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Rough Pythagorean fuzzy ideals in ternary semigroups

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Abstract

A ternary semigroup is a nonempty set equipped with an associative ternary operation. A Pythagorean fuzzy set is one of the generalizations of the fuzzy set. The aim of this paper is to study rough Pythagorean fuzzy ideals in ternary semigroups. This idea is extended to the lower and upper approximations of Pythagorean fuzzy ideals.

Keywords: Fuzzy sets, Pythagorean fuzzy sets, rough sets, ternary semigroups.

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1. Introduction

The notion of fuzzy sets was introduced by Zadeh [23] in 1965. Several research was conducted on the generalizations of the notion of fuzzy sets. The study of fuzzy algebraic structures started by Rosenfeld [19] in 1971. Rosenfeld introduced the notion of fuzzy groups and showed that many results in groups can be extended in an elementary manner to develop the theory of fuzzy groups. The concept of fuzzy ideals in semigroups was first developed by Kuroki (see [9, 10]). Pawlak [15] introduced the fundamental rough set concept in 1982. This concept has been developed and applied to computer science, particularly information systems. Kuroki [11] defined rough ideals in semigroups. He introduced the notion of a rough left (right, two-sided, bi-) ideal in semigroups and gave some properties of such ideals. Furthermore, the theory of rough ideals in other structures has also been studied by many authors, for example, rough ideals of Γ-semigroup studied in [2, 7, 8], Prasertpong and Siripitukdet [17] introduced a rough set in a universal set based on cores of successor classes with respect to level in a closed unit interval under a fuzzy relation, and some interesting properties were investigated, etc.. A Pythagorean fuzzy set [21, 22] is one of generalizations of the fuzzy set. After its existence, several researchers have studied. Peng and Yang presented some results on Pythagorean fuzzy sets in [16]. Garg [3, 4] proposed the weighted averaging and geometric aggregation operators using Einstein t-norm operators for solving

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the decision-making problems under Pythagorean fuzzy sets environment. Naz et al. proposed a new graph by using the concept of Pythagorean fuzzy sets, called Pythagorean fuzzy graph in [14].

A ternary semigroup is a nonempty set equipped with an associative ternary operation. Every semigroup can be considered to be ternary semigroup. The existence of ternary operations originated from the study of a ternary analogue of Abelian group in 1932 by Lehmer [12]. The notion of ternary semigroups came from the problem of Banach who created an example of ternary semigroups which is not reducible to a semigroup. In addition, he conjectured that every ternary semigroup may be extended to reducible to a semigroup (cf. [13]). Los [13] exposed Banach's problem and showed that the operation in the ternary semigroup is an extension of the binary operation satisfying associative law on some nonempty set. Then many properties of ternary semigroups have been extensively studied by many authors. For example, Iampan conceptualized ideal extensions in ternary semigroups in [6], Thongkam and Changphas introduced the concept of two-sided bases of a ternary semigroup in [20], Ansari and Yaqoob [1] discussed the concept of T-rough ternary subsemigroups, T-rough ideals, T-rough bi-ideals, and T-rough interior ideals in ternary semigroups, straddles in ternary semigroups were studied in [18], etc.. The research on ternary semigroups in many aspects has been creative and interesting. It is significant to note that analogues of the results of ternary semigroups can be obtained from semigroup theory. Recently, Hussain et al. [5] presented the idea of rough Pythagorean fuzzy ideals in semigroups and discussed lower and upper approximations of Pythagorean fuzzy ideals, bi-ideals, and interior ideals in semigroups. In this paper, we introduce rough Pythagorean fuzzy ideals in ternary semigroups and give some remarkable properties.

2. Preliminaries

2.1. Ternary semigroups

Definition 2.1. A *ternary semigroup* is a nonempty set T together with a ternary operation $(a, b, c) \mapsto [abc]$ satisfying the associative law

$$[[abc]uv] = [a[bcu]v] = [ab[cuv]] \text{ for all } a, b, c, u, v \in T.$$

The following examples show that a ternary semigroup does not necessarily reduce an ordinary semigroup.

Example 2.2 (Banach's example)**.** Let $T = \{-i, 0, i\}$ be a ternary semigroup under multiplication over complex numbers. We have that T is not a binary semigroup under multiplication over complex numbers.

Example 2.3. Let $\mathbb{Z}^-$ be a ternary semigroup under multiplication over integer numbers. We have that $\mathbb{Z}^-$ is not a binary semigroup under multiplication over integer numbers.

Let T be a ternary semigroup. A function $f : T \to [0, 1]$ is called a *fuzzy subset* of T.

Definition 2.4. Let f and g be fuzzy subsets of a ternary semigroup T.

(1) $f \subseteq g$ if $f(y) \leqslant g(y)$ for all $y \in T$.
(2) $(f \cap g)(y) = \min\{f(y), g(y)\}$ for all $y \in T$.
(3) $(f \cup g)(y) = \max\{f(y), g(y)\}$ for all $y \in T$.

Definition 2.5. A fuzzy subset f of a ternary semigroup T is called

(1) a *fuzzy ternary subsemigroup* of T if $f([xyz]) \geqslant \min\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$;
(2) a *fuzzy left ideal* of T if $f([xyz]) \geqslant f(z)$ for all $x, y, z \in T$;
(3) a *fuzzy right ideal* of T if $f([xyz]) \geqslant f(x)$ for all $x, y, z \in T$;
(4) a *fuzzy lateral ideal* of T if $f([xyz]) \geqslant f(y)$ for all $x, y, z \in T$;
(5) a *fuzzy ideal* of T if $f([xyz]) \geqslant \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Note that a fuzzy subset f is a fuzzy ideal of a ternary semigroup T if and only if f is a fuzzy left ideal, fuzzy right ideal and fuzzy lateral ideal of T.

Definition 2.6. For any three fuzzy sets f_1, f_2, and f_3 of a ternary semigroup T, the *product* $[f_1 \circ f_2 \circ f_3]$ of f_1, f_2 and f_3 is defined by

$$[f_1 \circ f_2 \circ f_3](y) = \sup_{y=[y_1y_2y_3]} \min\{f_1(y_1), f_2(y_2), f_3(y_3)\}.$$

Let $\mathcal{F}(T)$ be the set of all fuzzy subset of a ternary semigroup T. Note that $\mathcal{F}(T)$ is a ternary semigroup under the product defined in Definition 2.6.

2.2. *Pythagorean fuzzy sets*

Yager [21] and Yager and Abbasov [22] initiated the notion of Pythagorean fuzzy sets as follows.

Definition 2.7 ([21, 22])**.** Let U be a universal set. A *Pythagorean fuzzy set* $\mathcal{P} := \{< y, \mu_{\mathcal{P}}(y), \nu_{\mathcal{P}}(y) >| \, y \in U\}$ where $\mu_{\mathcal{P}} : U \to [0,1]$ and $\nu_{\mathcal{P}} : U \to [0,1]$ represents the degree of membership and the degree of non-membership of $y \in U$ to a set $\mathcal{P}$ with condition that $0 \leqslant (\mu_{\mathcal{P}}(y))^2 + (\nu_{\mathcal{P}}(y))^2 \leqslant 1$. For the sake of simplicity a Pythagorean fuzzy set $\mathcal{P}$ is denoted by $\mathcal{P} = (\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$.

Definition 2.8 ([21, 22])**.** Let $\mathcal{P}_1 = (\mu_{\mathcal{P}_1}, \nu_{\mathcal{P}_1})$ and $\mathcal{P}_2 = (\mu_{\mathcal{P}_2}, \nu_{\mathcal{P}_2})$ be Pythagorean fuzzy sets on a set U. Then for all $y \in U$,

(1) $\mathcal{P}_1 \subseteq \mathcal{P}_2 \Leftrightarrow \mu_{\mathcal{P}_1} \leqslant \mu_{\mathcal{P}_2}$ and $\nu_{\mathcal{P}_2} \leqslant \nu_{\mathcal{P}_1}$;
(2) $\mathcal{P}_1 = \mathcal{P}_2 \Leftrightarrow \mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{P}_2 \subseteq \mathcal{P}_1$;
(3) $\mathcal{P}_1 \cup \mathcal{P}_2 = (\mu_{\mathcal{P}_1} \cup \mu_{\mathcal{P}_2}, \nu_{\mathcal{P}_1} \cap \nu_{\mathcal{P}_2})$;
(4) $\mathcal{P}_1 \cap \mathcal{P}_2 = (\mu_{\mathcal{P}_1} \cap \mu_{\mathcal{P}_2}, \nu_{\mathcal{P}_1} \cup \nu_{\mathcal{P}_2})$.

Note that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are Pythagorean fuzzy sets on a set U, then $\mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2$ are also Pythagorean fuzzy sets on U.

3. Main results

3.1. *Pythagorean fuzzy sets in ternary semigroups*

Definition 3.1. A Pythagorean fuzzy set $\mathcal{P} = (\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ is called

(1) a *Pythagorean fuzzy ternary subsemigroup* of T if for all $y_1, y_2, y_3 \in T$,

$$\mu_{\mathcal{P}}([y_1y_2y_3]) \geqslant \min\{\mu_{\mathcal{P}}(y_1), \mu_{\mathcal{P}}(y_2), \mu_{\mathcal{P}}(y_3)\} \quad \text{and} \quad \nu_{\mathcal{P}}([y_1y_2y_3]) \leqslant \max\{\nu_{\mathcal{P}}(y_1), \nu_{\mathcal{P}}(y_2), \nu_{\mathcal{P}}(y_3)\},$$

(2) a *Pythagorean fuzzy left ideal* of T if for all $y_1, y_2, y_3 \in T$,

$$\mu_{\mathcal{P}}([y_1y_2y_3]) \geqslant \mu_{\mathcal{P}}(y_3) \text{ and } \nu_{\mathcal{P}}([y_1y_2y_3]) \leqslant \nu_{\mathcal{P}}(y_3),$$

(3) a *Pythagorean fuzzy right ideal* of T if for all $y_1, y_2, y_3 \in T$,

$$\mu_{\mathcal{P}}([y_1y_2y_3]) \geqslant \mu_{\mathcal{P}}(y_1) \text{ and } \nu_{\mathcal{P}}([y_1y_2y_3]) \leqslant \nu_{\mathcal{P}}(y_1),$$

(4) a *Pythagorean fuzzy lateral ideal* of T if for all $y_1, y_2, y_3 \in T$,

$$\mu_{\mathcal{P}}([y_1y_2y_3]) \geqslant \mu_{\mathcal{P}}(y_2) \text{ and } \nu_{\mathcal{P}}([y_1y_2y_3]) \leqslant \nu_{\mathcal{P}}(y_2),$$

(5) a *Pythagorean fuzzy ideal* of T if for all $y_1, y_2, y_3 \in T$,

$$\mu_{\mathcal{P}}([y_1y_2y_3]) \geqslant \max\{\mu_{\mathcal{P}}(y_1), \mu_{\mathcal{P}}(y_2), \mu_{\mathcal{P}}(y_3)\} \quad \text{and} \quad \nu_{\mathcal{P}}([y_1y_2y_3]) \leqslant \min\{\nu_{\mathcal{P}}(y_1), \nu_{\mathcal{P}}(y_2), \nu_{\mathcal{P}}(y_3)\}.$$

Theorem 3.2. *Let* f *be a fuzzy subset of a ternary semigroup* T. *Let* $\mu_{\mathcal{P}} = f$ *and* $\nu_{\mathcal{P}} = 1 - f$. *Then* $\mathcal{P} = (\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ *is a Pythagorean fuzzy set on* T. *Moreover, we have*

(1) f *is a fuzzy ternary subsemigroup of* T *if and only if* $\mathcal{P}$ *is a Pythagorean fuzzy ternary subsemigroup of* T*; and*
(2) f *is a fuzzy left ideal (respectively right ideal, lateral ideal, and ideal) of* T *if and only if* $\mathcal{P}$ *is a Pythagorean fuzzy left (respectively right ideal, lateral ideal, ideal) ideal of* T.

Proof. Let f be a fuzzy subset of T and $y \in T$. Clearly,

$$0 \leqslant (f(y))^2 + ((1-f)(y))^2 \leqslant f(y) + (1-f)(y) = 1.$$

Then $0 \leqslant (\mu_{\mathcal{P}}(y))^2 + (\nu_{\mathcal{P}}(y))^2 \leqslant 1$. Therefore $\mathcal{P} = (\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ is a Pythagorean fuzzy set on T.

(1) Let f be a fuzzy ternary subsemigroup of T and $y_1, y_2, y_3 \in T$. Then

$$\mu_{\mathcal{P}}([y_1y_2y_3]) = f([y_1y_2y_3]) \geqslant \min\{f(y_1), f(y_2), f(y_3)\} = \min\{\mu_{\mathcal{P}}(y_1), \mu_{\mathcal{P}}(y_2), \mu_{\mathcal{P}}(y_3)\}$$

and

$$\begin{aligned}\nu_{\mathcal{P}}([y_1y_2y_3]) = (1-f)([y_1y_2y_3]) &= 1 - f([y_1y_2y_3]) \\ &\leqslant 1 - \min\{f(y_1), f(y_2), f(y_3)\} \\ &= \max\{(1-f)(y_1), (1-f)(y_2), (1-f)(y_3)\} = \max\{\nu_{\mathcal{P}}(y_1), \nu_{\mathcal{P}}(y_2), \nu_{\mathcal{P}}(y_3)\}.\end{aligned}$$

This implies that $\mathcal{P}$ is a Pythagorean fuzzy ternary subsemigroup of T. Conversely, assume that $\mathcal{P}$ is a Pythagorean fuzzy ternary subsemigroup of T. Since $f = \mu_{\mathcal{P}}$ and property of $\mu_{\mathcal{P}}$, this implies that f is a fuzzy ternary subsemigroup of T.

The proof of (2) is similar to the proof of (1). □

Definition 3.3. Let $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ be any three Pythagorean fuzzy sets on a ternary semigroup T. The *product* $[\mathcal{P}_1 \circ \mathcal{P}_2 \circ \mathcal{P}_3]$ of $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$ is defined by

$$[\mathcal{P}_1 \circ \mathcal{P}_2 \circ \mathcal{P}_3] = ([\mu_{\mathcal{P}_1} \circ \mu_{\mathcal{P}_2} \circ \mu_{\mathcal{P}_3}], [\nu_{\mathcal{P}_1} \circ \nu_{\mathcal{P}_2} \circ \nu_{\mathcal{P}_3}]),$$

where

$$[\mu_{\mathcal{P}_1} \circ \mu_{\mathcal{P}_2} \circ \mu_{\mathcal{P}_3}](y) = \sup_{y=[y_1y_2y_3]} \min\{\mu_{\mathcal{P}_1}(y_1), \mu_{\mathcal{P}_2}(y_2), \mu_{\mathcal{P}_3}(y_3)\}$$

and

$$[\nu_{\mathcal{P}_1} \circ \nu_{\mathcal{P}_2} \circ \nu_{\mathcal{P}_3}](y) = \inf_{y=[y_1y_2y_3]} \max\{\nu_{\mathcal{P}_1}(y_1), \nu_{\mathcal{P}_2}(y_2), \nu_{\mathcal{P}_3}(y_3)\}.$$

Let $\mathcal{PFS}(T)$ be the set of all Pythagorean fuzzy sets on a ternary semigroup T. Note that $\mathcal{PFS}(T)$ is a ternary semigroup under the product defined in Definition 3.3. Let $\mathcal{T} := (\mu_{\mathcal{T}}, \nu_{\mathcal{T}})$ be a Pythagorean fuzzy set on T defined by $\mu_{\mathcal{T}}(y) = 1$ and $\nu_{\mathcal{T}}(y) = 0$ for all $y \in T$. The following theorem holds.

Theorem 3.4. *Let* $\mathcal{P} = (\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ *be a Pythagorean fuzzy set on a ternary semigroup* T.

(1) $\mathcal{P}$ *is a Pythagorean fuzzy ternary subsemigroup of* T *if and only if* $[\mathcal{P} \circ \mathcal{P} \circ \mathcal{P}] \subseteq \mathcal{P}$.
(2) $\mathcal{P}$ *is a Pythagorean fuzzy left ideal of* T *if and only if* $[\mathcal{T} \circ \mathcal{T} \circ \mathcal{P}] \subseteq \mathcal{P}$.
(3) $\mathcal{P}$ *is a Pythagorean fuzzy right ideal of* T *if and only if* $[\mathcal{P} \circ \mathcal{T} \circ \mathcal{T}] \subseteq \mathcal{P}$.
(4) $\mathcal{P}$ *is a Pythagorean fuzzy lateral ideal of* T *if and only if* $[\mathcal{T} \circ \mathcal{P} \circ \mathcal{T}] \subseteq \mathcal{P}$.

Proof.

(1) Assume that $\mathcal{P}$ is a Pythagorean fuzzy ternary subsemigroup of T. We have

$$[\mu_{\mathcal{P}} \circ \mu_{\mathcal{P}} \circ \mu_{\mathcal{P}}](y) = \sup_{y=[y_1y_2y_3]} \min\{\mu_{\mathcal{P}}(y_1), \mu_{\mathcal{P}}(y_2), \mu_{\mathcal{P}}(y_3)\} \leqslant \mu_{\mathcal{P}}(y)$$

and
$$[\nu_{\mathcal{P}} \circ \nu_{\mathcal{P}} \circ \nu_{\mathcal{P}}](y) = \inf_{y=[y_1y_2y_3]} \max\{\mu_{\mathcal{P}}(y_1), \mu_{\mathcal{P}}(y_2), \mu_{\mathcal{P}}(y_3)\} \geqslant \nu_{\mathcal{P}}(y).$$
Hence $[\mathcal{P} \circ \mathcal{P} \circ \mathcal{P}] \subseteq \mathcal{P}$. Conversely, let $y_1, y_2, y_3 \in T$.
$$\begin{aligned}\mu_{\mathcal{P}}([y_1y_2y_3]) &\geqslant [\mu_{\mathcal{P}} \circ \mu_{\mathcal{P}} \circ \mu_{\mathcal{P}}]([y_1y_2y_3]) \\ &= \sup_{[abc]=[y_1y_2y_3]} \min\{\mu_{\mathcal{P}}(a), \mu_{\mathcal{P}}(b), \mu_{\mathcal{P}}(c)\} \geqslant \min\{\mu_{\mathcal{P}}(y_1), \mu_{\mathcal{P}}(y_2), \mu_{\mathcal{P}}(y_3)\}\end{aligned}$$
and
$$\begin{aligned}\nu_{\mathcal{P}}([y_1y_2y_3]) &\leqslant [\nu_{\mathcal{P}} \circ \nu_{\mathcal{P}} \circ \nu_{\mathcal{P}}]([y_1y_2y_3]) \\ &= \inf_{[abc]=[y_1y_2y_3]} \max\{\mu_{\mathcal{P}}(a), \mu_{\mathcal{P}}(b), \mu_{\mathcal{P}}(c)\} \leqslant \max\{\mu_{\mathcal{P}}(y_1), \mu_{\mathcal{P}}(y_2), \mu_{\mathcal{P}}(y_3)\}.\end{aligned}$$
This implies that $\mathcal{P}$ is a Pythagorean fuzzy ternary subsemigroup of T.

(2) Assume that $\mathcal{P}$ is a Pythagorean fuzzy left ideal of T. We have
$$[\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{P}}](y) = \sup_{y=[y_1y_2y_3]} \min\{\mu_{\mathcal{T}}(y_1), \mu_{\mathcal{T}}(y_2), \mu_{\mathcal{P}}(y_3)\} = \sup_{y=[y_1y_2y_3]} \mu_{\mathcal{P}}(y_3) \leqslant \mu_{\mathcal{P}}(y)$$
and
$$[\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{P}}](y) = \inf_{y=[y_1y_2y_3]} \max\{\nu_{\mathcal{T}}(y_1), \nu_{\mathcal{T}}(y_2), \nu_{\mathcal{P}}(y_3)\} = \inf_{y=[y_1y_2y_3]} \nu_{\mathcal{P}}(y_3) \geqslant \nu_{\mathcal{P}}(y).$$
Hence $[\mathcal{T} \circ \mathcal{T} \circ \mathcal{P}] \subseteq \mathcal{P}$. Conversely, let $y_1, y_2, y_3 \in T$.
$$\begin{aligned}\mu_{\mathcal{P}}([y_1y_2y_3]) &\geqslant [\mu_{\mathcal{T}} \circ \mu_{\mathcal{T}} \circ \mu_{\mathcal{P}}]([y_1y_2y_3]) \\ &= \sup_{[abc]=[y_1y_2y_3]} \min\{\mu_{\mathcal{T}}(a), \mu_{\mathcal{T}}(b), \mu_{\mathcal{P}}(c)\} = \sup_{[abc]=[y_1y_2y_3]} \mu_{\mathcal{P}}(c) \geqslant \mu_{\mathcal{P}}(y_3)\end{aligned}$$
and
$$\begin{aligned}\nu_{\mathcal{P}}([y_1y_2y_3]) &\leqslant [\nu_{\mathcal{T}} \circ \nu_{\mathcal{T}} \circ \nu_{\mathcal{P}}]([y_1y_2y_3]) \\ &= \inf_{[abc]=[y_1y_2y_3]} \max\{\nu_{\mathcal{T}}(a), \nu_{\mathcal{T}}(b), \nu_{\mathcal{P}}(c)\} = \inf_{[abc]=[y_1y_2y_3]} \nu_{\mathcal{P}}(c) \leqslant \nu_{\mathcal{P}}(y_3).\end{aligned}$$
Then $\mu_{\mathcal{P}}([y_1y_2y_3]) \geqslant \mu_{\mathcal{P}}(y_3)$ and $\nu_{\mathcal{P}}([y_1y_2y_3]) \leqslant \nu_{\mathcal{P}}(y_3)$. This implies that $\mathcal{P}$ is a Pythagorean fuzzy left ideal of T.

The proofs of (3) and (4) are similar to the proof of (2). □

3.2. Rough Pythagorean fuzzy sets in ternary semigroups

Definition 3.5. An equivalence relation ρ on a ternary semigroup T is called a *congruence* if for all $x_1, x_2, x_3, y_1, y_2, y_3 \in S$
$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \rho \Rightarrow ([x_1x_2x_3], [y_1, y_2, y_3]).$$
The congruence class of $x \in S$ is denoted by $[x]_\rho$. A congruence ρ of T is called *complete* if $[y_1]_\rho[y_2]_\rho[y_3]_\rho = [y_1y_2y_3]_\rho$ for all $y_1, y_2, y_3 \in T$.

Definition 3.6. Let ρ be a congruence and $\mathcal{P} = \{< y, \mu_{\mathcal{P}}(y), \nu_{\mathcal{P}}(y) >| \ y \in T\}$ be the Pythagorean fuzzy set on a ternary semigroup T.

(1) The *lower approximation* is defined as
$$\underline{\mathrm{App}}(\mathcal{P}) = \{< y, \underline{\mu_{\mathcal{P}}}(y), \underline{\nu_{\mathcal{P}}}(y) >| \ y \in T\},$$
where $\underline{\mu_{\mathcal{P}}}(y) = \inf_{y' \in [y]_\rho} \mu_{\mathcal{P}}(y')$ and $\underline{\nu_{\mathcal{P}}}(y) = \sup_{y' \in [y]_\rho} \nu_{\mathcal{P}}(y')$ with the condition that
$$0 \leqslant (\underline{\mu_{\mathcal{P}}}(y))^2 + (\underline{\nu_{\mathcal{P}}}(y))^2 \leqslant 1.$$

(2) The *upper approximation* is defined as

$$\overline{App}(\mathcal{P}) = \{< y, \overline{\mu_{\mathcal{P}}}(y), \overline{\nu_{\mathcal{P}}}(y) > \mid y \in T\},$$

where $\overline{\mu_{\mathcal{P}}}(y) = \sup_{y' \in [y]_\rho} \mu_{\mathcal{P}}(y')$ and $\overline{\nu_{\mathcal{P}}}(y) = \inf_{y' \in [y]_\rho} \nu_{\mathcal{P}}(y')$ with the condition that

$$0 \leqslant (\overline{\mu_{\mathcal{P}}}(y))^2 + (\overline{\nu_{\mathcal{P}}}(y))^2 \leqslant 1.$$

(3) *Rough Pythagorean fuzzy set* of T is defined by

$$App(\mathcal{P}) = (\underline{App}(\mathcal{P}), \overline{App}(\mathcal{P})).$$

Example 3.7. Define a relation ρ on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication by

$$x\rho y \Leftrightarrow 2 \mid x - y \text{ for all } x, y \in \mathbb{Z}^-.$$

It is easy to show that ρ is a congruence on $\mathbb{Z}^-$. Let $\mu_{\mathcal{P}}(y) = -\frac{1}{y}$ and $\nu_{\mathcal{P}}(y) = 1 + \frac{1}{y}$ for all $y \in \mathbb{Z}^-$. Then

$$0 \leqslant (\mu_{\mathcal{P}}(y))^2 + (\nu_{\mathcal{P}}(y))^2 = (-\frac{1}{y})^2 + (1 + \frac{1}{y})^2 = 1 + 2(\frac{1}{y} + \frac{1}{y^2}) \leqslant 1$$

for all $y \in \mathbb{Z}^-$, this implies that $\mathcal{P}$ is a Pythagorean fuzzy set of T. We have

$$\underline{App}(\mathcal{P}) = \{< y, \underline{\mu_{\mathcal{P}}}(y), \underline{\nu_{\mathcal{P}}}(y) > \mid y \in T\} = \{< y, 0, 1 > \mid y \in T\}$$

and

$$\overline{App}(\mathcal{P}) = \{< y, \overline{\mu_{\mathcal{P}}}(y), \overline{\nu_{\mathcal{P}}}(y) > \mid y \in T\} = \{< y, 1, 0 > \mid y \text{ is odd}\} \cup \{< y, \frac{1}{2}, \frac{1}{2} > \mid y \text{ is even}\}.$$

Theorem 3.8. *Let ρ be a congruence relation on a ternary semigroup T and $\mathcal{P}$ be a Pythagorean fuzzy set of T.*

(1) *$\overline{App}(\mathcal{P})$ is a Pythagorean fuzzy set of T.*
(2) *$\underline{App}(\mathcal{P})$ is a Pythagorean fuzzy set of T.*

Proof.

(1) Let $y \in T$. Then

$$\begin{aligned}(\overline{\mu_{\mathcal{P}}}(y))^2 + (\overline{\nu_{\mathcal{P}}}(y))^2 &= (\sup_{y' \in [y]_\rho} \mu_{\mathcal{P}}(y'))^2 + (\inf_{y' \in [y]_\rho} \nu_{\mathcal{P}}(y'))^2 \\ &= \sup_{y' \in [y]_\rho} (\mu_{\mathcal{P}}(y'))^2 + \inf_{y' \in [y]_\rho} (\nu_{\mathcal{P}}(y'))^2 \\ &\leqslant \sup_{y' \in [y]_\rho} (\mu_{\mathcal{P}}(y'))^2 + \inf_{y' \in [y]_\rho} (1 - (\mu_{\mathcal{P}}(y'))^2) \\ &\leqslant \sup_{y' \in [y]_\rho} (\mu_{\mathcal{P}}(y'))^2 + 1 - \sup_{y' \in [y]_\rho} (\mu_{\mathcal{P}}(y'))^2 = 1.\end{aligned}$$

This implies that $0 \leqslant (\overline{\mu_{\mathcal{P}}}(y))^2 + (\overline{\nu_{\mathcal{P}}}(y))^2 \leqslant 1$. Therefore, $\overline{App}(\mathcal{P})$ is a Pythagorean fuzzy set of T.

The proof of (2) is similar to the proof of (1). □

Theorem 3.9. *Let ρ be a congruence on a ternary semigroup T and $\mathcal{P}_1 = (\mu_{\mathcal{P}_1}, \nu_{\mathcal{P}_1})$ and $\mathcal{P}_2 = (\mu_{\mathcal{P}_2}, \nu_{\mathcal{P}_2})$ be Pythagorean fuzzy sets on T. The following statements hold.*

(1) *If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\overline{App}(\mathcal{P}_1) \subseteq \overline{App}(\mathcal{P}_2)$ and $\underline{App}(\mathcal{P}_1) \subseteq \underline{App}(\mathcal{P}_2)$.*
(2) *$\overline{App}(\mathcal{P}_1 \cap \mathcal{P}_2) \subseteq \overline{App}(\mathcal{P}_1) \cap \overline{App}(\mathcal{P}_2)$.*

(3) $\overline{App}(\mathcal{P}_1 \cup \mathcal{P}_2) = \overline{App}(\mathcal{P}_1) \cup \overline{App}(\mathcal{P}_2)$.
(4) $\underline{App}(\mathcal{P}_1 \cap \mathcal{P}_2) = \underline{App}(\mathcal{P}_1) \cap \underline{App}(\mathcal{P}_2)$.
(5) $\underline{App}(\mathcal{P}_1) \cup \underline{App}(\mathcal{P}_2) \subseteq \underline{App}(\mathcal{P}_1 \cup \mathcal{P}_2)$.

Proof.

(1) Assume that $\mathcal{P}_1 \subseteq \mathcal{P}_2$. Then $\mu_{\mathcal{P}_1} \leqslant \mu_{\mathcal{P}_2}$ and $\nu_{\mathcal{P}_2} \leqslant \nu_{\mathcal{P}_1}$. Thus for all $y \in T$, we have

$$\overline{\mu_{\mathcal{P}_1}}(y) = \sup_{y' \in [y]_\rho} \mu_{\mathcal{P}_1}(y') \leqslant \sup_{y' \in [y]_\rho} \mu_{\mathcal{P}_2}(y') = \overline{\mu_{\mathcal{P}_2}}(y)$$

and

$$\overline{\nu_{\mathcal{P}_1}}(y) = \inf_{y' \in [y]_\rho} \nu_{\mathcal{P}_1}(y') \geqslant \inf_{y' \in [y]_\rho} \nu_{\mathcal{P}_2}(y') = \overline{\nu_{\mathcal{P}_2}}(y).$$

This implies that $\overline{App}(\mathcal{P}_1) \subseteq \overline{App}(\mathcal{P}_2)$. Similarly, we have $\underline{App}(\mathcal{P}_1) \subseteq \underline{App}(\mathcal{P}_2)$.

(2) Since $\mathcal{P}_1 \cap \mathcal{P}_2 \subseteq \mathcal{P}_1$ and $\mathcal{P}_1 \cap \mathcal{P}_2 \subseteq \mathcal{P}_2$, $\overline{App}(\mathcal{P}_1 \cap \mathcal{P}_2) \subseteq \overline{App}(\mathcal{P}_1) \cap \overline{App}(\mathcal{P}_2)$ by (1).

(3) Note that

$$\overline{App}(\mathcal{P}_1) \cup \overline{App}(\mathcal{P}_2) = (\overline{\mu_{\mathcal{P}_1}} \cup \overline{\mu_{\mathcal{P}_2}}, \overline{\nu_{\mathcal{P}_1}} \cap \overline{\nu_{\mathcal{P}_2}})$$

and

$$\overline{App}(\mathcal{P}_1 \cup \mathcal{P}_2) = (\overline{\mu_{\mathcal{P}_1 \cup \mathcal{P}_2}}, \overline{\nu_{\mathcal{P}_1 \cup \mathcal{P}_2}}).$$

Let $y \in T$. Then

$$\begin{aligned}
(\overline{\mu_{\mathcal{P}_1}} \cup \overline{\mu_{\mathcal{P}_2}})(y) &= \max\{\overline{\mu_{\mathcal{P}_1}}(y), \overline{\mu_{\mathcal{P}_2}}(y)\} \\
&= \max\{\sup_{y' \in [y]_\rho} \mu_{\mathcal{P}_1}(y'), \sup_{y' \in [y]_\rho} \mu_{\mathcal{P}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \max\{\mu_{\mathcal{P}_1}(y'), \mu_{\mathcal{P}_2}(y')\} = \sup_{y' \in [y]_\rho} \mu_{\mathcal{P}_1 \cup \mathcal{P}_2}(y') = \overline{\mu_{\mathcal{P}_1 \cup \mathcal{P}_2}}(y)
\end{aligned}$$

and

$$\begin{aligned}
(\overline{\nu_{\mathcal{P}_1}} \cap \overline{\nu_{\mathcal{P}_2}})(y) &= \min\{\overline{\nu_{\mathcal{P}_1}}(y), \overline{\nu_{\mathcal{P}_2}}(y)\} \\
&= \min\{\inf_{y' \in [y]_\rho} \mu_{\mathcal{P}_1}(y'), \inf_{y' \in [y]_\rho} \mu_{\mathcal{P}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \min\{\nu_{\mathcal{P}_1}(y'), \nu_{\mathcal{P}_2}(y')\} = \inf_{y' \in [y]_\rho} \nu_{\mathcal{P}_1 \cup \mathcal{P}_2}(y') = \overline{\nu_{\mathcal{P}_1 \cup \mathcal{P}_2}}(y).
\end{aligned}$$

(4) Note that

$$\underline{App}(\mathcal{P}_1) \cap \underline{App}(\mathcal{P}_2) = (\underline{\mu_{\mathcal{P}_1}} \cap \underline{\mu_{\mathcal{P}_2}}, \underline{\nu_{\mathcal{P}_1}} \cup \underline{\nu_{\mathcal{P}_2}})$$

and

$$\underline{App}(\mathcal{P}_1 \cap \mathcal{P}_2) = (\underline{\mu_{\mathcal{P}_1 \cap \mathcal{P}_2}}, \underline{\nu_{\mathcal{P}_1 \cap \mathcal{P}_2}}).$$

Let $y \in T$. Then

$$\begin{aligned}
(\underline{\mu_{\mathcal{P}_1}} \cap \underline{\mu_{\mathcal{P}_2}})(y) &= \min\{\underline{\mu_{\mathcal{P}_1}}(y), \underline{\mu_{\mathcal{P}_2}}(y)\} \\
&= \min\{\inf_{y' \in [y]_\rho} \mu_{\mathcal{P}_1}(y'), \inf_{y' \in [y]_\rho} \mu_{\mathcal{P}_2}(y')\} \\
&= \inf_{y' \in [y]_\rho} \min\{\mu_{\mathcal{P}_1}(y'), \mu_{\mathcal{P}_2}(y')\} = \inf_{y' \in [y]_\rho} \mu_{\mathcal{P}_1 \cap \mathcal{P}_2}(y') = \underline{\mu_{\mathcal{P}_1 \cap \mathcal{P}_2}}(y)
\end{aligned}$$

and

$$\begin{aligned}
(\underline{\nu_{\mathcal{P}_1}} \cup \underline{\nu_{\mathcal{P}_2}})(y) &= \max\{\underline{\nu_{\mathcal{P}_1}}(y), \underline{\nu_{\mathcal{P}_2}}(y)\} \\
&= \max\{\sup_{y' \in [y]_\rho} \mu_{\mathcal{P}_1}(y'), \sup_{y' \in [y]_\rho} \mu_{\mathcal{P}_2}(y')\} \\
&= \sup_{y' \in [y]_\rho} \max\{\nu_{\mathcal{P}_1}(y'), \nu_{\mathcal{P}_2}(y')\} = \sup_{y' \in [y]_\rho} \nu_{\mathcal{P}_1 \cap \mathcal{P}_2}(y') = \underline{\nu_{\mathcal{P}_1 \cap \mathcal{P}_2}}(y).
\end{aligned}$$

(5) Since $\mathcal{P}_1 \subseteq \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \subseteq \mathcal{P}_1 \cup \mathcal{P}_2$, $\underline{\mathrm{App}}(\mathcal{P}_1) \cup \underline{\mathrm{App}}(\mathcal{P}_2) \subseteq \underline{\mathrm{App}}(\mathcal{P}_1 \cup \mathcal{P}_2)$ by (1). $\square$

Example 3.10. Let ρ be a congruence on a ternary semigroup $\mathbb{Z}^-$ under the usual multiplication defined by

$$x\rho y \Leftrightarrow 2 \mid x - y \text{ for all } x, y \in \mathbb{Z}^-.$$

(1) Let $\mathcal{P}_1 = (\mu_{\mathcal{P}_1}, \nu_{\mathcal{P}_1})$ and $\mathcal{P}_2 = (\mu_{\mathcal{P}_2}, \nu_{\mathcal{P}_2})$ be Pythagorean fuzzy sets on T defined by

$$\begin{aligned} \mu_{\mathcal{P}_1}(-1) = \mu_{\mathcal{P}_1}(-4) = 1, \mu_{\mathcal{P}_1}(-2) = \mu_{\mathcal{P}_1}(-3) = 0, \\ \mu_{\mathcal{P}_2}(-1) = \mu_{\mathcal{P}_2}(-4) = 0, \mu_{\mathcal{P}_2}(-2) = \mu_{\mathcal{P}_2}(-3) = 1, \\ \mu_{\mathcal{P}_1}(x) = \mu_{\mathcal{P}_2}(y) = 0 \text{ for all } y \in \mathbb{Z}^- \setminus \{-1, -2, -3, -4\}, \end{aligned}$$

and

$$\nu_{\mathcal{P}_1}(x) = \nu_{\mathcal{P}_2}(y) = 0 \text{ for all } y \in \mathbb{Z}^-.$$

We have

$$\overline{\mathrm{App}}(\mathcal{P}_1 \cap \mathcal{P}_2) = \{< y, 0, 0 >\mid y \in T\}$$

and

$$\overline{\mathrm{App}}(\mathcal{P}_1) \cap \overline{\mathrm{App}}(\mathcal{P}_2) = \{< y, 1, 0 >\mid y \in T\}.$$

This implies that the converse of (2) in Theorem 3.9 is not true in general.

(2) Let $\mathcal{P}_1 = (\mu_{\mathcal{P}_1}, \nu_{\mathcal{P}_1})$ and $\mathcal{P}_2 = (\mu_{\mathcal{P}_2}, \nu_{\mathcal{P}_2})$ be Pythagorean fuzzy sets on T defined by for all $x \in \mathbb{Z}^-$,

$$\mu_{\mathcal{P}_1}(y) = \begin{cases} 0, & \text{if } y = -1 \text{ or } y = -2, \\ 1, & \text{otherwise}, \end{cases} \quad \text{and} \quad \mu_{\mathcal{P}_2}(x) = \begin{cases} 1, & \text{if } y = -1 \text{ or } y = -2, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\nu_{\mathcal{P}_1}(y) = \nu_{\mathcal{P}_2}(y) = 0 \text{ for all } y \in \mathbb{Z}^-.$$

We have

$$\underline{\mathrm{App}}(\mathcal{P}_1) \cup \underline{\mathrm{App}}(\mathcal{P}_2) = \{< y, 0, 0 >\mid y \in T\}$$

and

$$\underline{\mathrm{App}}(\mathcal{P}_1 \cup \mathcal{P}_2) = \{< y, 1, 0 >\mid y \in T\}.$$

This implies that the converse of (5) in Theorem 3.9 is not true in general.

Theorem 3.11. *Let ρ be a complete congruence on a ternary semigroup T and $\mathcal{P}_1 = (\mu_{\mathcal{P}_1}, \nu_{\mathcal{P}_1})$, $\mathcal{P}_2 = (\mu_{\mathcal{P}_2}, \nu_{\mathcal{P}_2})$ and $\mathcal{P}_3 = (\mu_{\mathcal{P}_3}, \nu_{\mathcal{P}_3})$ be Pythagorean fuzzy sets on T. Then*

$$\overline{\mathrm{App}}([\mathcal{P}_1 \circ \mathcal{P}_2 \circ \mathcal{P}_3]) \subseteq [\overline{\mathrm{App}}(\mathcal{P}_1) \circ \overline{\mathrm{App}}(\mathcal{P}_2) \circ \overline{\mathrm{App}}(\mathcal{P}_3)].$$

Proof. Obviously,

$$\overline{\mathrm{App}}(\mathcal{P}_1) \circ \overline{\mathrm{App}}(\mathcal{P}_2) \circ \overline{\mathrm{App}}(\mathcal{P}_3) = ([\overline{\mu_{\mathcal{P}_1}} \circ \overline{\mu_{\mathcal{P}_2}} \circ \overline{\mu_{\mathcal{P}_3}}], [\overline{\nu_{\mathcal{P}_1}} \circ \overline{\nu_{\mathcal{P}_2}} \circ \overline{\nu_{\mathcal{P}_3}}])$$

and

$$\overline{\mathrm{App}}([\mathcal{P}_1 \circ \mathcal{P}_2 \circ \mathcal{P}_3]) = ([\overline{\mu_{\mathcal{P}_1} \circ \mu_{\mathcal{P}_2} \circ \mu_{\mathcal{P}_3}}], [\overline{\nu_{\mathcal{P}_1} \circ \nu_{\mathcal{P}_2} \circ \nu_{\mathcal{P}_3}}]).$$

Let $y \in S$. Since $[y_1]_\rho [y_2]_\rho [y_3]_\rho = [y_1 y_2 y_3]_\rho$, we have

$$\begin{aligned} [\overline{\mu_{\mathcal{P}_1}} \circ \overline{\mu_{\mathcal{P}_2}} \circ \overline{\mu_{\mathcal{P}_3}}](y) &= \sup_{y = [y_1 y_2 y_3]} \min\{\overline{\mu_{\mathcal{P}_1}}(y_1), \overline{\mu_{\mathcal{P}_2}}(y_2), \overline{\mu_{\mathcal{P}_3}}(y_3)\} \\ &= \sup_{y = [y_1 y_2 y_3]} \min\{ \sup_{y_1' \in [y_1]_\rho} (\mu_{\mathcal{P}}(y_1'), \sup_{y_2' \in [y_2]_\rho} (\mu_{\mathcal{P}}(y_2'), \sup_{y_3' \in [y_3]_\rho} (\mu_{\mathcal{P}}(y_3')\} \end{aligned}$$

$$
\begin{aligned}
&\geqslant \sup_{y=[y_1y_2y_3]} \sup_{[y_1'y_2'y_3']\in[y_1y_2y_3]_\rho} \min\{\mu_{\mathcal{P}}(y_1'), \mu_{\mathcal{P}}(y_2'), \mu_{\mathcal{P}}(y_3')\}\\
&= \sup_{[y_1'y_2'y_3']\in[y]_\rho} \min\{\mu_{\mathcal{P}}(y_1'), \mu_{\mathcal{P}}(y_2'), \mu_{\mathcal{P}}(y_3')\}\\
&= \sup_{x\in[y]_\rho, x=[y_1'y_2'y_3']} \min\{\mu_{\mathcal{P}}(y_1'), \mu_{\mathcal{P}}(y_2'), \mu_{\mathcal{P}}(y_3')\}\\
&= \sup_{x\in[y]_\rho} \left\{ \sup_{x=[y_1'y_2'y_3']} \min\{\mu_{\mathcal{P}}(y_1'), \mu_{\mathcal{P}}(y_2'), \mu_{\mathcal{P}}(y_3')\} \right\}\\
&= \sup_{x\in[y]_\rho} [\mu_{\mathcal{P}_1} \circ \mu_{\mathcal{P}_2} \circ \mu_{\mathcal{P}_3}](x) = [\overline{\mu_{\mathcal{P}_1} \circ \mu_{\mathcal{P}_2} \circ \mu_{\mathcal{P}_3}}](y).
\end{aligned}
$$

This implies that $[\overline{\mu_{\mathcal{P}_1} \circ \mu_{\mathcal{P}_2} \circ \mu_{\mathcal{P}_3}}](y) \leqslant [\overline{\mu_{\mathcal{P}_1}} \circ \overline{\mu_{\mathcal{P}_2}} \circ \overline{\mu_{\mathcal{P}_3}}](y)$.

$$
\begin{aligned}
[\overline{\nu_{\mathcal{P}_1}} \circ \overline{\nu_{\mathcal{P}_2}} \circ \overline{\nu_{\mathcal{P}_3}}](y) &= \inf_{y=[y_1y_2y_3]} \max\{\overline{\nu_{\mathcal{P}_1}}(y_1), \overline{\nu_{\mathcal{P}_2}}(y_2), \overline{\nu_{\mathcal{P}_3}}(y_3)\}\\
&= \inf_{y=[y_1y_2y_3]} \max\{ \inf_{y_1'\in[y_1]_\rho} (\nu_{\mathcal{P}}(y_1'), \inf_{y_2'\in[y_2]_\rho} (\nu_{\mathcal{P}}(y_2'), \inf_{y_3'\in[y_3]_\rho} (\nu_{\mathcal{P}}(y_3')\}\\
&\leqslant \inf_{y=[y_1y_2y_3]} \inf_{[y_1'y_2'y_3']\in[y_1y_2y_3]_\rho} \max\{\nu_{\mathcal{P}}(y_1'), \nu_{\mathcal{P}}(y_2'), \nu_{\mathcal{P}}(y_3')\}\\
&= \inf_{[y_1'y_2'y_3']\in[y]_\rho} \max\{\nu_{\mathcal{P}}(y_1'), \nu_{\mathcal{P}}(y_2'), \nu_{\mathcal{P}}(y_3')\}\\
&= \inf_{x\in[y]_\rho, x=[y_1'y_2'y_3']} \max\{\nu_{\mathcal{P}}(y_1'), \nu_{\mathcal{P}}(y_2'), \nu_{\mathcal{P}}(y_3')\}\\
&= \inf_{x\in[y]_\rho} \left\{ \inf_{x=[y_1'y_2'y_3']} \max\{\nu_{\mathcal{P}}(y_1'), \nu_{\mathcal{P}}(y_2'), \nu_{\mathcal{P}}(y_3')\} \right\}\\
&= \inf_{x\in[y]_\rho} [\nu_{\mathcal{P}_1} \circ \nu_{\mathcal{P}_2} \circ \nu_{\mathcal{P}_3}](x) = [\overline{\nu_{\mathcal{P}_1} \circ \nu_{\mathcal{P}_2} \circ \nu_{\mathcal{P}_3}}](y).
\end{aligned}
$$

This implies that $[\overline{\nu_{\mathcal{P}_1} \circ \nu_{\mathcal{P}_2} \circ \nu_{\mathcal{P}_3}}](y) \geqslant [\overline{\nu_{\mathcal{P}_1}} \circ \overline{\nu_{\mathcal{P}_2}} \circ \overline{\nu_{\mathcal{P}_3}}](y)$.

Therefore, $\overline{\mathrm{App}}([\mathcal{P}_1 \circ \mathcal{P}_2 \circ \mathcal{P}_3]) \subseteq [\overline{\mathrm{App}}(\mathcal{P}_1) \circ \overline{\mathrm{App}}(\mathcal{P}_2) \circ \overline{\mathrm{App}}(\mathcal{P}_3)]$. □

3.3. Rough Pythagorean fuzzy ideals in ternary semigroups

Theorem 3.12. *Let ρ be a congruence relation on a ternary semigroup T and $\mathcal{P}$ be a Pythagorean fuzzy set of T.*

(1) *If $\mathcal{P}$ is a Pythagorean fuzzy left ideal (respectively Pythagorean fuzzy right ideal and Pythagorean fuzzy lateral idea) of T, then $\overline{\mathrm{App}}(\mathcal{P})$ is a Pythagorean fuzzy left ideal (respectively Pythagorean fuzzy right ideal and Pythagorean fuzzy lateral ideal) of T.*

(2) *If ρ is complete and $\mathcal{P}$ is a Pythagorean fuzzy left ideal (respectively Pythagorean fuzzy right ideal and Pythagorean fuzzy lateral ideal) of T, then $\underline{\mathrm{App}}(\mathcal{P})$ is a Pythagorean fuzzy left ideal (respectively Pythagorean fuzzy right ideal and Pythagorean fuzzy lateral ideal) of T.*

Proof.

(1) Let $y_1, y_2, y_3 \in T$.

$$
\begin{aligned}
\overline{\mu_{\mathcal{P}}}([y_1y_2y_3]) = \sup_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{P}}(y) &\geqslant \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{P}}(y)\\
&= \sup_{[abc]\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{P}}([abc])\\
&\geqslant \sup_{[abc]\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{P}}(c) = \sup_{c\in[y_3]_\rho} \mu_{\mathcal{P}}(c) = \overline{\mu_{\mathcal{P}}}(y_3)
\end{aligned}
$$

and

$$\begin{aligned}\overline{\nu_{\mathcal{P}}}([y_1y_2y_3]) = \inf_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{P}}(y) &\leqslant \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{P}}(y)\\ &= \inf_{[abc]\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{P}}([abc])\\ &\leqslant \inf_{c\in[y_3]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{P}}(c) = \inf_{c\in[y_3]_\rho} \nu_{\mathcal{P}}(c) = \overline{\nu_{\mathcal{P}}}(y_3).\end{aligned}$$

This implies that $\overline{\mu_{\mathcal{P}}}([y_1y_2y_3]) \geqslant \overline{\mu_{\mathcal{P}}}(y_3)$ and $\overline{\nu_{\mathcal{P}}}([y_1y_2y_3]) \leqslant \overline{\nu_{\mathcal{P}}}(y_3)$. Then $\overline{\mathrm{App}}(\mathcal{P})$ is a Pythagorean fuzzy left ideal of T. The proofs of other cases are similar.

(2) Let $y_1, y_2, y_3 \in T$.

$$\begin{aligned}\underline{\mu_{\mathcal{P}}}([y_1y_2y_3]) = \inf_{y\in[y_1y_2y_3]_\rho} \mu_{\mathcal{P}}(y) &= \inf_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{P}}(y)\\ &= \inf_{[abc]\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{P}}([abc])\\ &\geqslant \inf_{[abc]\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \mu_{\mathcal{P}}(c) = \inf_{c\in[y_3]} \mu_{\mathcal{P}}(c) = \underline{\mu_{\mathcal{P}}}(y_3)\end{aligned}$$

and

$$\begin{aligned}\underline{\nu_{\mathcal{P}}}([y_1y_2y_3]) = \sup_{y\in[y_1y_2y_3]_\rho} \nu_{\mathcal{P}}(y) &= \sup_{y\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{P}}(y)\\ &= \sup_{[abc]\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{P}}([abc])\\ &\leqslant \sup_{[abc]\in[y_1]_\rho[y_2]_\rho[y_3]_\rho} \nu_{\mathcal{P}}(c) = \sup_{c\in[y_3]} \nu_{\mathcal{P}}(c) = \underline{\nu_{\mathcal{P}}}(y_3).\end{aligned}$$

This implies that $\underline{\mu_{\mathcal{P}}}([y_1y_2y_3]) \geqslant \underline{\mu_{\mathcal{P}}}(y_3)$ and $\underline{\nu_{\mathcal{P}}}([y_1y_2y_3]) \leqslant \underline{\nu_{\mathcal{P}}}(y_3)$. Then $\underline{\mathrm{App}}(\mathcal{P})$ is a Pythagorean fuzzy left ideal of T. The proofs of other cases are similar. □

Corollary 3.13. *Let ρ be a congruence relation on a ternary semigroup T and $\mathcal{P}$ be a Pythagorean fuzzy set of T.*

(1) *If $\mathcal{P}$ is a Pythagorean fuzzy ideal of T, then $\overline{\mathrm{App}}(\mathcal{P})$ is a Pythagorean fuzzy ideal of T.*
(2) *If ρ is complete and $\mathcal{P}$ is a Pythagorean fuzzy ideal of T, then $\underline{\mathrm{App}}(\mathcal{P})$ is a Pythagorean fuzzy ideal of T.*

Proof. This follows from Theorem 3.12. □

4. Conclusion

In this paper, the notion of rough Pythagorean fuzzy sets in ternary semigroups is studied. The idea of rough Pythagorean fuzzy sets is extended to the lower and upper approximations of Pythagorean fuzzy ideals in ternary semigroups and some important properties related to these notions are presented.

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