} = \binom{r-1}{0}a_{r-1,r}$. Since the subscript r does not appear anywhere in $\binom{r-1}{0}$, $a_{r-1,r}$ does not divide $\binom{r-1}{0}$ (observed in Section 3); but $a_{r-1,r}$ does divide $\binom{r}{0}$, so by the observation of the previous paragraph we have that $\binom{r}{0}$ is the lcm of $\binom{r-1}{0}$ and $a_{r-1,r}$. But being a common multiple of $\{a_{pq} | 0 \le p < q < r+1\}$ implies that it is in fact the least common multiple of the whole set.

Thus in particular, $x := \binom{n-1}{0}$ is the least common multiple of $\{a_{pq} | 0 \le p < q < n\}$. By the same argument, since $a_0^{(1)}$ does not divide x (the only relations which could be applied being

$(\mathcal{R}_1), (\mathcal{R}_2), (\mathcal{R}_3)$ which contain no $a_p^{(i)}$) but $xa_{n-1}^{(1)} = a_0^{(1)}x$, we have that $a_0^{(1)}x$ is the lcm of $a_0^{(1)}$ and $\{a_{pq}|0 \leq p < q < n\}$.

Finally, since the lcm of $a_0^{(1)}$ and $a_0^{(0)}$ is $a_0^{(1)}a_0^{(0)} = a_0^{(0)}a_0^{(e-1)}$, we have that any common multiple of $a_0^{(1)}$ and $a_0^{(0)}$ must be divisible by $a_0^{(1)}a_0^{(0)}$; since x is not divisible by $a_0^{(0)}$ then $a_0^{(1)}x$ is not either. But since $a_0^{(1)}xa_{n-1}^{(0)} = \alpha$ is divisible by all the generators, then it is divisible by $a_0^{(0)}$, and hence by the observation of the first paragraph of this proof, is the lcm of $\{a_{pq}|0 \leq p < q < n\}$, $a_0^{(1)}$ and $a_0^{(0)}$. Thus α is the lcm of all the generators. □

Theorem 16 *The least common multiple α of the generators $\mathcal{A}$ is a Garside element for the braid monoid.*

Proof We have that α is the least common multiple of the generators and that $\alpha = a\alpha_a = \alpha_a\overline{a}$ for each a. Thus $a\alpha = a\alpha_a\overline{a} = \alpha\overline{a}$, for each a, and we can invoke Lemma 12. □

3 The presentation *does* present the braid group of $G(e, e, r)$

In this section we show that the group defined by the presentation introduced here is indeed the braid group of type $G(e, e, r)$.

The presentation of [BMR]

From [BMR] we know there is a presentation of the braid group of $G(e, e, n + 1)$ as follows:

Generators: $\mathcal{T} = \{\tau_2, \tau_2', \tau_3, \ldots, \tau_n, \tau_{n+1}\}$

Relations: The commuting relations are: $\tau_i\tau_j = \tau_j\tau_i$ whenever $|i-j| \geq 2$, together with $\tau_2'\tau_j = \tau_j\tau_2'$ for all $j \geq 4$. The others are:

$$\begin{aligned} \langle\tau_2\tau_2'\rangle^e &= \langle\tau_2'\tau_2\rangle^e \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} \qquad \text{for } i = 2, \ldots, n \\ \tau_2'\tau_3\tau_2' &= \tau_3\tau_2'\tau_3 \\ \tau_3\tau_2\tau_2'\tau_3\tau_2\tau_2' &= \tau_2\tau_2'\tau_3\tau_2\tau_2'\tau_3 \end{aligned}$$

where the expression $\langle ab\rangle^k$ denotes the alternating product $aba\cdots$ of length k.

We call these the BMR-generators and the BMR-relations. Recall that we started by looking for an answer to the following:

(1) Is the natural morphism $M \to B$ injective?

(2) Do we have

$$B = \{ \ \alpha^n b \mid (n \in \mathbb{Z}) \ (b \in M) \ \}?$$

where M denotes the monoid defined by the above presentation.

For ease on the eye, we will write 1 for τ_2', 2 for τ_2 and i for τ_i for $i = 3, \ldots, n-1$. Observe that

$$\begin{aligned} 2\ 13213\langle 21\rangle^{e-2} &= 321321\langle 21\rangle^{e-2} = 3213\langle 21\rangle^{e} = 3213\langle 12\rangle^{e} \\ &= 3213\ 1\langle 21\rangle^{e-1} = 32313\langle 21\rangle^{e-1} = 2\ 3213\langle 21\rangle^{e-1}. \end{aligned}$$

However $13213\langle 21\rangle^{e-2}$ and $3213\langle 21\rangle^{e-1}$ have no subwords appearing in $\mathcal{R}_0$, so are in singleton (hence distinct) $\mathcal{R}_0$-equivalence classes. Thus the monoid M is not cancellative, and so does not embed in B.

Proposition 17 *The monoid defined by the presentation for the braid group B of $G(e,e,r)$ given in [BMR] is not cancellative, and hence does not embed in the braid group. Denote by $\mathcal{S}$ the set of relations obtained by adjoining the relation*

$$13213\langle 21\rangle^{e-2} = 3213\langle 21\rangle^{e-1}$$

to the set of BMR relations for B. (Since this relation holds in B, adjoining the relation does not change the group thus presented.) Then $21^n 3213\langle 21\rangle^{e-2} =_{\mathcal{S}} 32132^n 1\langle 21\rangle^{e-2}$ for every n, while $21^n 3213$ and $32132^n 1$ are in distinct $\mathcal{S}$-classes for every $n > 1$. Thus $M(\mathcal{T}, \mathcal{S})$ is not cancellative either.

Proof The new relation can be written $1w = wx$ where w is $3213\langle 21\rangle^{e-2}$ and x is 2 if e is even, and 1 if e is odd. For the same letter x, $\langle 21\rangle^e x \equiv \langle 21\rangle^{e+1} \equiv 2\langle 12\rangle^e$ hence for any n we have that

$$21^n 3213\langle 21\rangle^{e-2} \equiv 21^n w =_{\mathcal{S}} 21wx^{n-1} =_{\mathcal{R}} 3213\langle 21\rangle^e x^{n-1} =_{\mathcal{S}} 32132^{n-1}\langle 21\rangle^e \equiv 32132^n 1\langle 21\rangle^{e-2}.$$

However for every $n > 1$, $21^n 3213$ and $32132^n 1$ are in singleton $\mathcal{R}$-equivalence classes; so the monoid $M(\mathcal{T}, \mathcal{S})$ is not cancellative, and so does not embed in B. □

In the next subsection we show that the new presentation we have is a presentation for the braid group of $G(e,e,r)$, giving a monoid answering affirmatively to all the desired properties. Using the facts that with the new presentation we have a solution to the word problem and can calculate least common multiples, we show at the end of the section that there can be no finite presentation for the braid group of $G(e,e,r)$ on the BMR-generators for which the corresponding monoid embeds in the group.

The new presentation presents $G(e,e,r)$

Let B denote the group defined by the BMR-presentation, and G denote the group defined by our presentation, as given in Section 1.1. Define $\varphi : B \to G$ as follows: on generators,

$$\varphi : \begin{cases} \tau_2 & \mapsto\ a_0^{(1)} \\ \tau_2' & \mapsto\ a_0^{(0)} \\ \tau_i & \mapsto\ a_{i-3,i-2} \quad \text{for } i = 3, \ldots, n+1, \end{cases}$$

and extend homomorphically. To see this is well-defined it suffices to show that whenever $\rho_1 = \rho_2$ is a defining relation in B, then $\varphi(\rho_1)$ can be transformed into $\varphi(\rho_2)$ using the defining relations of G. This can be seen for the commuting relations simply by inspection – they appear as incarnations of $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ and $(\mathcal{R}_4)$.

Observe that $\langle a_0^{(1)} a_0^{(0)} \rangle^{e-1} = (a_0^{(e-1)} a_0^{(e-2)}) \ \cdots \ a_0^{(1)}$, so

$$\langle a_0^{(0)} a_0^{(1)} \rangle^e = a_0^{(0)} \langle a_0^{(1)} a_0^{(0)} \rangle^{e-1} = a_0^{(0)} (a_0^{(e-1)} a_0^{(e-2)}) \ \cdots \ a_0^{(1)} = \langle a_0^{(1)} a_0^{(0)} \rangle^e.$$

Thus we have $\langle \tau_2 \tau_2' \rangle^e \overset{\varphi}{\mapsto} \langle a_0^{(1)} a_0^{(0)} \rangle^e = \langle a_0^{(0)} a_0^{(1)} \rangle^e \overset{\varphi}{\leftarrow\!\shortmid} \langle \tau_2' \tau_2 \rangle^e$ as required.

From $(\mathcal{R}_3)$ we have, for $i \geq 3$,

$$\tau_i \tau_{i+1} \tau_i \overset{\varphi}{\mapsto} a_{j-1,j} a_{j,j+1} a_{j-1,j} \overset{\mathcal{R}_3}{=} a_{j,j+1} a_{j-1,j+1} a_{j-1,j} \overset{\mathcal{R}_3}{=} a_{j,j+1} a_{j-1,j} a_{j,j+1} \overset{\varphi}{\leftarrow\!\shortmid} \tau_{i+1} \tau_i \tau_{i+1},$$

where $j = i - 3$. Using $(\mathcal{R}_5)$, we get

$$\tau_2 \tau_3 \tau_2 \overset{\varphi}{\mapsto} a_0^{(1)} a_{01} a_0^{(1)} \overset{\mathcal{R}_5}{=} a_{01} a_1^{(1)} a_0^{(1)} \overset{\mathcal{R}_5}{=} a_{01} a_0^{(1)} a_{01} \overset{\varphi}{\leftarrow\!\shortmid} \tau_3 \tau_2 \tau_3,$$

and similarly replacing τ_2 with τ_2' and $a_0^{(1)}$ with $a_0^{(0)}$. Finally, using the equation after $(\mathcal{C}_6)$,

$$\tau_3 \tau_2 \tau_2' \tau_3 \tau_2 \tau_2' \overset{\varphi}{\mapsto} a_{01} t_0 a_{01} t_0 = t_0 a_{01} t_0 a_{01} \overset{\varphi}{\leftarrow\!\shortmid} \tau_2 \tau_2' \tau_3 \tau_2 \tau_2' \tau_3.$$

Thus we have that φ is well-defined.

For any $0 \leq p < q < n$, we use the notation

$$\tbinom{q}{p} := \begin{cases} \varepsilon \text{ (the empty word)} & \text{if } p = q, \\ \tau_{p+3} \tau_{p+4} \cdots \tau_{q+2} & \text{otherwise.} \end{cases}$$

Thus, for example, $\binom{n-1}{0} = \tau_3 \tau_4 \cdots \tau_{n+1}$. Observe that $\binom{i+1}{i} = \tau_{i+3}$ for any $0 \leq i < n$, and this maps to $a_{i,i+1}$ under φ. Thus $\binom{q}{p}$ maps under φ to the element $\binom{q}{p}$ of G defined in Section 2.2. Define $\beta := \tau_2 \tau_2' \binom{n-1}{0}$ (which is precisely the generator of the centre given in [BMR]); thus $\beta \overset{\varphi}{\to} \alpha$, the least common multiple of the generators of the braid monoid.

In Section 2.2 we observed that defining $\overline{x} := \alpha^{-1} x \alpha$ restricts to a permutation on $\mathcal{A}$, in particular such that

$$\overline{a_{pq}} = a_{p-1,q-1} \quad \text{and} \quad \overline{a_p^{(i)}} = \begin{cases} a_{n-1}^{(i-2)} & \text{if } i = 0, \\ a_{p-1}^{i-1} & \text{otherwise.} \end{cases}$$

We will mimic this in B.

For $0 \leq p < q < n$ define $b_{pq} := \binom{q}{p+1}^{-1} \binom{q}{p}$, $b_{qp} := \binom{p}{0}^{-1} \binom{q}{0}^{-1} 1^{-1} 2^{-1} 321 \binom{q}{0} \binom{p}{0}$, and

$$b_0^{(i)} = \begin{cases} 1 & \text{if } i = 0, \\ \langle 21 \rangle^i (\langle 21 \rangle^{i-1})^{-1} & \text{for } i = 1, \ldots e-1, \end{cases}$$

where we again write 1 for τ_2', 2 for τ_2. Lastly, define $b_p^{(i)} := \binom{p}{0}^{-1} b_0^{(i)} \binom{p}{0}$.

Lemma 18

$$\beta^{-1} b_{pq} \beta = b_{p-1,q-1} \quad \textit{and} \quad \beta^{-1} b_p^{(i)} \beta = \begin{cases} b_{n-1}^{(i-2)} & \textit{if } i = 0, \\ b_{p-1}^{i-1} & \textit{otherwise.} \end{cases}$$

Proof The proof is by calculation of the various cases.

1. We show $\beta^{-1}b_{pq}\beta = b_{p-1,q-1}$ for $p < q$; there are two cases, in the second, $p > 0$.

$$\begin{array}{llcl}
(i) & \beta b_{n-1,q-1}\beta^{-1} & = & 21\binom{n-1}{0}\binom{q-1}{0}^{-1}\binom{n-1}{0}^{-1}1^{-1}2^{-1}321\binom{n-1}{0}\binom{q-1}{0}\binom{n-1}{0}^{-1}1^{-1}2^{-1} \\
& & = & 21\binom{q}{1}^{-1}\binom{n-1}{0}\binom{n-1}{0}^{-1}1^{-1}2^{-1}321\binom{n-1}{0}\binom{n-1}{0}^{-1}\binom{q}{1}1^{-1}2^{-1} \\
& & = & \binom{q}{1}^{-1}211^{-1}2^{-1}3211^{-1}2^{-1}\binom{q}{1} \\
& & = & \binom{q}{1}^{-1}3\binom{q}{1} \;=\; \binom{q}{1}^{-1}\binom{q}{0} \;=\; b_{0,q},
\end{array}$$

$$\begin{array}{llcl}
(ii) & \beta b_{p-1,q-1}\beta^{-1} & = & 21\binom{n-1}{0}\binom{q-1}{p}^{-1}\binom{q-1}{p-1}\binom{n-1}{0}^{-1}1^{-1}2^{-1} \\
& & = & 21\binom{q}{p+1}^{-1}\binom{n-1}{0}\binom{n-1}{0}^{-1}\binom{q}{p}1^{-1}2^{-1} \\
& & = & \binom{q}{p+1}^{-1}211^{-1}2^{-1}\binom{q}{p} \;=\; \binom{q}{p+1}^{-1}\binom{q}{p} \;=\; b_{p,q}.
\end{array}$$

2. Next we show that $\beta^{-1}b_{qp}\beta = b_{q-1,p-1}$ for $p < q$; again two cases, and in the second, $p > 0$.

$$\begin{array}{llcl}
(i) & \beta b_{q-1,n-1}\beta^{-1} & = & 21\binom{n-1}{0}\binom{n-1}{q}^{-1}\binom{n-1}{q-1}\binom{n-1}{0}^{-1}1^{-1}2^{-1} \\
& & = & 21\binom{q}{0}\binom{q-1}{0}^{-1}1^{-1}2^{-1} \\
& & = & 21\binom{q}{1}^{-1}\binom{q}{0}1^{-1}2^{-1} \\
& & = & \binom{q}{1}^{-1}2131^{-1}2^{-1}\binom{q}{1} \\
& & = & \binom{q}{0}^{-1}32131^{-1}2^{-1}3^{-1}\binom{q}{0} \\
& & = & \binom{q}{0}^{-1}1^{-1}2^{-1}32133^{-1}\binom{q}{0} \;=\; \binom{q}{0}^{-1}1^{-1}2^{-1}321\binom{q}{0} \;=\; b_{q0}
\end{array}$$

$$\begin{array}{llcl}
(ii) & \beta b_{q-1,p-1}\beta^{-1} & = & 21\binom{n-1}{0}\binom{p-1}{0}^{-1}\binom{q-1}{0}^{-1}1^{-1}2^{-1}321\binom{q-1}{0}\binom{p-1}{0}\binom{n-1}{0}^{-1}1^{-1}2^{-1} \\
& & = & \binom{p}{1}^{-1}\binom{q}{1}^{-1}21\binom{n-1}{0}1^{-1}2^{-1}321\binom{n-1}{0}^{-1}1^{-1}2^{-1}\binom{q}{1}\binom{p}{1} \\
& & = & \binom{p}{1}^{-1}\binom{q}{1}^{-1}21341^{-1}2^{-1}3214^{-1}3^{-1}1^{-1}2^{-1}\binom{q}{1}\binom{p}{1} \\
& & = & \binom{p}{1}^{-1}\binom{q}{1}^{-1}3^{-1}4^{-1}1^{-1}2^{-1}32143\binom{q}{1}\binom{p}{1} \\
& & = & \binom{p}{0}^{-1}\binom{q}{0}^{-1}1^{-1}2^{-1}321\binom{q}{0}\binom{p}{0} \;=\; b_{qp}
\end{array}$$

where in the second last line we used

$$\begin{array}{rcl}
2134\bar{1}\,\bar{2}321\bar{4}\,\bar{3}\,\bar{1}\,\bar{2} = 213\bar{1}\,\bar{2}43\bar{4}21\bar{3}\,\bar{1}\,\bar{2} & = & 213\bar{1}\,\bar{2}\,\bar{3}4321\bar{3}\,\bar{1}\,\bar{2} = \bar{3}\,\bar{1}\,\bar{2}3214\bar{1}\,\bar{2}\,\bar{3}213 \\
& = & \bar{3}\,\bar{1}\,\bar{2}34\,\bar{3}213 = \bar{3}\,\bar{1}\,\bar{2}\,\bar{4}34213 = \bar{3}\,\bar{4}\,\bar{1}\,\bar{2}32143,
\end{array}$$

where the overlining indicates inverse.

3. Finally we consider the generators $b_q^{(i)}$:

$$\begin{array}{llcl}
(i) & \beta b_{n-1}^{(i-2)}\beta^{-1} & = & 21\binom{n-1}{0}\binom{n-1}{0}^{-1}b_0^{(i-2)}\binom{n-1}{0}\binom{n-1}{0}^{-1}1^{-1}2^{-1} \;=\; 21b_0^{(i-2)}1^{-1}2^{-1} = b_0^{(i)}
\end{array}$$

$$\begin{array}{llcl}
(ii) & \beta b_{q-1}^{(i-1)}\beta^{-1} & = & 21\binom{n-1}{0}\binom{q-1}{0}^{-1}b_0^{(i-1)}\binom{q-1}{0}\binom{n-1}{0}^{-1}1^{-1}2^{-1} \\
& & = & \binom{q}{1}^{-1}21\binom{n-1}{0}b_0^{(i-1)}\binom{n-1}{0}^{-1}1^{-1}2^{-1}\binom{q}{1} \\
& & = & \binom{q}{0}^{-1}3213b_0^{(i-1)}3^{-1}1^{-1}2^{-1}3^{-1}\binom{q}{0} \;=\; \binom{q}{0}^{-1}b_0^{(i)}\binom{q}{0} \;=\; b_q^{(i)},
\end{array}$$

where in the last line we use $3213b_0^{(i-1)} = b_0^{(i)}3213$; to see this, observe that the defining relations for B imply $21(3213) = (3213)21$ and $2^{-1}(3213) = (3213)1^{-1}$. Firstly, if $i = 0$ we have

$$b_0^{(i)}(3213) = 1(3213) = 2^{-1}21(3213) = 2^{-1}(3213)21 = (3213)1^{-1}21 = (3213)b_0^{(e-1)}.$$

Next, suppose that i is even and larger than 0. Then

$$\begin{aligned} b_0^{(i)}\ (3213) = \langle 21\rangle^i(\langle 21\rangle^{i-1})^{-1}(3213) &= \langle 21\rangle^{(i)} 2^{-1}(3213)\left(\langle 21\rangle^{i-2}\right)^{-1} \\ &= \langle 21\rangle^{(i)}(3213)1^{-1}\left(\langle 21\rangle^{i-2}\right)^{-1} \\ &= (3213)\langle 21\rangle^{(i)}1^{-1}\left(\langle 21\rangle^{i-2}\right)^{-1} \\ &= (3213)\langle 21\rangle^{(i-1)}\left(\langle 21\rangle^{i-2}\right)^{-1} = (3213)b_0^{(i-1)}. \end{aligned}$$

Finally, suppose that i is odd. Then we have

$$b_0^{(i)}\ (3213) = 21\left(b_0^{(i-1)}\right)^{-1}(3213) = 21(3213)\left(b_0^{(i-2)}\right)^{-1} = (3213)21\left(b_0^{(i-2)}\right)^{-1} = (3213)b_0^{(i-1)}.$$

□

Proposition 19 *The elements b_{pq}, b_{qp} and $b_0^{(i)}$ map under φ to the corresponding generators a_{pq}, a_{qp} and $a_0^{(i)}$ of G respectively. Thus φ is surjective.*

Proof Firstly, for each $0 \leq i < e-1$ we have $b_{i,i+1} = \binom{i+1}{i} \overset{\varphi}{\mapsto} a_{i,i+1}$. We prove the general case $b_{pq} \overset{\varphi}{\mapsto} a_{pq}$ by induction on $q-p$: we use the fact that $\binom{j+1}{i} = \binom{j}{i}\binom{j+1}{j}$, so that $b_{p,q+1} = \binom{q+1}{q}^{-1} b_{pq}\binom{q+1}{q}$, and hence

$$b_{p,q+1} = \overset{\varphi}{\mapsto} a_{q,q+1}^{-1}a_{pq}a_{q,q+1} = a_{q,q+1}^{-1}a_{q,q+1}a_{p,q+1} = a_{p,q+1}.$$

Secondly, we observe that $b_0^{(0)}(=\tau_1)$ and $b_0^{(1)}(=\tau_2)$ by definition map to $a_0^{(0)}$ and $a_0^{(1)}$ respectively. Then we continue inductively on i: clearly $b_0^{(i+1)}b_0^{(i)} = 21$ for $i > 1$, so

$$b_0^{(i+1)} = 21\left(b_0^{(i)}\right)^{-1} \overset{\varphi}{\mapsto} a_0^{(1)}a_0^{(0)}(a_0^{(i)})^{-1} = a_0^{(i+1)}a_0^{(i)}(a_0^{(i)})^{-1} = a_0^{(i+1)}.$$

Thus the generators a_{pq} with $p < q$ and $a_0^{(i)}$ for all i are in the image of φ. For $p < q$, we have that $a_{qp} = \alpha^{-(n-q)}a_{0p-q}\alpha^{n-q} = \varphi(\beta^{-(n-q)}b_{0p-q}\beta^{n-q})$, so is in $\varphi(B)$. Finally, writing X_p for $\{a_p^{(i)} \mid 0 \leq i < n\}$, we have $\alpha^{-1}X_p\alpha = X_{p-1}$ for any p, so $a_p^{(i)}$ will appear in

$$\varphi\left(\beta^{-(n-p)}\ \{b_0^{(i)} \mid 0 \leq i < n\}\ \beta^{n-p}\right) = X_p.$$

Thus all the generators of G lie in the image of φ. □

To show that B and G are isomorphic, it remains to show that φ is one-to-one. This is the case if the elements b_{pq}, b_{qp} and $b_p^{(i)}$ satisfy the defining relations for the a_{pq}, a_{qp} and $a_p^{(i)}$.

Proposition 20 *The defining relations for G in terms of a_{pq}, etc., hold in B in terms of b_{pq}, etc.*

Proof The defining relations are stated in terms of cyclically ordered sequences $(p_1, \ldots, p_f)$. Notice that such a sequence is cyclically ordered precisely when $(p_1-1, \ldots, p_f-1)$ is cyclically ordered; that is, the action of integer shifting preserves cyclic order.

Using conjugation by β, it thus suffices to check the relations for but one sequence in any orbit under integer shifting: for example, suppose that (p,q,r,s) is cyclically ordered; then so is $(0,q-p,r-p,s-p)$, and $b_{pq}b_{rs} = b_{rs}b_{pq}$ if and only if $b_{0,q-p}b_{r-p,s-p} = b_{0,q-p}b_{r-p,s-p}$ (conjugation by β^p); thus if the relation holds with respect to cyclically ordered quadruples of the form $(0,s,t,u)$, then it holds for all cyclically ordered quadruples.

Furthermore, the subgroup of B generated by $b_{i,i+1}$ is isomorphic to the braid group of type A_n; the b_{pq} for $0 \leq p < q < n$ are precisely the BKL-generators (see [BKL]) and for linearly ordered sequences, the relations $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are precisely the BKL-relations these generators; thus they are already known to hold. Since every cyclically ordered sequence can be integer shifted to a linearly ordered one, we thus have that $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ hold in B.

Now consider any cyclically ordered triple (p,q,r); we want to check $\mathcal{R}_4$ holds: that is, for each i, $b_{pq}b_r^{(i)} = b_r^{(i)}b_{pq}$. Now conjugating by β^r, this holds if and only if $b_{p-r,q-r}b_0^{(j)} = b_0^{(j)}b_{p-r,q-r}$, for some j. Now $0 < p-r < q-r$, and for any $0 < s < t$ and any j,

$$b_{st}b_0^{(j)} = \binom{t}{s+1}^{-1}\binom{t}{s}b_0^{(j)} = b_0^{(j)}\binom{t}{s+1}^{-1}\binom{t}{s} = b_0^{(j)}b_{st}$$

since τ_2 and τ_2' commute with $\binom{t}{s}$ whenever $s \geq 1$.

Consider $0 < q < n$, and any i; we will verify $\mathcal{R}_5$ when $p = 0$. Firstly, $b_{0q}b_q^{(i)} = \binom{q}{1}^{-1}b_0^{(i)}\binom{q}{0} = b_0^{(i)}\binom{q}{1}^{-1}\binom{q}{0}$, since $\binom{t}{s}$ and $b_0^{(i)}$ commute whenever $s \geq 1$. This is $b_0^{(i)}b_{0q}$ by definition. On the other hand, $b_q^{(i)}b_0^{(i)} = \binom{q}{0}^{-1}b_0^{(i)}3\binom{q}{1}b_0^{(i)} = \binom{q}{0}^{-1}b_0^{(i)}3b_0^{(i)}\binom{q}{1}$. We will show that $b_0^{(i)}3b_0^{(i)} = 3b_0^{(i)}3$ for all i; for $i = 0,1$ this is precisely the defining relations $\tau_2'\tau_3\tau_2' = \tau_3\tau_2'\tau_3$ and $\tau_2\tau_3\tau_2 = \tau_3\tau_2\tau_3$ respectively. We continue inductively: let $2 \leq i+1 < e$. Then

$$\begin{aligned} 3b_0^{(i+1)^{-1}}213 \;=\; 3b_0^{(i)}3 \;=\; b_0^{(i)}3b_0^{(i)} \;=\; b_0^{(i+1)^{-1}}213b_0^{(i)} &= b_0^{(i+1)^{-1}}3^{-1}3213b_0^{(i)} \\ &= b_0^{(i+1)^{-1}}3^{-1}b_0^{(i+1)}3213, \end{aligned}$$

where we use $3213b_0^{(i)} = b_0^{(i+1)}3213$, shown in the proof of Lemma 18 (see page 18). Cancelling 213 from the right, and rearranging, gives $3b_0^{(i+1)}3 = b_0^{(i+1)}3b_0^{(i+1)}$ as required. Thus we have $b_q^{(i)}b_0^{(i)} = \binom{q}{0}^{-1}3b_0^{(i)}3\binom{q}{1} = \binom{q}{1}^{-1}b_0^{(i)}\binom{q}{0}$, as required.

Now take any $0 < p \neq q < n$ and any i; a relation of type $\mathcal{R}_5$ is of the form $b_{pq}b_q^{(j)} = b_q^{(j)}b_p^{(i)} = b_p^{(i)}b_{pq}$ for a fixed j. Conjugating by β^p, this holds if and only if $b_{0,q-p}b_{q-p}^{(k)} = b_{q-p}^{(k)}b_0^{(l)} = b_0^{(l)}b_{0,q-p}$ holds, for certain k and l. Since the image of the first equation holds in G, its conjugate does as well, and since we know $b_{0,x}b_x^{(k)} = b_x^{(k)}b_0^{(k)}$ in G, for all x and k, then we have $k = l$. Thus the second equation in terms of the b-generators is of the form shown in the previous paragraph to hold.

Finally, we show that all relations of the form $\mathcal{R}_6$ hold. But for any $0 \leq p < n$ and $0 \leq i < e$, $b_p^{(i)}b_p^{(i-1)} = \binom{p}{0}^{-1}b_0^{(i)}b_0^{(i-1)}\binom{p}{0}$. We observed earlier that $b_0^{(i)}b_0^{(i-1)} = 21$ for all $0 < i < e$; and if $i = 0$, we have $b_0^{(0)}b_0^{(e-1)} = 1\langle 21\rangle^{e-1}\left(\langle 21\rangle^{e-2}\right)^{-1} = \langle 12\rangle^{e}\left(\langle 21\rangle^{e-2}\right)^{-1} = \langle 21\rangle^{e}\left(\langle 21\rangle^{e-2}\right)^{-1} = 21$. Thus $b_p^{(i)}b_p^{(i-1)} = \binom{p}{0}^{-1}21\binom{p}{0}$, which is independent of i, so $\mathcal{R}_6$ holds. □

Thus we can complete the main result.

Theorem 21 *The presentation of Section 1.1 gives rise to a Garside structure for the braid group B of the (complex) reflection group $G(e,e,r)$, where the Garside element is the least common multiple of the generators in the Garside monoid; and furthermore, for which the generators map to reflections under the natural map $\nu : B \to G(e,e,r)$.*

Proof The only part of the assertion remaining unproved is the last phrase. We know from [BMR] that the images of the BMR-generators under ν are reflections in $G(e,e,r)$. From the definitions given immediately before the statement of Lemma 18, we see that each of the new generators is a conjugate of a BMR-generator. Thus its image under ν is a conjugate of a reflection, and hence is a reflection. □

4 No finite nice presentation on BMR-generators exists

We will call a presentation of a braid group "nice" if it is of the form given in the above result.

In this section, we will show that there is no finite presentation of the braid group of type $G(e,e,r)$ on the BMR-generators, where the relations are all positive words over these generators, such that the corresponding monoid embeds in the braid group. Recall that we have an infinite collection of equations in the braid group of the form $21^n3213 = 32132^n1$. In the monoid, however, none of these implies another, and for all $n > 1$, none of these is implied by the defining relations. Thus it suffices to show that there is no finite collection of relations on the BMR-generators of the required form which imply all of these equations.

In order to derive $21^n3213 = 32132^n1$ we need to be able to rewrite 21^n3213; thus we need to rewrite some word of the form 21^k or of the form 1^k3213 in terms of the generators $\{1,2,3\}$, where "rewrite w_1 in terms of the generators $\{1,2,3\}$" means: find an equation $w_1 = w_2$ in the braid group B where w_2 is a (positive) word over $\{1,2,3\}$. Let M denote the monoid generated by the b_{pq}, b_{qp} and $b_p^{(i)}$; since M embeds in B this is exactly the same as finding an equation $w_1 = w_2$ which holds in M where w_2 is a word over $\{1,2,3\}$. Since M is a Garside monoid, we have solutions to the word problem and can calculate lcm's, which will do the work for us here. Recall that $1 = a_0^{(0)}, 2 = a_0^{(1)}$ and $3 = a_{01}$.

Proposition 22 *There is no finite presentation for the braid group of type $G(e,e,r)$ with generating set that given in [BMR] and with all relations positive words, for which the corresponding monoid embeds in the braid group.*

Proof There is a surjection $B(e,e,n+1) \to B_n$, the braid group of type A on n strings, given by $1,2 \mapsto \sigma_1$ and $i \mapsto \sigma_{i-1}$ for $i \geq 3$. Thus $21^k \overset{\varphi}{\mapsto} \sigma_1^{k+1}$; so 21^k can only be rewritten in terms of $\{2,1\}$. We may suppose that $21^k = 1w$ for some word w over $\{2,1\}$; we have that $21^k = 211^{k-1} = 1b_0^{(e-1)}\ 1^{k-1}$; so by left cancellation, $w = b_0^{(e-1)}\ 1^{k-1}$. However $b_0^{(e-1)}\ 1^{k-1} \equiv b_0^{e-1}\ (b_0^{(0)})^{k-1}$ is in a singleton equivalence class in M, so can never be rewritten in terms of $\{1,2\}$. Thus it remains to show that 1^k3213 cannot be rewritten in terms of $\{1,2,3\}$.

Let U denote the word 1^k3213; it suffices to show that for any $k \geq 0$, we cannot rewrite U in terms of $\{1,2,3\}$. Since M is Garside, then it has the 'reduction property', and so we can quickly determine divisibility in M by a generator using the method of a-chains (see [C2]).

Observe that $U = 3213u$ where $u = (a_0^{(e-2)})^k$; clearly u is not divisible by 1, 2, or 3, so is certainly not rewriteable over these letters. Also, neither 1 nor 2 divides $3u$ so U cannot be rewritten over $\{1, 2, 3\}$ to begin with 321. Since $13 = 3a_1^{(0)}$, we have $U = 323v$ where $v = a_1^{(0)}u$, which is not divisible by $1, 2$ or 3; furthermore, $3v$ is not divisible by 2, so we see that U cannot be rewritten $\{1, 2, 3\}$ to begin with 32. Next, we write $U = 313w$ where $w = a_1^{(e-1)}u$; this is not divisible by 1,2 or 3; and $3w$ is not divisible by 1 or 2. Thus U cannot be rewritten over $\{1, 2, 3\}$ to start with 31. Finally, $U = 3^2x$ where $x = a_1^{(1)}a_1^{(0)}u$; x is not divisible by 1,2 or 3, so we have that U cannot be rewritten to begin with 33 either. This exhausts all the possibilites for rewriting U over $\{1, 2, 3\}$ starting with 3.

On the other hand, we can write $U = 213y$ where $y = a_{10}u$, which is not divisible by $1, 2$ or 3; so not rewriteable over these letters. Moreover, $3y$ is not divisible by 1 or 2; and $13y$ is not divisible by 2. Thus we have that U cannot be rewritten to begin with 21 or 22. So now we write $U = 232v$ where $v = a_1^{(0)}u$, which is not divisible by $1, 2$ or 3; furthermore, $2v$ is not divisible by 1 or 3. In this way we have exhausted all the possibilites for rewriting U over $\{1, 2, 3\}$ starting with 2.

Finally, we want to show that U cannot be rewritten to begin with 1. By left cancellation, we may assume that $k = 0$; then we have that $3213 = 131a_1^{-1}$. But $31a_1^{-1}$ (being in a singleton equivalence class in M) cannot be rewritten over $\{1, 2, 3\}$. The result follows. □

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