Australian Journal of Basic and Applied Sciences, 4(4): 633-638, 2010
ISSN 1991-8178


# Solving a System of Linear Fredholm Fractional Integro-differential Equations Using Homotopy Perturbation Method

[1]Rostam K. Saeed and [2]Hassan M. Sdeq

[1,2]Salahaddin University/Erbil - College of Science - Department of Mathematics, Erbil, Iraq
[1]rostamkarim64@uni-sci.org [2]hassan@uni-sci.org

**Abstract:** Homotopy perturbation method has been employed to obtain a solution of a system of linear Fredholm fractional integro-differential equations:

$$[D^{q_{n_i}}+\sum_{s=0}^{n-1}p_{is}(x)D^{q_{s_i}}]u_i(x)=f_i(x)+\sum_{j=1}^{m}\int_a^b k_{ij}(x,t)u_j(t)dt,\ i=1,2,\ldots,m$$

where $D^{q_{n_i}}$ denotes Remann - Leiouville fractional derivatives.

**Keywords:** System of linear Fredholm Fractional integro-differential equations, Homotopy perturbation method, fractional derivatives

## INTRODUCTION

The concept of fractional order derivatives and integration can be traced back to the genesis of integer order calculus itself. The use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earth quake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties (Arikoglu and Ozkal (2007)).

There are only a few techniques for the solution of fractional integro-differential equations, since it is relatively a new subject in mathematics. These methods are: Adomian decomposition method (Momani and Noor (2006)), (Momani and Qaralleh, (2006)), the collocation method (Rawashdeh (2006)) and fractional differential trams form method (Arikoglu and Ozkol (2007)). In this study presented, fractional differentiations and integration are understood in Remann - Liouville sense.

In this paper, homotopy perturbation method is used for finding a solution of the system of linear Fredholm fractional integro-differential equations. Homotopy perturbation method was proposed by He (1999) and systematical description also by He (2000) which is, in fact, a coupling of the traditional perturbation method and homotopy in topology. This new method was further developed and improved by He and applied to nonlinear oscillators with discontinuities, (He (2004)), asymptotology (He (2004)), nonlinear wave equations (He (2005)) and boundary value problem (He (2006)).

This paper has been organized as follows: section 2 gives notations and basic definitions. Section 3 consists of main results of this paper, in which homotopy perturbation method has been applied on the system of linear Fredholm fractional integro-differential equations. Some illustrative examples are given in section 4 followed by the conclusions presented in section 5.

***Preliminaries:***

In this section, we give some basic definitions which are needed in the sequel.

**Definition 1** Let $f:[a,b]\rightarrow R,$ and $f\in L^1[a,b]$. The left sided Riemann-Liouville fractional integral

*I* of order $\mu\geq 0,$ (Daftardar-Gejji (2005)) of function $f\in C_\alpha,\ \alpha\geq -1,$ is defined as

**Corresponded Author:** Rostam K. Saeedand, Salahaddin University/Erbil - College of Science - Department ofMathematics, Erbil, Iraq
E-mail: rostamkarim64@uni-sci.org

$$I^{\mu}f(x)=\frac{1}{\Gamma(\mu)}\int_a^x\frac{f(t)}{(x-t)^{1-\mu}}dt,\mu\rangle 0,a\langle x\langle b, \text{ where } I^0f(x)=f(x). \quad (1)$$

**Definition 2** The left sided Riemann-Liouville fractional derivative of a function $f:[a,b]\to R,$ is defined $D^{\mu}f(x)=D^{m}I^{m-\mu}f(x),$ as where $m=[\mu]+1,$ $D^{m}=\frac{d^{m}}{dt^{m}},a\langle x\langle b.$

Hereunder, $D\mu$ denotes $D_0^{\mu}$ and $I\mu$ denotes $I_0^{\mu}$ If the fractional derivative $D_a^{\mu}f(t)$ is integrable, then (Podlubny (1999))

$$I_a^{\mu}(D_a^{\alpha}f(t))=I_a^{\mu-\alpha}f(t)-[I_a^{1-\alpha}f(t)]_{t=a}\frac{(t-a)^{\mu-1}}{\Gamma(\mu)},\ 0\langle\alpha\le\mu\langle 1. \quad (2)$$

If $f\in C[a,b],$ then, $[I_a^{1-\alpha}f(t)]_{t=a}=0$ and Eq.(2) takes the form

$$I_a^{\mu}(D_a^{\alpha}f(t))=I_a^{\mu-\alpha}f(t)\quad 0\langle\alpha\le\mu\langle 1. \quad (3)$$

***Homotopy perturbation method***

Consider the following system of linear Fredholm fractional integro-differential equations:

$$[D^{q_{n_i}}+\sum_{s=0}^{n-1}p_{is}(x)D^{q_{s_i}}]u_i(x)=f_i(x)+\sum_{j=1}^{m}\int_a^b k_{ij}(x,t)u_j(t)dt\, i=1,2,\ldots,m \quad (4)$$

For each $i$ where $i$=1,2...,$m$ applying $I^{q_{n_i}}$ to both sides of (4) where $q_{n_i}$ is maximum fractional derivatives in the equation number $i$, we get:

$$u_i(x)=I^{q_{n_i}}[f_i(x)]-I^{q_{n_i}}\left[\sum_{s=0}^{n-1}p_{is}(x)D^{q_{s_i}}]u_i(x)\right]+I^{q_{n_i}}\left[\int_a^b\sum_{j=1}^{m}k_{ij}(x,t)u_j(t))dt\right],\ \text{i=1,2,...,m.} \quad (5)$$

By the homotopy technique given in (He (1999), Momani and Noor (2006), Momani and Qaralleh (2006) and Rawashdeh (2006)), we construct a homotopy for Eq (5) which satisfies:

$$H_i(U,p)=(1-p)F_i(U)+pL_i(U)=0, \quad (6)$$

$$H_i(U,0)=F_i(U),\quad H_i(U,p)=L_i(U),$$

where

$$F_i(U)=u_i-I^{q_{n_i}}[f_i]$$

$$L_i(U)=u_i(x)-I^{q_{n_i}}[f_i(x)]+I^{q_{n_i}}\left[\sum_{s=0}^{n-1}p_{is}(x)D^{q_{s_i}}]u_i(x)\right]-I^{q_{n_i}}\left[\int_a^b\sum_{j=1}^{m}k_{ij}(x,t)u_j(t))dt\right],$$

$i$=1,2,...,$m$

and $p\in[0,1]$ is an embedding parameter. The embedding parameter $p$ monotonically increases from zero to unit as $F_i\,(U)$ is continuously deformed to the $L_i\,(U)$. According to the homotopy perturbation method (see He (1999)), we assume that the solution of Eq. (5) can be expressed in a series of $p$

$$u_i(x) = u_{i0} + pu_{i1} + p^2u_{i2} + p^3u_{i3} + \ldots \qquad (7)$$

Substituting Eq (7) in to Eq (6), we have

$$\begin{aligned} H_i(U,p) &= (1-p)(u_{i0} + pu_{i1} + p^2u_{i2} + p^3u_{i3} + \ldots - I^{q_{n_i}}[f_i]) + \\ & p(u_{i0} + pu_{i1} + p^2u_{i2} + p^3u_{i3} + \ldots - I^{q_{n_i}}[f_i] + \\ & I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}](u_{i0} + pu_{i1} + p^2u_{i2} + p^3u_{i3} + \ldots)\right] - \\ & I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t)(u_{i0} + pu_{i1} + p^2u_{i2} + p^3u_{i3} + \ldots)\,dt\right]. \end{aligned} \qquad (8)$$

Rewrite Eq (8) in the following form:

$$\begin{aligned} & p^0\left[u_{i0} - I^{q_{n_i}}[f_i]\right] + p^1\left[u_{i1} + I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}]u_{i0}(x)\right] - I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t) u_{i0}(t)dt\right] + \\ & p^2\left[u_{i2} + I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}]u_{i1}(x)\right] - I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t) u_{i1}(t)dt\right]\right] + \\ & p^3\left[u_{i3} + I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}]u_{i2}(x)\right] - I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t) u_{i2}(t)dt\right] + \ldots \end{aligned}$$

Equating coefficients of like powers of $p$ in the above equation, yields

$$u_{i0} = I^{q_{n_i}}[f_i],$$

$$u_{i1} = I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}]u_{i0}(x)\right] - I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t) u_{i0}(t)dt\right],$$

$$u_{i2} = I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}]u_{i1}(x)\right] - I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t) u_{i1}(t)dt\right],$$

$$u_{i3} = I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}]u_{i2}(x)\right] - I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t) u_{i2}(t)dt\right]$$

$\vdots$

$$u_{im} = I^{q_{n_i}}\left[\sum_{s=0}^{n-1} p_{is}(x) D^{q_{s_i}}]u_{i,m-1}(x)\right] - I^{q_{n_i}}\left[\int_a^b \sum_{j=1}^{m} k_{ij}(x,t) u_{i,m-1}(t)dt\right],$$

$m$=0,1,...

***Numerical Examples:***

In this section, to demonstrate the effectiveness of the proposed method, two systems of linear Fredholm fractional integro-differential equations of the second kind was considered. To show the efficiency of the present method for our problem in comparison with the exact solution we report absolute error which defined by

$$\left\| u_i^N(x) \right\| = \left| u_i(x) - u^N{}_i(x) \right|, i = 1, 2, \ldots, n,$$

where $u_i^N(x) = \sum_{m=0}^{N} u_{im}$ and $u_i(x), i = 1, 2, \ldots, n$ are the exact solutions.

***Example1:***

Consider the following system of linear Fredholm fractional integro-differential equations:

$$D^{0.5}u_1(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{x}{2} + \int_0^1 x u_2(t)dt$$

$$D^{0.5}u_2(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{3} + \int_0^1 t u_1(t)dt$$

The exact solution of this system is: $u_1(x) = x$ and $u_2(x) = x$ .

***Solution***

By using homotopy perturbation method we have

$$u_{10}(x) = x - \frac{2}{3\sqrt{\pi}}x^{3/2} \qquad u_{20}(x) = x - \frac{2}{3\sqrt{\pi}}x^{1/2}$$

$$\begin{aligned}
&u_{11}(x) = .187498308x^{3/2} && u_{21}(x) = .254865480x^{1/2}\\
&u_{12}(x) = .127815510x^{3/2} && u_{22}(x) = .0604483386x^{1/2}\\
&u_{13}(x) = .0303149537x^{3/2} && u_{23}(x) = .0412069597x^{1/2}\\
&u_{14}(x) = .0206653666x^{3/2} && u_{24}(x) = .00977336064x^{1/2}\\
&u_{15}(x) = .00490135846x^{3/2} && u_{25}(x) = .00666239119x^{1/2}\\
&u_{16}(x) = .00334120152x^{3/2} && u_{26}(x) = .00158016879x^{1/2}
\end{aligned} \qquad (9)$$

Substituting (9) into $u^2{}_i(x) = \sum_{m=0}^{2} u_{im}(x)$ and $u^6{}_i(x) = \sum_{m=0}^{6} u_{im}(x)$ for $i$=1, 2 we get

$u^2{}_1(x) = x - .06081257046x^{3/2}$ $\qquad$ $u^6{}_1(x) = x - .00158969011x^{3/2}$

$u^2{}_1(x) = x - .06081257046^{1/3}$ $\qquad$ $u^6{}_1(x) = x - .00158969011x^{1/2}$

The numerical results obtained for Example 1 are shown in Table 1.

***Example2***.consider the following system of linear Fredholm fractional integro-differential equations:

$$D^{0.5}u_1(x) = f_1(x) + \int_0^1 (u_1(t) + u_2(t))dt$$

$$D^{15}u_2(x) = f_2(x) + \int_0^1 x(u_1(t) - u_2(t))dt$$

with $f_1(x) = \frac{2\sqrt{x}}{\sqrt{\pi}} - \frac{5}{6}$ and $f_2(x) = \frac{4\sqrt{x}}{\sqrt{\pi}} - \frac{x}{6}$ and with exact solution $u_1(x) = x$ and $u_2(x) = x^2$. By using homotopy perturbation method we have

$$u_{10}(x) = x - \frac{5}{3\sqrt{\pi}}x^{1/2} \qquad u_{20}(x) = x^2 - \frac{1}{6\Gamma(7/2)}x^{5/2}$$

$u_{11}(x) = .216792549x^{1/2}$ $\qquad u_{21}(x) = -.134166397x^{5/2}$
$u_{12}(x) = .119828349x^{1/2}$ $\qquad u_{21}(x) = .0550232653x^{5/2}$
$u_{13}(x) = .107880382x^{1/2}$ $\qquad u_{21}(x) = .0193072095x^{5/2}$ (10)
$u_{14}(x) = .0873778464x^{1/2}$ $\qquad u_{21}(x) = .0199810100x^{5/2}$
$u_{15}(x) = .0721719864x^{1/2}$ $\qquad u_{21}(x) = .0158102584x^{5/2}$
$u_{16}(x) = .0593887105x^{1/2}$ $\qquad u_{21}(x) = .0131185184x^{5/2}$

Substituting (10) into $u^2_i(x) = \sum_{m=0}^{2} u_{im}(x)$, and $u^6_i(x) = \sum_{m=0}^{6} u_{im}(x)$, for $i$=1, 2, we have

$u^2_1(x) = x-.6036950744x^{1/2}$ $\qquad u^6_1(x) =-.2768761491x^{1/2}$
$u^2_2(x) = x^2-.1292933165x^{5/2}$ $\qquad u^6_1(x) =-.061076320\ 13x^{5/2}$

The numerical results obtained for Example 2 are shown in Table 2.

**Table 1:** Numerical results for Example 1

| x | $\|u_i^n(x)\|$ | | | |
|---|---|---|---|---|
| | $\|u_1^2(x)\|$ | $\|u_2^2(x)\|$ | $\|u_1^6(x)\|$ | $\|u_2^6(x)\|$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | .54392e-5 | .2700e-2 | .14219e-6 | .71093e-6 |
| 0.4 | .1500e-2 | .3870e-1 | 40217e-6 | .10054e-5 |
| 0.6 | .2800e-2 | .4690e-1 | .74485e-6 | .12314e-5 |
| 0.8 | .4360e-1 | .5480e-1 | .11375e-5 | .14219e-5 |
| 1 | .6080e-1 | .6080e-1 | .15897e-5 | .15897e-5 |

**Table 2:** Numerical results for Example 2

| x | $\|u_i^n(x)\|$ | | | |
|---|---|---|---|---|
| | $\|u_1^2(x)\|$ | $\|u_2^2(x)\|$ | $\|u_1^6(x)\|$ | $\|u_2^6(x)\|$ |
| 0 | 0 | 0 | 0 | 0 |
| 0.2 | .1703 | .23129e-5 | .14219e-6 | .10926e-5 |
| 0.4 | .3818 | .1300e-2 | .40217e-6 | .61805e-5 |
| 0.6 | .4677 | .3610e-1 | .74485e-6 | .1700e-2 |
| 0.8 | .5400 | .7420e-1 | .11375e-5 | .3460e-1 |
| 1 | .6037 | .12920 | .15897e-5 | .6080e-1 |

***Conclusion***

In this article, homotopy perturbation method has been successfully applied to find the solution of a system of linear Fredholm fractional integro-differential equations are presented in Table 1 and 2, for differential result of $x$ to show the stability of the method. The approximate solution obtained by homotopy perturbation method are compared with exact solution.

## REFERENCES

Arikoglu, A. and I. Ozkol, 2007. Solution of fractional integro-differential equations by using fractional differential transform method, Choas, Solitons and Fractals, Article in press, doi:10.101/j.

Dftardar-Gejji, V., 2005. positive solutions of a non-autonomous fractional differential equations J. Math. Anal., 302: 56-64.

He, J.H., 1999. Homotopy perturbation technique, Comu. Methods Appl. Mech. Engrg., 178: 257-262.

He, J.H., 2000. A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Non-Linear Mech., 35(1): 37–43.

He, J.H., 2004. The homotopy perturbation method for non-linear oscillators with discontinuities, Applied Mathematics and Computation, 151(1): 287–292.

He, J.H., 2004. Asymptotology by homotopy perturbation method, Applied Mathematics and Computation, 156(3): 591–596.

He, J.H., 2005. Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons and Fractals, 26(3): 695–700.

He, J.H., 2006. Homotopy perturbation method for solving boundary problems, Phys. Lett., 350(1–2): 87–88.

Momani, S. and M.A. Noor, 2006. Numerical methods for fourth order fractional integro-differential equations, Applied Mathematics and Computation, 182: 754-760.

Momani, S. and A. Qaralleh, 2006. An efficient method for solving systems of fractional integro-differential equations, Applied Mathematics and Computation, 52: 459-470.

Podlubny, I., 1999. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. New York: Academic press.

Rawashdeh, E.A., 2006. Numerical solution of fractional integro-differential equations by collocation method, Applied Mathematics and Computation, 176: 1-6.