A TECHNIQUE FOR ANALYZING FINITE ELEMENT METHODS FOR VISCOUS INCOMPRESSIBLE FLOW

(Revised manuscript, May 4, 1990)

Rolf Stenberg
Faculty of Mechanical Engineering
Helsinki University of Technology
02150 Espoo, Finland

ABSTRACT. We give a self-contained presentation of our macroelement technique for verifying the stability of finite element discretizations of Navier-Stokes equations in the velocity-pressure formulation. The results are also presented in form suitable for the non-mathematical reader.

KEY WORDS. Stokes equations, Mixed finite elements, stability, Patch test
1. INTRODUCTION

In this paper we will consider some aspects of the approximate solution of the incompressible Navier-Stokes equations by finite element methods. The class of methods to be discussed consists of methods where independent finite element spaces are used for the velocity and the pressure.

There are two difficult problems connected with this approach. The first is the approximation of the convection term. Recently considerable progress has been done for this field of problem, as can be seen from some other papers of this issue.

The problem we are going to discuss stems from the incompressibility condition. It is well known that this implies that the finite element spaces for the velocity and pressure cannot be chosen arbitrarily. Instead, the velocity pressure pair has to satisfy a stability inequality, the famous ”Babuška-Brezzi” or ”inf-sup” condition.

The basic theory for mixed methods was developed in the fundamental papers by Babuška [1,2] and Brezzi [7]. Later this theory has been applied to mixed finite element methods for a number of problems in continuum mechanics.

With regards to the discretization of Navier-Stokes equations, a technique for proving the Babuška-Brezzi condition was introduced in a basic paper by Crouzeix and Raviart [10]. In the same paper a widely used technique for designing stable discretizations, using ”bubble functions”, was introduced.

The drawback of the technique of [10] is that it consists of an explicit construction of the stability inequality and this involves some technical scaling arguments. In particular, the technique is difficult to apply to so called Taylor-Hood methods in which continuous approximations are used for the pressure.

The problem of analyzing Taylor-Hood methods was partially resolved by Bercovier and Pironneau [5], who showed that the convergence can be proved by altering the norms used in the stability inequality. Later Verfürth [18] considerably simplified their analysis by showing that the modified stability inequality implies the inequality with the natural norms.

We refer to the book by Girault and Raviart [14] for a rather complete survey of methods which are proven to be stable.

The purpose of this chapter is to give a self-contained review of a technique developed by us [15,17] for the analysis of mixed methods (cf. also [6] for very related ideas). Our results show that the Babuška-Brezzi inequality can be proved by verifying similar local inequalities posed over ”macroelements” consisting of a finite number of elements. Furthermore, these local stability estimates are equivalent with a simple algebraic condition.
which often can easily be checked. As a result, many technical arguments previously needed when analyzing a method can be avoided. In engineering language our technique consists of a "patch test" that has to be verified. Another related, but non-rigorous (cf. Remark 3.2 below), "patch test" has recently been advocated in Reference [19].

The plan of the paper is the following. In the next section be briefly recall some background results and definitions. Section 3 is devoted to our analysis technique which is applied in some examples in section 4.

We only treat conforming methods, but the technique can also be applied to nonconforming approximations. This has recently been done in [11].

We would like to emphasize that the results of the paper also covers the analysis of mixed methods for (nearly) incompressible elasticity.

Let us also point out that the same technique can be applied for the analysis for mixed finite element methods for other problems such as e.g. the equations of elasticity with the displacement and the stress tensor as independent variables.

2. PRELIMINARIES

Since we are not concerned with the discretization of the convection term in the Navier-Stokes equations, it will be sufficient to consider the approximation of the Stokes equations with viscosity equal to one: Find the velocity \( u = (u_1, ..., u_d) \) and the pressure \( p \) such that

\[
-\Delta u + \nabla p = f \quad \text{in} \quad \Omega,
\]

\[
\nabla \cdot u = 0 \quad \text{in} \quad \Omega,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where as usual nonhomogeneous boundary conditions are included in the body force vector \( f \). The domain \( \Omega \subset \mathbb{R}^d, \quad d = 2 \text{ or } 3, \) is assumed to be bounded and, for simplicity, polygonal or polyhedral.

The mathematical formulation of the problem is: Find \( u \in [H^1_0(\Omega)]^d \) and \( p \in L^2_0(\Omega) \) such that

\[
(\nabla u, \nabla v) - (\nabla \cdot v, p) = (f, v), \quad v \in [H^1_0(\Omega)]^d,
\]

\[
(\nabla \cdot u, q) = 0, \quad q \in L^2_0(\Omega).
\]

Our notation is standard (cf. [8]): \([H^s(D)]^\alpha\), with \( \alpha = 1 \text{ or } d \), and integer \( s \), denotes the standard \( L^2(D) \)-based Sobolev spaces. \( L^2_0(D) \) denotes the subspace of \( L^2(D) \) of functions with zero mean value:

\[
L^2_0(D) = \{ \ p \in L^2(D) \ | \ \int_D p \ dx = 0 \ \}.
\]
The norms and seminorms in \([H^s(D)]^\alpha\) are denoted by \(\|\cdot\|_{s,D}\) and \(|\cdot|_{s,D}\), respectively. Furthermore, \((\cdot, \cdot)_D\) denotes the inner product in \(L^2(D)\), \([L^2(D)]^d\) or \([L^2(D)]^{d\times d}\). As usual the subscripts are dropped for the case \(D = \Omega\). By \(C\) and \(C_j, j \in \mathbb{N}\), we denote various positive constants which not necessarily take the same values at each instance. Furthermore, these constants are independent of the element and macroelement partitionings \(C_h\) and \(M_h\) to be introduced.

The problem (2.2) is a typical example of a saddle point problem, and the existence and uniqueness of the solution is a consequence of the inequality

\[
\sup_{0 \neq v \in [H^1_0(\Omega)]^d} \frac{(\nabla \cdot v, p)}{\|v\|_1} \geq C \|p\|_0, \quad p \in L^2_0(\Omega). \tag{2.3}
\]

A simple proof of this in the case when \(\Omega\) has a smooth boundary can be found in [3, pp. 172-174]. For the general case we refer to [14].

The class of methods we are concerned with are formulated as follows. We choose two subspaces \(V_h \subset [H^1_0(\Omega)]^d\) and \(P_h \subset L^2_0(\Omega)\) and pose the problem: Find \(u_h \in V_h\) and \(p_h \in P_h\) such that

\[
(\nabla u_h, \nabla v) - (\nabla \cdot v, p_h) = (f, v), \quad v \in V_h,
\]

\[
(\nabla \cdot u_h, q) = 0, \quad q \in P_h. \tag{2.4}
\]

Now, in order to have a good finite element method, the finite element spaces have to be chosen so that they inherit the property (2.3), i.e. they should satisfy

\[
\sup_{0 \neq v \in V_h} \frac{(\nabla \cdot v, p)}{\|v\|_1} \geq C \|p\|_0, \quad p \in P_h. \tag{2.5}
\]

Then the theory of mixed method states that the following optimal error estimate is valid

\[
\|u - u_h\|_1 + \|p - p_h\|_0 \leq C\{ \inf_{v \in V_h} \|u - v\|_1 + \inf_{q \in P_h} \|p - q\|_0 \}. \tag{2.6}
\]

### 3. THE ANALYSIS TECHNIQUE

In order to be able to give precise and general results, we have to define our concepts properly. This will unfortunately burden the presentation, but the main idea should, however, not be difficult to grasp.

The reader mainly interested in applications of finite element methods may skip the technical details in the proofs of the stability and error estimates, and instead go to Theorem 3.2 after getting acquainted with our definitions.
We let \( \mathcal{C}_h \) be a partitioning of \( \bar{\Omega} \) into elements which all are assumed to be either triangles or convex quadrilaterals in the two-dimensional case, and tetrahedrons or convex hexahedrons for a three-dimensional problem. Naturally, the partitioning is assumed to satisfy the usual compatibility and regularity conditions \[8\]. As an example we recall the definition of regularity for a triangular partitioning. Given an element \( K \in \mathcal{C}_h \), let \( h_K \) denote the diameter of \( K \), and let \( \rho_K \) be the maximum diameter of all circles contained in \( K \). \( \mathcal{C}_h \) is then regular if there is a constant \( \sigma > 1 \) such that
\[
 h_K \leq \sigma \rho_K \quad \text{for all} \ K \in \mathcal{C}_h. \tag{3.1}
\]
For the other type of elements the regularity is defined analogously; cf. \[8\].

Let us further assume that the finite element spaces can uniquely be defined using a reference element \( \hat{K} \) (i.e. the unit triangle, tetrahedron, square or cube) and two finite dimensional polynomial spaces \( \hat{V} \) and \( \hat{P} \) defined on \( \hat{K} \). For \( K \in \mathcal{C}_h \) we let \( F_K \) be the affine, bi- or trilinear mapping from \( \hat{K} \) onto \( K \). We then define
\[
 V_h = \{ \, v \in [H^1_0(\Omega)]^d \mid v(x) = \hat{v}(F_K^{-1}(x)), \ \hat{v} \in \hat{V}, \ K \in \mathcal{C}_h \, \}, \tag{3.2}
\]
and
\[
 P_h = \{ \, p \in L^2_0(\Omega) \mid p(x) = \hat{p}(F_K^{-1}(x)), \ \hat{p} \in \hat{P}, \ K \in \mathcal{C}_h \, \} \tag{3.3a}
\]
or
\[
 P_h = \{ \, p \in C(\Omega) \cap L^2_0(\Omega) \mid p(x) = \hat{p}(F_K^{-1}(x)), \ \hat{p} \in \hat{P}, \ K \in \mathcal{C}_h \, \}. \tag{3.3b}
\]

The choice \(3.3b\) gives a method of the Taylor-Hood type.

Next, let us introduce the concept of a macroelement, i.e. a connected set which is the union of at least two elements. For the elements of a macroelement we also impose the usual compatibility and regularity conditions.

Given a macroelement \( M \), we define finite element spaces in consistency with (3.2) and (3.3):
\[
 V_{0,M} = \{ \, v \in [H^1_0(M)]^d \mid v(x) = \hat{v}(F_K^{-1}(x)), \ \hat{v} \in \hat{V}, \ x \in K, \ K \subset M \, \} \tag{3.4}
\]
and
\[
 P_M = \{ \, p \in L^2(M) \mid p(x) = \hat{p}(F_K^{-1}(x)), \ \hat{p} \in \hat{P}, \ x \in K, \ K \subset M \, \} \tag{3.5a}
\]
or
\[
 P_M = \{ \, p \in C(M) \mid p(x) = \hat{p}(F_K^{-1}(x)), \ \hat{p} \in \hat{P}, \ x \in K, \ K \subset M \, \} \tag{3.5b}
\]
By $\Gamma_h$ we denote the collection of edges or faces (for a three-dimensional problem), of the elements of $C_h$, not lying on the boundary of $\Omega$.

The following norm defined in $P_h$ turns out to be very useful

$$\|p\|_h^2 = \sum_{K \in C_h} h_K^2 \|\nabla p\|_{0,K}^2 + \sum_{T \in \Gamma_h} h_T \int_T [[p]]^2 \, ds.$$ 

Here and in the sequel $T$ stands for an edge or a face of an element and $h_T$ denotes the diameter of $T$. $([[p]])_T$ denotes the jump in $p$ along $T$.

In $P_M$ we similarly define

$$|p|^2_M = \sum_{K \subseteq M} h_K^2 \|\nabla p\|_{0,K}^2 + \sum_{T \in \Gamma_M} h_T \int_T [[p]]^2 \, ds,$$

where $\Gamma_M$ denotes the interior edges (faces) of $M$, i.e. $\Gamma_M = \{ \, T \subseteq M \mid T \not\subseteq \partial M \, \}$. The usefulness of the macroelement concept and the above mesh dependent norm is that it enables us to build a global stability estimate by simply adding together analogous local estimates:

**Lemma 3.1.** Suppose that we can define a macroelement partitioning $\mathcal{M}_h$ such that:

(i) Each $T \in \Gamma_h$ is an interior edge (face) of at least one and not more than $L$ macroelements of $\mathcal{M}_h$.

(ii) There is a positive constant $C$ such that

$$\sup_{0 \neq v \in V_{0,M}} \frac{(\nabla \cdot v, p)_M}{|v|_{1,M}} \geq C |p|_M, \quad p \in P_M, \quad (3.6)$$

holds for all $M \in \mathcal{M}_h$.

Then the stability inequality

$$\sup_{0 \neq v \in V_h} \frac{(\nabla \cdot v, p)}{||v||_1} \geq C ||p||_h, \quad p \in P_h \quad (3.7)$$

is valid.

**Proof:** Let $p \in P_h$ be arbitrary. The local stability estimates imply that for every $M \in \mathcal{M}_h$ there exists $v_M \in V_h$, vanishing outside of $M$, such that

$$(\nabla \cdot v_M, p) = (\nabla \cdot v_M, p)_M \geq C |p|^2_M$$

and

$$|v_M|_1 = |v_M|_{1,M} \leq |p|_M.$$
Let us define
\[ v = \sum_{M \in \mathcal{M}_h} v_M. \]

Since each \( T \in \Gamma_h \) is an interior edge (face) of at least one \( M \in \mathcal{M}_h \), each element \( K \in \mathcal{C}_h \) is contained in one macroelement of \( \mathcal{M}_h \). Hence, we have
\[ (\nabla \cdot v, p) = \sum_{M \in \mathcal{M}_h} (\nabla \cdot v_M, p) \geq C \sum_{M \in \mathcal{M}_h} |p|_M^2 \geq C\|p\|_h^2. \]

Furthermore, since each \( T \in \Gamma_h \) is contained in at most \( L \) macroelements, each element \( K \in \mathcal{C}_h \) is contained in at most \( 6L \) macroelements (with the maximum obtained for a hexahedral partitioning). This gives
\[ \|v\|_1 \leq C \|v\|_1 \leq C \sum_{M \in \mathcal{M}_h} |v_M|_1 \leq C \sum_{M \in \mathcal{M}_h} |p|_M \leq 6CL\|p\|_h, \]

which together with the earlier estimate prove the assertion. \[ \square \]

The following two results, essentially due to Verfürth [18], provide the link between the stability in the mesh dependent norm and in the desired \( L^2 \)-norm. We formulate them as separate lemmas, since the same reasoning can be used in other contexts as well; cf. [12,13]. We remark that the lemma below is valid for arbitrary spaces \( V_h \) and \( P_h \).

**Lemma 3.2.** There are two positive constants \( C_1, C_2 \), such that
\[ \sup_{0 \neq v \in V_h} \frac{(\nabla \cdot v, p)}{\|v\|_1} \geq C_1 \|p\|_0 - C_2 \|p\|_h, \quad p \in P_h. \]

**Proof:** Let \( p \in P_h \) be arbitrary. The condition (2.3) then implies the existence of \( w \in [H^1_0(\Omega)]^d \) such that
\[ (\nabla \cdot w, p) \geq C_3 \|p\|_0^2 \]
and
\[ \|w\|_1 \leq \|p\|_0. \]

We now interpolate \( w \) with \( \tilde{w} \in V_h \) defined by the technique of Clemént [9], so that we have the estimates (cf. [4, Lemma 3] and [14, pp. 109-111])
\[ \left( \sum_{K \in \mathcal{C}_h} h_K^{-2} \|w - \tilde{w}\|_{0,K}^2 + \sum_{T \in \Gamma_h} h_T^{-1} \int_T |w - \tilde{w}|^2 \, ds \right)^{1/2} \leq C_4 \|w\|_1 \]
and
\[ \|\tilde{w}\|_1 \leq C_5 \|w\|_1. \]
Integrating by parts on each $K \in \mathcal{C}_h$ and using (3.8), (3.10) we get

\[
(\nabla \cdot \tilde{w}, p) = (\nabla \cdot (\tilde{w} - w), p) + (\nabla \cdot w, p)
\geq (\nabla \cdot (\tilde{w} - w), p) + C_3\|p\|_0^2
\]

\[
= \sum_{K \in \mathcal{C}_h} (w - \tilde{w}, \nabla p)_K + \sum_{T \in \mathcal{T}_h} \int_T ((\tilde{w} - w) \cdot n)([p]) \, ds + C_3\|p\|_0^2
\geq -\left( \sum_{K \in \mathcal{C}_h} h_K^{-2}\|w - \tilde{w}\|_{0,K}^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \int_T |w - \tilde{w}|^2 \, ds \right)^{1/2} \cdot \|p\|_h + C_3\|p\|_0^2
\]

(3.12) and (3.9) now give

\[
\|\tilde{w}\|_1 \leq C_5\|w\|_1 \leq C_5\|p\|_0.
\]

Hence, (3.12) and (3.13) give the asserted estimate.

LEMMA 3.3. Suppose that the stability estimate (3.7) is valid. Then the desired stability condition (2.5) also holds.

PROOF: Let $C_1$, $C_2$ be the constants in Lemma 3.2 and denote by $C_3$ that of (3.7). For $0 < \xi < 1$ we then have

\[
\sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}, p)}{\|\mathbf{v}\|_1} = \xi \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}, p)}{\|\mathbf{v}\|_1} + (1 - \xi) \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}, p)}{\|\mathbf{v}\|_1}
\geq C_1\xi\|p\|_0 + [(1 - \xi)C_3 - \xi C_2]\|p\|_h
\geq C\|p\|_0,
\]

when choosing $\xi < C_3(C_2 + C_3)^{-1}$.

By Lemmas 3.1 to 3.3, the problem of proving the stability inequality is reduced to proving the local estimates (3.6).

An immediate observation is that a necessary condition for the inequality (3.6) to be valid is that the subspace

\[
\mathcal{N}_M = \{ \mathbf{v} \in \mathbf{P}_M \mid (\nabla \cdot \mathbf{v}, p)_M = 0, \mathbf{v} \in \mathbf{V}_{0,M} \}
\]

only consists of the functions which are constant on $M$.

We call this the "macroelement condition".

8
In [15] we showed, roughly speaking, that if this condition is satisfied independent of the geometrical shape of the macroelement, then the local stability estimate, with a constant independent of the particular macroelement, is valid. For stating the exact result we need one more definition.

A macroelement $M$ is said to be equivalent with a reference macroelement $\hat{M}$ if one can define a continuous one-to-one mapping $F_M: \hat{M} \to M$ such that:

(i) $F_M(\hat{M}) = M$.

(ii) If $\hat{M} = \bigcup_{j=1}^m \hat{K}_j$, where $\hat{K}_j$, $j = 1, 2, ..., m$, are the elements of $\hat{M}$, then $K_j = F_M(\hat{K}_j)$, $j = 1, 2, ..., m$, are the elements of $M$.

(iii) $F_M|_{\hat{K}_j} = F_{K_j} \circ F_{\hat{K}_j}^{-1}$, $j = 1, 2, ..., m$, where $F_{K_j}$ and $F_{\hat{K}_j}$ are the mappings from the reference element $\hat{K}$ onto $K_j$ and $\hat{K}_j$, respectively.

**Lemma 3.4.** Let $\mathcal{E}$ be a class of equivalent macroelements. Suppose that for every $M \in \mathcal{E}$ the space $N_M$ is one-dimensional consisting of functions constant on $M$. Then there is a constant $C$ such that

$$\sup_{0 \neq v \in V_{0,M}} \frac{(\nabla \cdot v, p)_M}{|v|_{1,M}} \geq C|p|_M, \quad p \in P_M,$$

holds for all $M \in \mathcal{E}$.

**Proof:** For $M \in \mathcal{E}$ define

$$\beta_M = \inf_{p \in P_M} \sup_{|v|_{1,M} = 1} (\nabla \cdot v, p)_M.$$

Since $N_M$ is assumed to consist of only the constant functions, we have $\beta_M > 0$.

We thus have to show that there exists a constant $\beta_{\mathcal{E}} > 0$, such that $\beta_M \geq \beta_{\mathcal{E}} > 0$ for every $M \in \mathcal{E}$.

To prove this we use a kind of generalized scaling argument. Denote by $\hat{x}^1, \hat{x}^2, ..., \hat{x}^k$ the vertices of $\hat{M}$. Then $M$ is uniquely defined by its vertices $x^i = F_M(\hat{x}^i)$, $i = 1, 2, ..., k$. In particular, this means that we can write $\beta_M = \beta(x^1, x^2, ..., x^k) = \beta(X)$, with $X = (x^1, x^2, ..., x^k)$ considered as a point in $\mathbb{R}^{dk}$. Without loss of generality we can assume that $x^1$ coincides with the origin, and $h_M = 1$, with $h_M = \max_{K \subset M} h_K$, since the general case can handled by changing variables from $x$ to $h_M^{-1}(x - x^1)$. By this every vertex will be within a given distance from the origin. Furthermore, every $K \subset M$ has a diameter less or equal to unity and satisfies some regularity conditions of the type (3.1). This means that $X$ belongs to a compact set in $\mathbb{R}^{dk}$. If now the function $\beta$ can be proven to be continuous, we have

$$\inf_{p \in P_M} \sup_{|v|_{1,M} = 1} (\nabla \cdot v, p)_M = \beta_M \geq \beta_{\mathcal{E}} > 0, \quad M \in \mathcal{E},$$
which is equivalent with the asserted estimate.

It is not difficult to see that $\beta$ is continuous. Let $\tilde{M}$ be another macroelement in $\mathcal{E}$ and denote by $\tilde{X}$ the corresponding point in $\mathbb{R}^{dk}$. Define $G : M \to \tilde{M}$ through $G = F_{\tilde{M}} \circ F_{M}^{-1}$ where $F_{\tilde{M}}$ and $F_{M}$ are the mappings from the reference macroelement onto $\tilde{M}$ and $M$, respectively. For arbitrary $v \in V_{0,M}$ and $p \in P_{M}$, we define $\tilde{v} \in V_{0,\tilde{M}}$ and $\tilde{p} \in P_{\tilde{M}}$ through 

$$\tilde{v}(\tilde{x}) = v(G^{-1}(\tilde{x})), \quad \tilde{p}(\tilde{x}) = p(G^{-1}(\tilde{x})), \quad \tilde{x} \in \tilde{M}.$$ 

Let $J_{G}$ be the Jacobian of $G$. By transforming integrals posed over $\tilde{M}$ to integrals over $M$, and using the fact that $J_{G}$ converges towards the identity when $\tilde{X} \to X$, we now get estimates of the type 

$$\left| (\nabla \cdot v, p)_{M} - (\nabla \cdot \tilde{v}, \tilde{p})_{\tilde{M}} \right| \leq C_{1}(X, \tilde{X})|v|_{1,M}|p|_{M},$$

$$\left| |v|_{1,M} - |\tilde{v}|_{1,\tilde{M}} \right| \leq C_{2}(X, \tilde{X})|v|_{1,M},$$

$$\left| |p|_{M} - |\tilde{p}|_{\tilde{M}} \right| \leq C_{3}(X, \tilde{X})|p|_{M},$$

with $C_{i}(X, \tilde{X}) \to 0$, $i = 1, 2, 3$, when $\tilde{X} \to X$.

The continuity of $\beta$ is now a simple consequence of these three estimates.  

By combining Lemmas 3.1, 3.3 and 3.4 we arrive at our technique for the analysis of mixed methods.

THEOREM 3.1. Suppose that there is a fixed set of equivalence classes $\mathcal{E}_{i}$, $i = 1, 2, ..., l$, of macroelements, a positive integer $L$ and a macroelement partitioning $\mathcal{M}_{h}$ such that:

(M1) For each $M \in \mathcal{E}_{i}$, $i = 1, 2, ..., l$, the space $N_{M}$ is one-dimensional consisting of functions that are constant on $M$.

(M2) Each $M \in \mathcal{M}_{h}$ belongs to one of the classes $\mathcal{E}_{i}$, $i = 1, 2, ..., l$.

(M3) Each $T \in \Gamma_{h}$ is an interior edge (face) of at least one and not more than $L$ macroelements of $\mathcal{M}_{h}$.

Then the stability inequality (2.5) is valid.

PROOF: Lemma 3.4 shows that (3.6), with a constant $C_{i}$, holds for each class $\mathcal{E}_{i}$, $i = 1, 2, ..., l$. Letting $C = \min\{C_{1}, C_{2}, ..., C_{l}\}$, the assumptions of Lemma 3.1 are valid and the assertion then follows from Lemma 3.3.  

10
REMARK 3.1. Here and in [17] we have chosen to define a macroelement to consist of at least two elements and defined the partitioning $\mathcal{M}_h$ to consist of overlapping macroelements.

In [15] (and also [6,14]) the partitioning is defined such that every element belongs to one and only one macroelement. Furthermore, the local stability estimates, used to build the global one, where

$$
\sup_{0 \neq v \in \mathbf{V}_{0,M}} \frac{\langle \nabla \cdot v, p \rangle_M}{|v|_{1,M}} \geq C \|p\|_{0,M}, \quad p \in P_M \cap L^2_0(M).
$$

This estimate also follows from the condition that $N_M$ consists of the constant functions. Since the macroelements where nonoverlapping, some additional assumptions had to be made on $\mathcal{M}_h$ in order to assure the stability of the velocity-pressure pair $(\mathbf{V}_h, \bar{P}_h)$, with

$$
\bar{P}_h = \{ \ p \in P_h \mid p|_M \text{ is constant for all } M \in \mathcal{M}_h \}.
$$

For introducing the present modification of our technique there are two reasons. First, by using nonoverlapping macroelements often far more classes are needed, and it can even sometimes be difficult to see how a macroelement partitioning, satisfying all the conditions required, should be constructed. This is particularly true for some three-dimensional methods. A good example is provided by the quadratic tetrahedral Taylor-Hood method. The macroelement condition is easily proven to be satisfied for a macro consisting of tetrahedrons which have exactly one common vertex in the interior of the macroelement (cf. [16]). An arbitrary finite element partitioning $\mathcal{C}_h$ cannot, however, be regrouped into nonoverlapping macroelements of this type. Also, for those cases for which this would be possible, the condition that the pair $(\mathbf{V}_h, \bar{P}_h)$ is stable, is not necessarily valid.

Second, by using overlapping macroelements and the local estimates in the form (3.6), the analysis more clearly shows that it is the condition that $N_M$ consists of the constant functions which is the condition that has to be verified for the macroelements chosen.

REMARK 3.2. The patch test introduced in [19] consists simply of checking that

$$
\dim \mathbf{V}_{0,M} \geq \dim P_M - 1.
$$

Hence, that test is merely the first thing that has to be checked when choosing a candidate for a class satisfying the our macroelement condition.

Therefore the patch test of [19] is far from a sufficient condition for the stability of the method, and, in fact, it does not even guarantee that the solution is unique. A good
example is the Q9/Q4 element (with the notation of [19]) which satisfies the test for a patch of $2 \times 2$ elements. However, it is a simple exercise to show that there are meshes for which this method does not yield a unique solution.

For the practitioners engineer it is clearly desirable to clearly understand the problem with mixed methods. Let us therefore present the following simplified version of our results.

**Theorem 3.2.** Suppose that there is a fixed set of equivalence classes $\mathcal{E}_i$, $i = 1, 2, ..., l$, of macroelements, a positive integer $L$ and a macroelement partitioning $\mathcal{M}_h$ such that:

(M1) For each $M \in \mathcal{E}_i$, $i = 1, 2, ..., l$, the space $N_M$ is one-dimensional consisting of functions that are constant on $M$.

(M2) Each $M \in \mathcal{M}_h$ belongs to one of the classes $\mathcal{E}_i$, $i = 1, 2, ..., l$.

(M3) Each $T \in \Gamma_h$ is an interior edge (face) of at least one macroelement of $\mathcal{M}_h$.

Then the problem (2.4) has a unique solution.

**Proof:** By the linearity we have to show that if

\[
(\nabla u_h, \nabla v) - (\nabla \cdot v, p_h) = 0, \quad v \in V_h,
\]

\[
(\nabla \cdot u_h, q) = 0, \quad q \in P_h,
\]

then $u_h = 0$ and $p_h = 0$.

To this end we choose $v = u_h$ and $q = p_h$ above. This gives

\[
0 = (\nabla u_h, \nabla u_h) = \int_{\Omega} |\nabla u_h|^2 \, dx,
\]

i.e. $u_h$ is a constant vector in $\Omega$, and since it vanish on the boundary of $\Omega$, we have $u_h = 0$.

The first equation above then reduces to

\[
(\nabla \cdot v, p_h) = 0, \quad v \in V_h.
\] (3.14)

Now, the conditions M1-M3 ensures that for each interior edge (face) $T \in \Gamma_h$ there is a macroelement $M$ with $T$ in its interior, and a $v_M \in V_{0,M} \subset V_h$ such that choosing $v = v_M$ in (3.14) forces $p_h$ to be constant on $M$. Since each element has at least one edge (face) in $\Gamma_h$ this shows that $p_h$ is constant in the whole of $\Omega$. Due to the requirement of zero mean value we have $p_h = 0$.

We see that the proof above is extremely simply, and hence it could be presented in elementary engineering education. Compared to non-rigorous ”theories” like the well-known ”constraint counting” or the patch test of [19], it has the advantage of giving a
simple and completely rigorous condition by which the uniqueness of the approximate solution can be assured.

For the practitioner it should be comfortable to know, that the condition also implies the strict mathematical stability condition.

4. APPLICATIONS

Let us here illustrate our technique by applying it in some concrete examples.

We will first consider a non-standard method introduced by us in [17]. The method has at least some pedagogical interest, since it is ideally suited for demonstrating the technique.

EXAMPLE 1. Let $C_h$ be a triangulation of the two-dimensional domain. Define

$$V_h = \{ \mathbf{v} = (v_1, v_2) \in [H^1_0(\Omega)]^2 \mid v_1|_K \in P_1(K), \ v_2|_K \in P_2(K), \ K \in C_h \},$$

$$P_h = \{ p \in L^2_0(\Omega) \mid p|_K \in P_0(K), \ K \in C_h \}.$$

As macroelements we take the union of elements which all have exactly one common vertex in the interior of the macroelement, see the figure below. Let us impose the slight restriction on the mesh that every element has at least one vertex in the interior of $\Omega$. $M_h$ is then constructed by taking for each interior vertex of the mesh one macroelement with this vertex as its interior vertex.

With this the conditions M2 and M3 of Theorem 3.1 are satisfied, and it remains to check the condition M1.

Let $M$ be an arbitrary macroelement of this type and let $K_i, \ i = 1, 2, ..., \kappa,$ be the elements of $M$. The midpoints and the normals to the interior edges of $M$ we denote by $x^i$ and $n_i, \ i = 1, 2, ..., \kappa,$ respectively. $x^0$ is the vertex common to all elements of $M$. For $p \in P_M$ we let $p_i = p|_{K_i}, \ i = 1, 2, ..., \kappa.$
The degrees of freedom for \( u \in V_{0,M} \) are the values of both components of \( u \) at \( x^0 \) and the values \( u_2(x^i), \ i = 1, 2, \ldots, \kappa \). Choosing \( u \in V_{0,M} \) such that the only nonvanishing degree of freedom is \( u_2(x^i) \), the condition \( (\nabla \cdot u, p)_M = 0 \) implies that \( p_i = p_{i+1} \) (with \( p_{\kappa+1} = p_0 \)) if \( n_i \cdot e_2 \neq 0 \) where \( e_2 = (0, 1) \). Hence, the space \( N_M \) can be at most two-dimensional, and this happens only if two of the edges are parallel to \( e_2 \). But in this case one chooses \( u \) such that the only non-zero degree of freedom is \( u_1(x^0) \). The condition for \( N_M \) then forces \( p \) to be constant on the whole of \( M \).

The conditions of Theorem 3.1 are thus valid, and hence we get the error estimate

\[
\| u - u_h \|_1 + \| p - p_h \|_0 \leq Ch(|u|_2 + |p|_1).
\]

The corresponding method can naturally also be defined for a quadrilateral mesh or a mixing of triangles and quadrilaterals.

In the next examples we will consider the original Taylor-Hood methods.

**EXAMPLE 2 ([5,18]).** We again let \( C_h \) be a triangulation of \( \Omega \subset \mathbb{R}^2 \), and define

\[
V_h = \{ v \in [H_0^1(\Omega)]^2 \mid v_K \in [P_2(K)]^2, \ K \in C_h \},
\]

\[
P_h = \{ p \in C(\Omega) \cap L_0^2(\Omega) \mid p_K \in P_1(K), \ K \in C_h \}.
\]

For this method the macroelement condition is valid for a macroelement consisting of three elements. To prove this we consider an arbitrary macroelement \( M = \bigcup_{i=1}^3 K_i \) as in the figure below.

![Figure 2](image-url)
The degrees of freedom for \( u \in V_{0,M} \) are now the values of \( u \) at the midpoints \( x^{12} \) and \( x^{23} \) of the edges in the interior of \( M \). \( t_{12}, t_{23}, \) and \( n_{12}, n_{23}, \) denote the tangents and the normals, respectively, to the interior edges.

Let us choose \( u \) such that \( u(x^{12}) \cdot t_{12} = 1 \), \( u(x^{12}) \cdot n_{12} = 0 \) and \( u(x^{23}) = 0 \). Since \( \nabla p \cdot t_{12} \) is constant in \( K_1 \cup K_2 \), a simple calculation gives

\[
(\nabla \cdot u, p)_M = -u, \nabla p)_M = -\frac{1}{3} [\text{area}(K_1) + \text{area}(K_2)] (\nabla p \cdot t_{12})|_{K_1 \cup K_2}.
\]

Hence, if \( p \in N_M \), then

\[
\nabla p \cdot t_{12} = 0 \quad \text{in} \quad K_1 \cup K_2, \tag{4.1}
\]

and by the same argument

\[
\nabla p \cdot t_{23} = 0 \quad \text{in} \quad K_2 \cup K_3. \tag{4.2}
\]

In \( K_2 \) we thus have

\[
\nabla p = 0,
\]

i.e. \( p \in N_M \) is constant in \( K_2 \).

Next we choose \( u \) such that the only nonvanishing degrees of freedom are \( u(x^{12}) \cdot n_{12} \) and \( u(x^{23}) \cdot n_{23}, \) respectively. The condition for \( N_M \) then implies

\[
\nabla p \cdot n_{12} = 0 \quad \text{in} \quad K_1,
\]

and

\[
\nabla p \cdot n_{23} = 0 \quad \text{in} \quad K_3.
\]

Together with (4.1), (4.2) this shows that \( p \in N_M \) is also constant in \( K_1 \) and \( K_3 \). Since \( p \) is by definition continuous, it is a constant in the whole of \( M \). The macroelement condition is thus satisfied.

The construction of the macroelement partitioning \( \mathcal{M}_h \) is now simple; for each interior edge of \( C_h \) we take one macroelement with this edge in its interior.

The conditions of Theorem 3.1 are then valid and we get the error estimate

\[
\|u - u_h\|_1 + \|p - p_h\|_0 \leq Ch^2(|u|_3 + |p|_2).
\]

We would here like to remark that our analysis shows that the error estimates are valid for an arbitrary mesh \( C_h \). Hence, the restriction imposed in [5,18], that every \( K \in C_h \) has at least one vertex in the interior of \( \Omega \), is unnecessary.
EXAMPLE 3 ([5,15]). $C_h$ is now defined to be a partitioning of $\Omega$ into convex quadrilaterals and the finite element spaces are defined as

$$V_h = \{ \mathbf{v} \in [H_0^1(\Omega)]^2 \mid \mathbf{v}|_K \in [Q_2(K)]^2, \; K \in C_h \},$$

$$P_h = \{ p \in C(\Omega) \cap L_0^2(\Omega) \mid p|_K \in Q_1(K), \; K \in C_h \}.$$

For this method the macroelement condition is valid for a macroelement consisting of two elements.
To prove this we consider a macroelement $M = K_1 \cup K_2$ and the corresponding reference macroelement $\hat{M} = \hat{K}_1 \cup \hat{K}_2$ as in the figure below.

Figure 3

Let $F = (F_1, F_2)$ be the continuous piecewise bilinear mapping from $\hat{M}$ onto $M$. Defining $\hat{u}(\hat{x}) = u(F(\hat{x}))$ and $\hat{p}(\hat{x}) = p(F(\hat{x}))$ we can write

$$(\nabla \cdot u, p)_M = -(u, \nabla p)_M$$

$$= -\sum_{i=1}^2 \int_{K_i} \hat{u}(\hat{x})^T J_F^{-T}(\hat{x}) \nabla \hat{p}(\hat{x}) |J_F(\hat{x})| \, d\hat{x},$$

for $u \in \mathbf{V}_{0,M}$ and $p \in P_M$. Here $J_F$ is the Jacobian matrix of $F$, $J_F^{-T}$ is the transpose of $J_F^{-1}$ and $|J_F|$ denotes the determinant of $J_F$. $\hat{u}(\hat{x})$ and $\nabla \hat{p}(\hat{x})$ are considered as column vectors.

Since

$$|J_F(\hat{x})| J_F^{-T}(\hat{x}) = \begin{pmatrix} \partial_2 F_2(\hat{x}) & -\partial_1 F_2(\hat{x}) \\ -\partial_2 F_1(\hat{x}) & \partial_1 F_1(\hat{x}) \end{pmatrix},$$

and $F_1$ and $F_2$ are bilinear, we have

$$\left[ |J_F(\hat{x})| J_F^{-T}(\hat{x}) \nabla \hat{p}(\hat{x}) \right]_{\hat{K}_i} \in [Q_1(\hat{K}_i)]^2, \quad i = 1, 2.$$}

This gives

$$\left[ \hat{u}(\hat{x})^T J_F^{-T}(\hat{x}) \nabla \hat{p}(\hat{x}) |J_F(\hat{x})| \right]_{\hat{K}_i} \in Q_3(\hat{K}_i), \quad i = 1, 2,$$
and hence the composite Simpson rule gives the exact values for the integrals

\[ \int_{K_i} \hat{u}(\hat{x})^T J_F^{-1}(\hat{x}) \nabla \hat{p}(\hat{x}) | J_F(\hat{x}) | d\hat{x}, \quad i = 1, 2. \]

Let us now choose \( u \in V_{0,M} \) such that the only non-vanishing degrees of freedom are the values of both components at the midpoints \( x^7, \ x^9 \) of \( K_1 \) and \( K_2 \), respectively. Then using the above observation for calculating the integrals, we conclude that the condition \( (\nabla \cdot u, p)_M = 0 \) implies that (with \( x^i = F(\hat{x}^i) \))

\[ J_F^{-1}(\hat{x}^i) \nabla \hat{p}(\hat{x}^i) | J_F(\hat{x}^i) | = 0, \quad i = 7, 9. \]

Since \( |J_F(\hat{x}^i)| \neq 0, \ i = 7, 9 \), this shows that

\[ \nabla \hat{p}(\hat{x}^i) = \mathbf{0}, \quad i = 7, 9. \]

Let now \( p_i = p(x^i) = \hat{p}(\hat{x}^i), \ i = 1, 2, ..., 6 \), be the degrees of freedom for \( p \in P_M \). Then the above four condition for \( N_M \) implies that

\[ p_1 = p_3 = p_5 = a \quad \text{and} \quad p_2 = p_4 = p_6 = b, \]

where \( a \) and \( b \) are arbitrary real constants.

Next, when choosing \( u \in V_{0,M} \) such that the only non-vanishing degrees of freedom are \( u(x^8)(= \hat{u}(\hat{x}^8)) \), we conclude as before that the condition \( (\nabla \cdot u, p)_M = 0 \) implies

\[ \hat{u}(\hat{x}^8)^T \left\{ [J_F^{-1}(\hat{x}^8) \nabla \hat{p}(\hat{x}^8) | J_F(\hat{x}^8)] \big|_{\hat{K}_i} + [J_F^{-1}(\hat{x}^8) \nabla \hat{p}(\hat{x}^8) | J_F(\hat{x}^8)] \big|_{\hat{K}_j} \right\} = 0, \quad (4.3) \]

with the restriction to \( \hat{K}_i, \ i = 1, 2 \), denoting the limiting value when \( \hat{x} \to \hat{x}^8, \ \hat{x} \in \hat{K}_i \). If we now let \( u(x^8) = \hat{u}(\hat{x}^8) = x^6 x^3 \), with \( x^6, x^3 \) denoting the vector from \( x^6 \) to \( x^3 \), then we have

\[ [J_F^{-1}(\hat{x}^8) \hat{u}(\hat{x}^8)] \big|_{\hat{K}_i} = \hat{e}_2, \quad i = 1, 2, \]

with \( \hat{e}_2 = (0, 1) \). Since \( \partial_2 \hat{p} \) is continuous at \( \hat{x}^8 \), (4.3) reduces to

\[ \partial_2 \hat{p}(\hat{x}^8) [J_F(\hat{x}^8)]_{\hat{K}_1} + [J_F(\hat{x}^8)]_{\hat{K}_2} = 0. \]

This gives

\[ 0 = \partial_2 \hat{p}(\hat{x}^8) = p_3 - p_6 = a - b, \]

i.e. \( a = b \).

The macroelement condition is thus proved for a macroelement of two elements.
The partitioning $M_h$ is then obtained by taking one macroelement for each interior edge of the mesh.

We have thus proved the optimal convergence rate of the method:

$$\|u - u_h\|_1 + \|p - p_h\|_0 \leq Ch^2(|u|_3 + |p|_2).$$

Let us finally remark that the above arguments can be generalized for the whole family of quadrilateral Taylor-Hood methods

$$V_h = \{ v \in [H_0^1(\Omega)]^2 \mid v|_K \in [Q_k(K)]^2, \ K \in C_h \},$$

$$P_h = \{ p \in C(\Omega) \cap L^2_0(\Omega) \mid p|_K \in Q_{k-1}(K), \ K \in C_h \},$$

with $k \geq 2$, see Reference [17].

For some further applications of our technique we refer to [14,15,16,17].

REFERENCES

9. P. CLEMÉNT. Approximation by finite elements using local regularization. RAIRO Sér. Rouge 9 (1975) 77-84

10. M. CROUZEIX, P.A. RAVIART. Conforming and nonconforming finite element methods for the stationary Stokes equations. RAIRO R3 (1973) 33-76


