An efficient condition for a graph to be Hamiltonian

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Abstract

Let $G = (V, E)$ be a 2-connected simple graph and let $d_G(u, v)$ denote the distance between two vertices $u, v$ in $G$. In this paper, it is proved: if the inequality $d_G(u) + d_G(v) \geq |V(G)| - 1$ holds for each pair of vertices $u$ and $v$ with $d_G(u, v) = 2$, then $G$ is Hamiltonian, unless $G$ belongs to an exceptional class of graphs. The latter class is described in this paper. Our result implies the theorem of Ore [Note on Hamilton circuits, Amer. Math. Monthly 67 (1960) 55]. However, it is not included in the theorem of Fan [New sufficient conditions for cycles in graph, J. Combin. Theory Ser. B 37 (1984) 221–227].

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1. Introduction and main result

Let $G$ be a simple graph. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. A subgraph induced by a subset $X \subseteq V(G)$ is denoted by $G[X]$. In addition, $G - X = G[V(G) - X]$.

For a vertex $u$ of $G$, the set $N_G(u) = \{v \mid uv \in E(G)\}$ is called the neighborhood of $u$ in $G$. The degree of $u$ in $G$ is $|N_G(u)|$, denoted by $d_G(u)$ or $d(u)$. The graph $G$ is said to be $k$-regular, if $d(x) = k$ for all $x \in V(G)$.

Let $C$ be a cycle in $G$. A path $P$ between $x$ and $y$ is called a $C$-bypass if $|V(P)| \geq 3$ and $V(P) \cap V(C) = \{x, y\}$. The gap of $P$ with respect to $C$ is the length of the shortest path between $x$ and $y$ in the cycle $C$.

The graph $G$ is connected, if it contains a path between any two vertices. $G$ is said to be $k$-connected, if $|V(G)| \geq k + 1$ and $G - S$ is connected for each $S \subset V(G)$ with $|S| \leq k - 1$.

If $G$ is connected, then for two vertices $u, v \in V(G)$, the distance $d(u, v)$ between $u$ and $v$ is the length of a shortest path between $u$ and $v$ in $G$.

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A cycle of $G$ is called a Hamiltonian cycle, if it contains all vertices of $G$. The graph $G$ is said to be Hamiltonian, if it has a Hamiltonian cycle.

It is well-known that the Hamiltonian cycle problem is $\text{NP}$-complete, and many sufficient conditions, respect to various parameters, have been found, e.g., the following ones.

**Theorem 1.1 (Ore [6]).** Let $G$ be a graph with $n$ vertices. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u$ and $v$, then $G$ is Hamiltonian.

**Theorem 1.2 (Fan [3]).** Let $G$ be a 2-connected graph with $n \geq 3$ vertices. If for all vertices $u$ and $v$ with $d(u, v) = 2$, $\max\{d(u), d(v)\} \geq \frac{n}{2}$ holds, then $G$ is Hamiltonian.

The last well-known theorem is a corollary of the following statement.

**Theorem 1.3 (Fan [3]).** Let $G$ be a 2-connected graph with $n$ vertices and $c \in \mathbb{N}$, $3 \leq c \leq n$. If for all vertices $u$ and $v$ with $d(u, v) = 2$, $\max\{d(u), d(v)\} \geq \frac{c}{2}$ holds, then $G$ contains a cycle of length at least $c$.

Note that the conditions in Theorems 1.2 and 1.3 are related to degrees of vertices with distance exactly two. Moreover, the condition in Theorem 1.1 implies $N_G(u) \cap N_G(v) \neq \emptyset$ if $uv \notin E(G)$, i.e., the distance between any two nonadjacent vertices is exactly two.

Some further sufficient conditions related to degrees of vertices with distance exactly two for Hamiltonian graphs have been found in [1,2,4,5]. For example, the next theorem is an improvement of Theorem 1.2, where $\alpha(G)$ is the independence number of $G$ (i.e., the maximal number of vertices in $G$, any two of which are not adjacent), and $H \cup \mathcal{G}_n$ is defined according to [1].

**Theorem 1.4 (Benhocine and Wojda [1]).** Let $G$ be a 2-connected graph on $n$ vertices with $\alpha(G) \leq \frac{1}{2}n$. If $\max\{d(u), d(v)\} \geq \frac{1}{2}(n - 1)$ for all vertices $u$ and $v$ with $d(u, v) = 2$, then $G$ is Hamiltonian or $G \in H \cup \mathcal{G}_n$.

Clearly, it can be done in polynomial time to check whether a given graph satisfies the conditions in Theorems 1.1 and 1.3, respectively. Such conditions are said to be efficient.

In this paper, we give an efficient condition for graphs to be Hamiltonian. To present our main result, we consider a special class of such graphs, namely,

$$\mathcal{L}_{2m+1} = \{Z_m \lor (K_m^c + \{u\}) \mid Z_m \text{ is a graph with } m \text{ vertices}\},$$

where $K_m^c$ is a set of $m$ vertices (also as the complement of the complete graph $K_m$) and $u$ is another single vertex, furthermore, the edge set of $Z_m \lor (K_m^c + \{u\})$ consists of $E(Z_m)$ and $\{xy \mid x \in V(Z_m) \text{ and } y \in K_m^c \cup \{u\}\}$ (see Fig. 1).

It is easy to check that every graph $G \in \mathcal{L}_{2m+1}$ for $m \geq 2$ is non-Hamiltonian, but it satisfies the condition that $d(u) + d(v) \geq (2m + 1) - 1$ for each pair of vertices $u$ and $v$ with $d(u, v) = 2$.

In this paper, we prove the following:

**Theorem 1.5 (Main result).** Let $G$ be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of vertices $u$ and $v$ with $d(u, v) = 2$, then $G$ is Hamiltonian, unless $n$ is odd and $G \in \mathcal{L}_n$.

It is easy to see that the condition of Theorem 1.5 is weaker than the condition of Ore’s theorem (see Theorem 1.1). The next example shows that there are graphs, whose Hamiltonicity can be verified by Theorem 1.5, but neither by Ore’s theorem nor by Fan’s theorem (see Theorem 1.2).
Example 1.6. Let \( G \) be a 2-connected, \( k \)-regular graph with \( 2k + 1 \) vertices. It is easy to see that \( G \) does not satisfy the condition either of Theorem 1.1 or of Theorem 1.2. However, it is not difficult to check that \( G \) satisfies the condition of Theorem 1.5 and \( G \notin \mathcal{L}_{2k+1} \). Hence, \( G \) is Hamiltonian.

2. Proof of the main result

As a preparation, we firstly prove the following lemma.

**Lemma 2.1.** Let \( P = v_1v_2 \ldots v_s \) (\( s \geq 2 \)) and \( Q = w_1w_2 \ldots w_t \) (\( t \geq 1 \)) be two disjoint paths in a graph \( G \). If \( d_P(w_1) + d_P(w_t) \geq |V(P)| + 2 \), then \( Q \) can be inserted into \( P \) (i.e., \( v_1 \ldots v_kQv_{k+1} \ldots v_s \) is a path in \( G \) for some \( 1 \leq k < s \)).

**Proof.** From \( d_P(w_1) + d_P(w_t) \geq s + 2 \) and \( s \geq 2 \), we see that \( w_1 \) is adjacent with at least two vertices of \( P \). Let \( N_P(w_1) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_2} \} \) with \( i_1 < i_2 < \cdots < i_2 \). Suppose to the contrary that \( Q \) cannot be inserted into \( P \). Then \( w_t \) is not adjacent with \( v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_{s-1}} \). This implies that \( d_P(w_1) \leq s - (\gamma - 1) = s - \gamma + 1 \). Now, we see \( d_P(w_1) + d_P(w_t) \leq s + 1 \), a contradiction. \( \square \)

Note that for \( t = 1 \), Lemma 2.1 states the following: if \( d_P(w_1) \geq \lceil |V(P)|/2 \rceil + 1 \), then \( w_1 \) can be inserted into \( P \).

**Proof of Theorem 1.5.** Let \( G \) be a graph satisfying the condition of Theorem 1.5. It is obvious that if \( 3 \leq n \leq 4 \), then \( G \) is Hamiltonian. For \( n \geq 5 \), we prove that if \( G \) is non-Hamiltonian, then \( n \) is odd and \( G \in \mathcal{L}_n \).

It is easy to see that \( G \) satisfies the conditions of Theorem 1.3 for \( c = n - 1 \). Therefore, \( G \) contains a cycle of length \( n - 1 \), denoted by \( C = u_1u_2 \ldots u_{n-1}u_1 \). Let \( \{u\} = V - V(C) \). Since \( G \) is 2-connected, there is a \( C \)-bypass. Let \( P = v_1v_2v_3 \) be a \( C \)-bypass with minimum gap among all \( C \)-bypasses in \( G \), and assume without loss of generality that \( v_1 = u_1 \) and \( v_2 = u_\gamma \).

Suppose that \( G \) is non-Hamiltonian. Then, we have \( \gamma \geq 3 \). From the choice of \( P \), we see that \( v \notin N(u_i) \) for \( 2 \leq i \leq \gamma - 1 \). Let \( C' = u_2u_3 \ldots u_{\gamma-1} \) and \( C'' = u_\gamma u_{\gamma+1} \ldots u_{n-1}u_1 \). We consider the following two cases.

**Case 1:** \( \gamma \geq 4 \).
Because of \( d(u_2, v) = d(u_{\gamma-1}, v) = 2 \), we have
\[
\min\{d(u_2) + d(v), d(u_{\gamma-1}) + d(v)\} \geq n - 1.
\] (1)

Since \( P \) has the minimum gap among all \( C \)-bypasses, \( |V(C'')| \geq \gamma \geq 4 \) holds, and furthermore, we conclude that
\[
d(v) = d_{C''}(v) = d_C(v) \leq \frac{n - 1}{\gamma - 1}
\]
\[
= \frac{|V(C'')| + (\gamma - 2)}{\gamma - 1}
\]
\[
= \frac{|V(C'')|}{2} + \frac{2(\gamma - 3)(|V(C'')| - 2) - 2}{2(\gamma - 1)}
\]
\[
\leq \frac{|V(C'')|}{2}.
\] (2)

Moreover, the following holds:
\[
d(u_j) = d_{C'}(u_j) + d_{C''}(u_j) \leq (|V(C')| - 1) + d_{C''}(u_j) \quad \text{for } j = 2, \gamma - 1.
\] (3)

It follows from (1)–(3) that
\[
2(n - 1) \leq d(u_2) + d(v) + d(u_{\gamma-1}) + d(v)
\]
\[
\leq 2(|V(C')| - 1) + d_{C''}(u_2) + d_{C''}(u_{\gamma-1}) + 2 \left( \frac{|V(C'')|}{2} \right)
\]
\[
= 2n - 4 - |V(C'')| + d_{C''}(u_2) + d_{C''}(u_{\gamma-1}),
\]
and hence, we have \( d_{C''}(u_2) + d_{C''}(u_{\gamma-1}) \geq |V(C'')| + 2 \). By Lemma 2.1, the path \( C' = u_2u_3 \ldots u_{\gamma-1} \) can be inserted into \( C'' \). So, we obtain a Hamiltonian cycle of \( G \), a contradiction.
Case 2: $\gamma = 3$.

Since $u_2$ ($v$, respectively) cannot be inserted into the cycle $u_1v_3 \ldots u_{n-1}u_1$ ($C$, respectively), we have $\max\{d(u_2), d(v)\} \leqslant \frac{n-1}{2}$. By recalling $d(u_2)+d(v) \geqslant n-1$, we conclude that $d(u_2)=d(v)=\frac{n-1}{2}$. Clearly, $n$ is an odd integer. Now, it is easy to check that $NG(u_2)=NG(v)=\{u_1, u_3, \ldots, u_{n-2}\}$. Similarly, we can verify that $NG(u_2)k=NG(u_2)$ for $k = 2, \ldots, \frac{n-1}{2}$. Let $Z_{(n-1)/2} = G[\{u_1, u_3, \ldots, u_{n-2}\}]$. Then, we see that $G = Z_{(n-1)/2} \lor (K_{(n-1)/2}^{c} + \{v\}) \in \mathcal{L}_{n}$. The proof of the theorem is complete. □

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