Impacts of CFO, IQ Imbalance and Phase Noise on the system performance in OFDM systems

Rong Zhang*, Edward K. S. Au#, and Roger S. Cheng§

*Wireless IC System Design Center, HKUST Fok Ying Tung Graduate School, Nansha, China
#Modulation and Coding Department, Institute for Infocomm Research, A*STAR, Singapore
§Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology, Hong Kong SAR
Email: tyrzhr@ust.hk, ksau@i2r.a-star.edu.sg, eecheng@ust.hk

Abstract—OFDM systems are sensitive to front-end imperfections including CFO, IQ imbalance and phase noise. In practice, these imperfections are inevitable and would lead to significant performance degradation. In this paper, a general closed-form expression on average signal-to-interference-and-noise ratio (SINR) over multipath fading channels is derived for performance evaluation. When additive white Gaussian noise channels and flat fading channels are considered, the expression reduces to an exact expression and a tight upper bound, respectively. Our results show that the average SINR is inversely proportional to these front-end non-idealities.

I. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) has been considered as an effective modulation scheme and widely applied in many communication systems, such as wireless local area network and digital video broadcasting [1,2]. Unfortunately, it is well-known that OFDM is very sensitive to front-end non-idealities, in particular, carrier frequency offset (CFO), IQ imbalance, and phase noise. These imperfections are mainly caused by the fabrication process and channel variation that are hard to control, and they would cause serious performance degradation.

Conventional OFDM systems generally employ a super-heterodyne architecture in which the up/down converters operate in a digital domain, so that the IQ modulation/demodulation can be perfectly performed. In order to reduce the number of components and hence lower the cost, an alternative to the super-heterodyne architecture is a zero-intermediate frequency (Zero-IF) architecture in which the RF signal is directly converted to baseband, and vice versa, in the analog domain [3]. While this approach has the advantage of reduced computational complexity, its major drawback is the existence of the amplitude and phase mismatches between in-phase (I) and quadrature (Q) branches [3,4]. These mismatches not only attenuate the desired signal, but also introduce inter-carrier interference (ICI) to the other sub-carriers and amplify the noise.

For the CFO, it is mainly caused by the Doppler shift and/or the discrepancy of Local Oscillator (LO) frequencies at both the transmitter and the receiver. In practice, the offset value can be divided into an integer part and a fractional part. While the former only results in a cyclic shift of sub-carriers, the latter destroys the orthogonality among different sub-carriers, resulting in ICI [5,6].

Similarly, phase noise refers to the difference between the phase of a carrier signal and that of the LO, and it is characterized as a Gaussian random process with a constant mean and a variance that is a linear function of time and oscillator linewidth. Similar to CFO, phase noise causes DFT leakage, and its distortion to the system can be characterized by two different impairments, namely a common phase error (CPE) which refers to the same phase shift for all sub-carriers, and an ICI term in which the phase offset varies from each sub-carrier to the others [7,8].

In this paper, we analyzed the effect of these common front-end errors in a coded OFDM system, and derived an approximate closed-form expression on average SINR. In case of a flat fading channel and an additive white Gaussian noise (AWGN) channel, the expression reduces to a tight upper bound and an exact expression, respectively. Unlike the other papers that consider only one of these three RF impairments [6,8-11] for the SNR analysis, we consider not only all of these impairments, but also take into account the impact of the code rate. Monte Carlo simulations suggest that the results are tight enough to provide useful insight for system design.

II. SIGNAL MODEL WITH NON-IDEALITIES

Let \( N \) be the number of sub-carriers and \( X_n(k) \) be the \( k \)-th sub-carrier of the \( m \)-th OFDM symbol, where \( k=0, 1, \ldots, N-1 \). At the transmitter, the data is serial-to-parallel converted, coded with convolutional code, modulated, and fed-into an \( N \)-point inverse DFT that outputs
\[
x_m(n)=\frac{1}{N}\sum_{k=0}^{N-1} X_n(k) \exp\left(\frac{j2\pi k(n-N_y)}{N}\right), \quad n = 0, 1, \ldots, N+N_y-1,
\]
where \( N_y \) is the cyclic-prefix length in the number of samples. Throughout this paper, we assume statistical independence...
among different kinds of signals, channel responses, and front-end non-idealities.

With the transmit (TX) IQ imbalance, $x_{op}(n)$ suffers from an amplitude mismatch $\delta$ between its I and Q branches and a phase orthogonality mismatch $\theta$ [12]. Denote $\alpha$ as the carrier frequency. The resultant complex time-domain signal in the presence of these mismatches is given by

$$x_{im}(n) = (1 + \delta) \cdot \mathbb{R}[x_{r}(n)] \cdot \cos \left( \frac{\theta}{2} \right) + (1 - \delta) \cdot \mathbb{I}[x_{r}(n)] \cdot \sin \left( \frac{\theta}{2} \right)$$

where $\mathbb{R} \{ \cdot \}$ and $\mathbb{I} \{ \cdot \}$ refer to the real and imaginary parts, respectively. The corresponding baseband signal can be obtained by low-pass filtering and is given by

$$\tilde{x}_{im}(n) = (1 + \delta) \cdot \mathbb{R}[x_{im}(n)] \cdot \cos \left( \frac{\theta}{2} \right) + (1 - \delta) \cdot \mathbb{I}[x_{im}(n)] \cdot \sin \left( \frac{\theta}{2} \right) + j(1 + \delta) \cdot \mathbb{R}[x_{im}(n)] \cdot \sin \left( \frac{\theta}{2} \right) + j(1 - \delta) \cdot \mathbb{I}[x_{im}(n)] \cdot \cos \left( \frac{\theta}{2} \right).$$

For the ease of description, we denote $\alpha = \cos(\theta/2) + j/\sin(\theta/2)$ such that (1) is simplified as

$$\tilde{x}_{w}(n) = \alpha x_{w}(n) + \beta x_{r}(n)$$

with $\gamma$ being the complex conjugate operator. From (2), it is clear that the TX IQ imbalance not only attenuates the desired signal $x_{w}(n)$ by a factor of $\alpha$, but also creates a scaled ICI term $\beta x_{r}(n)$ that is an image signal based on the complex conjugate of the desired signal.

Denote $T$ as the sampling period. The channel $h = [h_0, h_1, \ldots, h_N]$ is defined as $T$-spaced complex gains, which are assumed to be zero-mean complex Gaussian random variables and are independent to each other. For the $m$-th OFDM symbol, we follow [6,13] and assume that the channel responses $h_n$ and $h_m$, where $n \neq m$, are uncorrelated with one another, such that the power of each sub-carrier channel gain $H_{im}(k)$ is expressed as

$$E\left[|H_{im}(k)|^2\right] = E \left[ \sum_{n=0}^{N-1} h_n[k] e^{j2\pi nk/N} \right]^2 = \sum_{n=0}^{N-1} |h_n|^2$$

Consider a channel that is stationary during each OFDM symbol, the received signal $y_{im}(n)$ is expressed as

$$y_{im}(n) = h_{im}(n) \cdot \tilde{x}_{w}(n) + z_{im}(n)$$

where () denotes the convolution operation and $z_{im}(n)$ is a zero-mean AWGN with variance $\sigma^2$.

In the presence of carrier frequency offset $\Delta f$, the received signal $h_k(n) \cdot \tilde{x}_{w}(n)$ is mixed with a LO signal which is $\Delta f$ above the carrier frequency $f_c$, i.e. $h_k(n) \cdot \tilde{x}_{w}(n) \cdot \exp(j2\pi nkf_{\Delta f})^1$. Define $\delta = \Delta f T$ as the normalized frequency offset and $\Delta = N/\delta$ being the ratio of the length of cyclic prefix and that of the data samples. The received signal in the frequency domain can be written as

$$y_{im}(n) = Nc_{m}(e,n) \cdot [h_{im}(n) \cdot \tilde{x}_{w}(n)] + z_{im}(n)$$

where $c_{m}(e,n) = (1/N) \cdot \exp(j2\pi neN/N) \cdot \exp(j2\pi nk(\Delta f)$. It varies with the normalized CFO $e$ and the number of sub-carriers $N$.

In contrast to the CFO whose value can be modeled as an arbitrary number, phase noise is interpreted as a random phase modulation in the signal of the local oscillator such that each sub-carrier is acquired with different phase offset $\omega_{im}(n)$. As indicated in [13, 14], the phase noise on the $n$-th sample of the $m$-th OFDM symbol $\omega_{im}(n)$ is given by the following recursive equation

$$\omega_{im}(n) = \omega_{im,1}(N-1) + \sum_{u=0}^{n-1} u \cdot \left[ m(N + N_g) + 1 \right] \tag{4}$$

where $u$ is a zero-mean Gaussian random variable with variance $\sigma_{\omega}^2 = 2\pi T_r/\Delta$, with $T_r$ being the symbol duration and $\nu$ being the oscillator linewidth. Hence, the received signal (3) is further re-expressed as

$$\tilde{y}_{m}(n) = Nc_{m}(e,\alpha,n) \cdot [h_{im}(n) \cdot \tilde{x}_{w}(n)] + z_{im}(n)$$

where $\tilde{c}_{m}(e,\alpha,n) = c_{m}(e,n) \cdot \exp(j\omega_{im}(n))$.

When the signal is down-converted from RF to baseband using the Zero-IF architecture [3], the hardware inaccuracy would cause the receive (RX) IQ imbalance in I and Q branches. Such effect is modeled by an amplitude mismatch $\alpha$ and a phase orthogonality mismatch $\theta$, and the resultant complex time-domain signal is expressed as

$$\tilde{y}_{m}(n) = \alpha \cdot \tilde{y}_{m}(n) + \beta \cdot \tilde{y}_{m}(n)$$

where $\alpha = \cos(\theta/2) - j\sin(\theta/2)$ and $\beta = \delta \cos(\theta/2) + j\sin(\theta/2)$. Similar to the TX counterpart, the RX IQ imbalance attenuates the desired received signal by a factor of $\alpha$, and create a scaled ICI term $\beta \cdot \tilde{y}_{m}(n)$ which is the image sub-carrier of $\tilde{y}_{m}(n)$.

Substituting (2), (3) and (5) into (6) and performing DFT yields the following frequency-domain received signal

$$F_{m}(k) = [\alpha \cdot c_{m}(e,\alpha,0) \cdot H_{im}(k) + \beta \cdot G_{im}(e,\alpha,0) \cdot H_{im}^*(k)] \cdot X_{m}(k)$$

$$= \sum_{m=0}^{N-1} [\alpha \cdot c_{m}(e,\alpha,0) \cdot H_{im}(k) + \beta \cdot G_{im}(e,\alpha,0) \cdot H_{im}^*(k)] \cdot X_{m}(k)$$

$$= \alpha \cdot Z_{im}(k) + \beta \cdot Z_{im}^*(k)$$

(7)

where $\alpha, \beta, c_{m}(e,\alpha,0), h_{im}(n), h_{m}(n)$ and $z_{im}(n)$, respectively, with $c_{m}(e,\alpha,0)$ being expressed as (8), and $r = 1,2,\ldots,N$ is the sub-carrier index.

From (7), it is found that the interference of a sub-carrier mainly comes from (i) the power leakage from neighboring sub-carriers due to the presence of the CFO, phase noise, and (ii) the mirror sub-carrier due to the TX and RX IQ imbalance. In summary, the desired signal of the $k$-th subcarrier $H_{im}(k)$ $X_{im}(k)$ is affected by these non-idealities in the following way.

- **Degraded desired signal.** The magnitude of the desired signal is seriously attenuated by a factor of $|\alpha \cdot c_{m}(e,\alpha,0)|$, which varies with the normalized CFO $e$, the phase noise $\omega_{im}(n)$, and both the TX and RX IQ imbalances $\alpha, \alpha$. Interestingly, there is an additional term with magnitude $|\beta \cdot \beta^* \cdot G_{im}(e,\alpha,0)|$ due to the image
\[
C_m(\epsilon, \omega, k) = \frac{1}{N} \exp \left( j 2 \pi m (1 + \alpha) \sum_{r=0}^{N-1} \sin \left( \frac{\pi (r - \epsilon)}{N} \right) \exp \left( j \pi (r - \epsilon) \left( \frac{1 - 1}{N} \right) \right) \exp \left( j \omega_m (n) - 2 \pi (k - r) n / N \right) \right)
\]

\[
\Xi(\epsilon, \omega) = \frac{1}{N} \sum_{r=0}^{N-1} \sin \left( \frac{\pi (r - \epsilon)}{N} \right) \sum_{r=0}^{N-1} \sin \left( \frac{\pi (r - \epsilon)}{N} \right) \exp \left( j \pi (r - \epsilon) \left( \frac{1 - 1}{N} \right) \right) \sum_{n=0}^{N-1} \exp \left( - j \pi \n \right) \exp \left( j \frac{2 \pi}{N} (n - n_r) \right)
\]

The average SINR of the \( k \)-th sub-carrier can be approximated by the following closed-form expression

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**A. General SINR Analysis**

**Proposition 1:** The average SINR of the \( k \)-th sub-carrier of an OFDM symbol is approximated by the following closed-form expression

\[
\text{SINR}(k) = \frac{\left[ \alpha^2 \cdot S_r \cdot \sigma_r^2 + \beta \cdot \alpha^2 \cdot C_r \cdot S_r \cdot \sigma_r^2 + \alpha \cdot \beta \cdot \alpha^2 \cdot C_r \cdot S_r \cdot \sigma_r^2 + \beta \cdot \alpha \cdot \beta \cdot \alpha^2 \cdot C_r \cdot S_r \cdot \sigma_r^2 + \beta \cdot \alpha \cdot \beta \cdot \beta \cdot \alpha^2 \cdot C_r \cdot S_r \cdot \sigma_r^2 + (1 + \delta^2) \sigma_r^2 \right]}{\left( \left( \Xi(\epsilon, \omega) \cdot \gamma \cdot \sigma_r^2 + \beta \cdot \beta \cdot \gamma \cdot \sigma_r^2 \right) \cdot (1 - \Xi(\epsilon, \omega)) \cdot \gamma \cdot \sigma_r^2 + \alpha \cdot \beta \cdot \gamma \cdot \sigma_r^2 + \beta \cdot \alpha \cdot \beta \cdot \gamma \cdot \sigma_r^2 + (1 + \delta^2) \sigma_r^2 \right)}
\]

where \( M \) is the modulation level, \( R \) is the code rate, \( \text{SNR}_S \) is the average SINR per symbol, \( \Psi(\alpha, \beta, \beta, \beta) \) is a function of TX and RX IQ imbalances in (10) and \( \Xi(\epsilon, \omega) \) is a function of the normalized CFO, phase noise, and the total number of sub-carriers given in (11).

\[
\Psi(\alpha, \beta, \beta, \beta) = \frac{(1 + \delta^2) (1 + \delta^2)}{\left[ \alpha \cdot \beta \cdot \gamma \cdot \sigma_r^2 \right]}
\]

**Proof of Proposition 1:**

For notational convenience, denote

\[
S_r = \sum_{r=0}^{N-1} C_r (\epsilon, \omega, r) H_m (k - r)
\]

The instantaneous SINR of the \( k \)-th sub-carrier can be computed from (7) and it is shown in (12), with \( \sigma_r^2 \) being the power of the transmitted signal. Then, the average SINR can be calculated by using the probability density functions (PDFs) of channel gains \( p_u(v) \) and phase noise \( p_o(v) \). However, such a computation involves multiple integrations which are too complex to provide any meaningful insight. Alternatively, we follow [6] and consider taking the expectations of the denominator and the numerator separately with respect to the PDFs. Due to the mutual independence of different kinds of signals, channel responses and front-end non-idealities as assumed in Section II, and take into account the fact that

\[
E \left\{ \sum_{r=0}^{N-1} C_r (\epsilon, \omega, r) \right\} = E \left\{ N \sum_{n=0}^{N-1} \Xi(\epsilon, \omega, n) \right\} = 1
\]

the resultant SINR expression is given in (13).
The expression of the average SINR (13) can be further simplified by defining $\psi(\alpha, \alpha, \beta, \beta)$ as in (10) and the average SNR per sub-carrier as

$$SNR_s = \frac{SNR_r}{R \cdot \log_2 M} = \frac{\sigma_s^2}{\sigma_f^2}$$

(14)

where $SNR_s$ is the SNR per symbol, $R$ is the convolutional code rate and $M$ is the modulation level. The approximate closed-form expression (9) is finally obtained by substituting (10) and (14) into (13).

**B. SINR Analysis for two special channel model**

For the special case of flat fading channels, the channel gain is the same for all sub-carriers and it is given by $H[k] = h[0]$, where $k = 0, 1, \ldots, N-1$. Assume that channel gain is $\gamma$, the instantaneous SINR becomes

$$SINR = \frac{\gamma^2 SNR_s}{R \log_2 M} \left( \frac{\psi(\alpha, \alpha, \beta, \beta)}{\Xi(\varepsilon, \omega)} \right)^{-1} \left( \frac{\Psi(\alpha, \alpha, \beta, \beta)}{R \log_2 M} \right) + \frac{1}{1+\delta^2} \frac{\psi(\alpha, \alpha, \beta, \beta)}{\Xi(\varepsilon, \omega)}$$

(15)

The approximate SINR can then be simplified by using Jensen’s inequality and is given as follows.

**Corollary 1:** In a flat fading channel, the average SINR of a coded OFDM system in presence of RF impairments is upper-bounded by

$$\overline{SINR} \leq \frac{\gamma^2 SNR_s}{R \log_2 M} \left( \frac{\psi(\alpha, \alpha, \beta, \beta)}{\Xi(\varepsilon, \omega)} \right)^{-1} \left( \frac{\Psi(\alpha, \alpha, \beta, \beta)}{R \log_2 M} \right) + \frac{1}{1+\delta^2} \frac{\psi(\alpha, \alpha, \beta, \beta)}{\Xi(\varepsilon, \omega)}$$

(16)

From (16), we can observe that the approximate SINR under a general multipath fading channel is an upper bound of the average SINR in a flat-fading channel, which is coherent with the conclusion in [6] in which only CFO is considered.

In case of an AWGN channel, the channel gains of all sub-carriers equal unity and the approximate closed-form expression (9) is reduced to the following exact expression.

**Corollary 2:** In an AWGN channel, the average SINR of a coded OFDM system in presence of front-end non-idealities is expressed as

$$\overline{SINR} = \frac{SNR_s}{R \log_2 M} \left( \frac{\psi(\alpha, \alpha, \beta, \beta)}{\Xi(\varepsilon, \omega)} \right)^{-1} \left( \frac{\Psi(\alpha, \alpha, \beta, \beta)}{R \log_2 M} \right) + \frac{1}{1+\delta^2} \frac{\psi(\alpha, \alpha, \beta, \beta)}{\Xi(\varepsilon, \omega)}$$

(17)

From the closed-form SINR expressions derived in Theorem 1 and its Corollaries above, we can observe that the average SINR is inversely proportional to the ratio of the parameters $\psi(\alpha, \alpha, \beta, \beta)$ and $\Xi(\varepsilon, \omega)$, which are the functions of all the RF impairments.

**IV. NUMERICAL RESULTS**

Monte Carlo simulations are provided to assess the accuracy and the tightness of the approximate closed-form expressions on the average SINR derived in the previous section. Unless otherwise stated, we consider a convolutional code with code rates of $R=1\/3$, two different modulation levels (4QAM and 16QAM), and $N = 128$ sub-carriers.

As the first example, we look at the impact of the modulation order on the average SINR and verify the tightness of the closed-form expression (9) with Monte Carlo simulations. In Figure 1, we consider the performance of an un-coded OFDM system in a flat fading channel and the configurations of the front-end non-idealities are set to $(\theta, \delta, \varepsilon, \nu) = (\pi/12, 0.05, 0.05, 10^3)$. Recall the approximate closed-form SINR is reduced to an upper bound (16) in the flat fading channel. Referring to the figure, we observe that the SINR decreases with the modulation order as explained in Proposition 1. In addition, it is coherent with the derivation in Corollary 1 that an upper bound on SINR exists at large SNRs.

Next, the effect of different levels of front-end non-idealities is investigated by using a 4QAM-modulated system in an AWGN channel as an illustrative example. Two different scenarios are considered, namely $(\theta, \delta, \varepsilon, \nu) \in \{(\pi/32, 0.05, 0.05, 10^3), (\pi/16, 0.1, 0.1, 10^3)\}$. Referring to Figure 2, one can observe that the theoretical expression (17) matches exactly with the Monte Carlo simulations in AWGN channel. In addition, it is found that a small change in the values of these RF impairments would lead to a significant degradation in the average SINR. For example, the average SINR decreases from 11 dB to 7 dB at $SNR_s = 20$ dB. Lastly, we also show in the same figure that the average SINR performance is significantly improved as the code rate increases.

Finally, Figures 3 and 4 show the performance of a coded 4QAM modulated system in two multipath frequency selective fading channels, namely the channel models 1 and 2 (a.k.a. CM1 and CM2) adopted in IEEE 802.15.3a [15] that represent the LOS and NLOS channels, respectively, with distances ranging from 0 to 4 meters. Following the same scenarios considered in Figure 4, it is observed from the figures that the analytical expressions derived for general multipath channels are tight enough (the greatest bias is less than 0.7 dB and 0.8 dB in CM1 and CM2, respectively) to provide useful insight on the system performance.

**V. CONCLUSION**

We have analyzed the average SINR of a coded OFDM system in presence of carrier frequency offset, phase noise and IQ imbalance. In particular, an approximate closed-form expression of the average SINR over multipath fading channels is derived, which can be reduced to an exact expression and a tight upper bound when AWGN and flat fading channels are considered, respectively. These results were used to provide insights regarding the effects of the CFO, the amplitude and phase mismatches of IQ imbalances, and the oscillator linewidth of the phase noise on the system performance. Our results have shown that the average SINR is inversely proportional to these front-end non-idealities, and the average SINR over multipath fading channels is actually an upper bound in flat fading channels.
REFERENCES


Figure 1. Impact of 4QAM and 16QAM in flat fading channel; ($\theta$, $\delta$, $\varepsilon$, $\nu$) = ($\pi$/32, 0.05, 0.05, 10$^4$).

Figure 2. Impact of different levels of front-end non-idealities on the average SINR in an AWGN channel; 4QAM, ($\theta$, $\delta$, $\varepsilon$, $\nu$) $\in$ {($\pi$/32, 0.05, 0.05, 10$^3$), ($\pi$/16, 0.1, 0.1, 10$^4$)}.

Figure 3. Impact of different levels of front-end non-idealities on the SINR in IEEE 802.15.3a CM1; 4QAM, code rate of 1/3, ($\theta$, $\delta$, $\varepsilon$, $\nu$) $\in$ {($\pi$/32, 0.05, 0.05, 10$^3$), ($\pi$/16, 0.1, 0.1, 10$^4$)}.

Figure 4. Impact of different levels of front-end non-idealities on the SINR in IEEE 802.15.3a CM2; 4QAM, code rate of 1/3, ($\theta$, $\delta$, $\varepsilon$, $\nu$) $\in$ {($\pi$/32, 0.05, 0.05, 10$^3$), ($\pi$/16, 0.1, 0.1, 10$^4$)}.