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Abstract. In this work we are study the Fuzzy Initial Value Problem (FIVP) with parameters and/or initial conditions given by fuzzy sets. Starting from the flow equation of the deterministic Initial Value Problem (IVP) associates to FIVP, we obtain the FIVP flow, through the principle of Zadeh. Follow, we introduce the concept of fuzzy equilibrium stability of FIVP and some examples are given.

Keywords: Fuzzy subset; Flow; Zadeh’s extension; stability.

1. Introduction

Consider the Initial Value Problem (IVP)

\[
\begin{align*}
x'(t) &= f(x(t)) \\
x(0) &= x_0.
\end{align*}
\]

One can suppose that the initial condition and/or parameters in (1) are uncertain and given by fuzzy subsets. Thus (1) becomes the Fuzzy Initial Value Problem (FIVP):

\[
\begin{align*}
x'(t) &\in \hat{f}(x, K) \\
x(0) &\in X_0,
\end{align*}
\]

where $K$ and $X_0$ are fuzzy subsets and $\hat{f}$ is Zadeh’s extension of $f$. 

According to Hüllermeier, (2) can be interpreted by the following family of differential inclusions:

\[
\begin{cases}
  x'(t) \in \hat{f}(x,K)\alpha \\
  x_0 \in [X_0]\alpha,
\end{cases}
\]

where \([\hat{f}(x,K)]\alpha\) and \([X_0]\alpha\) are the \(\alpha\)-levels of the fuzzy subsets \(\hat{f}(x,K)\) and \(X_0\), respectively, whose definition is given in the Section 2.

According to Mizukoshi et. al, for each \(t > 0\), the solution obtained under the Hüllermeier’s interpretation of (2), coincides with that obtained by finding the crisp solution of (1) and then fuzzifying this solution via the Zadeh extension principle.

In section 3, we define fuzzy flow and fuzzy equilibrium point for (2) using the extended deterministic flow. In other words, we obtain the fuzzy flow by applying Zadeh’s extension principle to the deterministic flow.

In Section 4 we establish a result relating the stability of the equilibrium points \(\bar{x}\) of (1) and the stability of \(\chi\{\bar{x}\}\) (characteristic of \(\bar{x}\)), the fuzzy equilibrium points of (2). We also present some results that relates attractors points of (1) and the fuzzy asymptotically stable equilibrium of (2).

Finally, we apply the studied methods to the malthusian and logistic models with the initial condition being a fuzzy set, then with a sharp initial condition and coefficient and a fuzzy coefficient and, at last both the coefficient and initial condition as a fuzzy sets.

2. Preliminaries

First of all, let us establish some notation and recall known results.

**Definition 1.** Let \(X\) be a complete metric space and \(\mathbb{R}_+ = [0, \infty)\). A family of mappings \(T_t : X \rightarrow X, t \geq 0\), is a semigroup, if:

(i) \(T_0 = I\), where \(I\) is the identity mapping on \(X\);

(ii) \(T_{t+s} = T_t \circ T_s, t, s \in \mathbb{R}_+\), where “\(\circ\)” is the composition operation.

A fuzzy subset \(U\) of \(\mathbb{R}^n\) is defined as a set of ordered couples \((x, \mu_U(x))\), \(x \in \mathbb{R}^n\) where \(\mu_U : \mathbb{R}^n \rightarrow [0, 1]\) is called membership function of \(U\). The number \(\mu_U(x)\) indicates the degree of \(x\) in \(U\). The membership values 0 and 1 represent, respectively, the non pertinence and the maximum pertinence of \(x\) in the fuzzy set \(U\).

To simplify the notation we will indicate the membership function \(\mu_U\) by \(U\).

For \(0 < \alpha \leq 1\) we will denote by \([U]_{\alpha} = \{x \in \mathbb{R}^n / U(x) \geq \alpha\}\) the \(\alpha\)-level of \(U\) and \([U]_0 = \text{supp}U = \{x \in \mathbb{R}^n / U(x) > 0\}\), the support of \(U\).

We denote by \(\mathcal{F}(\mathbb{R}^n)\) the space of all nonempty compact fuzzy subsets of \(\mathbb{R}^n\) and \(\mathcal{E}^n\) is a subset of \(\mathcal{F}(\mathbb{R}^n)\), whose \(\alpha\)-level is a convex set in \(\mathbb{R}^n\).

The addition and scalar product on \(\mathcal{F}(\mathbb{R}^n)\) are defined by

\[
(U + V)(x) = \sup_{y \in X} \{U(y) \wedge V(x - y)\} \quad \text{and} \quad (\lambda U)(x) = \begin{cases} U(\frac{x}{\lambda}) & \text{if } \lambda \neq 0 \\ \chi_{\{0\}} & \text{if } \lambda = 0, \end{cases}
\]
where \( \chi_{\{0\}} \) is the characteristic function of the zero.

It is well known\(^3,4\) that the following operations are true for all \( \alpha \)-levels

\[
[U + V]^{\alpha} = [U]^{\alpha} + [V]^{\alpha} \quad \text{and} \quad [\lambda U]^{\alpha} = \lambda[U]^{\alpha}, \quad \forall \alpha \in [0, 1] \quad \text{and} \quad \lambda \in \mathbb{R}.
\]

The metric on \( F(\mathbb{R}^n) \) is given by

\[
D(U, V) = \sup_{0 \leq \alpha \leq 1} h([U]^{\alpha}, [V]^{\alpha}),
\]

where \( h \) is the usual Hausdorff metric defined on compact subsets of \( \mathbb{R}^n \). We have that \( D(U, V) \) measures the largest difference in the membership degree of the two fuzzy sets \( U, V \in \mathcal{E}^n \) over all points \( x \) in the base space \( \mathbb{R}^n \).

Zadeh\(^6\) proposed the so called extension principle\(^7\) which became an important tool in fuzzy set theory. The idea is that each function \( f : X \rightarrow Y \) induces a corresponding function \( \hat{f} : F(X) \rightarrow F(Y) \) (i.e., \( \hat{f} \) is a function mapping fuzzy sets in \( X \) to fuzzy sets in \( Y \)) defined for each fuzzy set \( U \) in \( X \) by

\[
\hat{f}(U)(y) = \begin{cases} 
\sup_{u \in f^{-1}(y)} U(u) & \text{if} \ f^{-1}(y) \neq \emptyset \\
0 & \text{if} \ f^{-1}(y) = \emptyset.
\end{cases}
\]

(7)

The function \( \hat{f} \) is said to be obtained from \( f \) by the extension principle.

We finish this section with a result that characterizes the image levels of a fuzzy subset through \( \hat{f} \), where \( f \) is a continuous function.

**Theorem 1.**\(^7\) If \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous, then the Zadeh extension \( \hat{f} : F(\mathbb{R}^n) \rightarrow F(\mathbb{R}^n) \) is well-defined and

\[
[\hat{f}(U)]^{\alpha} = f([U]^{\alpha}), \forall \alpha \in [0, 1].
\]

(8)

Note that (8) continues to be valid for \( f : W \rightarrow \mathbb{R}^n \), where \( W \) is an open subset in \( \mathbb{R}^n \). Moreover, according with Román-Flores et al\(^3\) it is possible to prove that \( \hat{f} \) is also continuous in Hausdorff metric extended \( D \).

**Definition 2.**\(^4\) The cartesian product of two fuzzy subsets \( U \in F(X) \) and \( V \in F(Y) \) is defined by

\[
(U \times V)(u, v) = \min\{U(u), V(v)\}, (u, v) \in X \times Y.
\]

\section*{3. Fuzzy Dynamic Systems}

Consider the nonlinear system

\[
\begin{aligned}
x'(t) &= f(x(t)) \\
x(0) &= x_0,
\end{aligned}
\]

(9)

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function. Let \( \varphi_t(x_0) \) be the unique solution of (9) for each \( x_0 \) in time \( t \), defined on its maximal interval of existence \( I(x_0) \). For each \( t \in I(x_0) \), the family of mappings \( \varphi_t : U \rightarrow U \), defined by
ϕₜ(𝐱₀) = ϕ(𝐭, 𝐱₀),
satisfies the definition of semigroup (see Proposition 1) and it’s usually called flow⁸,⁹ of the differential equation (9).
Next, consider the Initial Value Problem
\[
\begin{cases}
y′(t) = f(y(t), k) \\
y(0) = y₀,
\end{cases}
\] (10)
where \( f: \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n \) is a continuous function, \( y₀ \in \mathbb{R}^n \), \( k \in \Omega \subset \mathbb{R}^p \), and \( t \in (0, \infty) \). The IVP (10) can be rewritten as
\[
\begin{cases}
x′(t) = F(x) = f((y, k), 0) \\
x(0) = x₀,
\end{cases}
\] (11)
where \( x′(t) = (y′(t), 0), x(0) = x₀ = (y(0), k) \in \mathbb{R}^n \times \mathbb{R}^p \) and \( x \) should be defined on \([0, T]\).

Then, supposing that the initial condition and the parameter \( k \) are given by fuzzy sets. We have the FIVP
\[
\begin{cases}
x′(t) ∈ F(x) \\
x(0) ∈ X₀,
\end{cases}
\] (12)
where \( X₀ = (Y₀, K) \) is a fuzzy set.

A fuzzy autonomous system, with fuzzy parameters and initial condition, can be treated as a system in which all the fuzziness lies only the initial condition².

In Mizukoshi et al² it was verified that for each \( t > 0 \), the solution of (12) obtained through a family of differential inclusion, coincides with the solution obtained through the extension of the deterministic solution of (10).

Hence, the fuzzy solution of (12) in this paper is obtained by Zadeh extension principle, more specifically through the extended deterministic flow. Besides, we will denote by \( \hat{ϕ}_t(X₀) \) the fuzzy solution of (12).

Below we verify that \( \hat{ϕ}_t \), the Zadeh’s extension of \( ϕ_t \), satisfies the properties of a semigroup in \( \mathcal{F}(W) \), where \( W \subseteq \mathbb{R}^n \) is an open set.

**Proposition 1.** Let \( ϕₜ(x₀) \) be the flow associated to the deterministic problem (10). Then the Zadeh extension \( \hat{ϕ}_t: \mathcal{F}(U) \rightarrow \mathcal{F}(U) \) is a flow in \( \mathcal{F}(W) \), that is, \( \hat{ϕ}_t \) satisfies the following properties:

\begin{itemize}
  \item \( \hat{ϕ}_0(X₀) = X₀; \)
  \item \( \hat{ϕ}_{t+s}(X₀) = (\hat{ϕ}_t \circ \hat{ϕ}_s)(X₀), \) for \( t, s \in \mathbb{R}_+, X₀ \in \mathcal{F}(W). \)
\end{itemize}

For \( α \in [0, 1] \), we have

\begin{itemize}
  \item (i) \( [\hat{ϕ}_0(X₀)]^α = [X₀]^α; \)
  \item (ii) \( [\hat{ϕ}_{t+s}(X₀)]^α = [(\hat{ϕ}_t \circ \hat{ϕ}_s)(X₀)]^α, \) for \( t, s \in \mathbb{R}_+, X₀ \in \mathcal{F}(W). \)
\end{itemize}

**Proof.**
This result can be proved using the \( α \)-level definition.
Then both (i) and (ii) are consequence of Theorem 1 applied to the flow $\varphi_t$.

(i) $\hat{\varphi}_0(X_0)^\alpha = \varphi_0([X_0]^\alpha) = [X_0]^\alpha$, since that $\varphi_t(x_0) = x_0, \forall x_0 \in \text{supp}\{X_0\}$.

(ii) $\hat{\varphi}_{t+s}(X_0)^\alpha = \varphi_{t+s}([X_0]^\alpha) = \varphi_t(\varphi_s([X_0]^\alpha)) = (\varphi_t \circ \varphi_s)([X_0]^\alpha) = [\varphi_t \circ \varphi_s](X_0)^\alpha$.

We will define now an equilibrium point for the FIVP (12) through the extended flow.

**Definition 3.** One can say $\bar{X} \in \mathcal{F}(U)$ is a fuzzy equilibrium point if

$$\hat{\varphi}_t(\bar{X}) = \bar{X}, \forall t \geq 0$$

or equivalently,

$$[\hat{\varphi}_t(\bar{X})]^\alpha = [\bar{X}]^\alpha, \forall \alpha \in [0, 1].$$

**4. Stability of the equilibrium point**

The classical definitions of stable and asymptotically stable points can be found, for instance, in Hale and Koçak or Hale. In this section we define them for fuzzy differential equations.

**Definition 4.** Let $\bar{X}$ be an equilibrium point for the FIVP

$$\begin{cases} x'(t) \in f(x(t)) \\ x_0 \in X_0, \end{cases}$$

(13)

where $X_0$ is a fuzzy set in $\mathcal{F}(\mathbb{R}^n)$. Then

(i) $\bar{X}$ is stable if and only if, for every $\varepsilon > 0$, there is a $\delta > 0$, such that, if then $X \in \mathcal{F}(\mathbb{R}^n)$,

$$D(\hat{\varphi}_t(X), \bar{X}) = \sup_{0 \leq \alpha \leq 1} h([\hat{\varphi}_t(X)]^\alpha, [\bar{X}]^\alpha) < \delta, \quad \forall t \geq 0.$$  

(ii) $\bar{X}$ is asymptotically stable if it is stable and, also, there exists $r > 0$ such that

$$\lim_{t \rightarrow +\infty} D(\hat{\varphi}_t(X_0), \bar{X}) = \lim_{t \rightarrow +\infty} \sup_{0 \leq \alpha \leq 1} h([\hat{\varphi}_t(X)]^\alpha, [\bar{X}]^\alpha) = 0,$$

for every $X$ satisfying $D(X, \bar{X}) = \sup_{0 \leq \alpha \leq 1} h([X]^\alpha, [\bar{X}]^\alpha) < r$.

**Proposition 2.** $\bar{x}$ is an equilibrium point of (10) if, and only if $\chi_{(\bar{x})}$ is an equilibrium point of (13), where $\chi_{(\bar{x})}$ is the characteristic function of $\bar{x}$.

**Proof.**
This result is a direct consequence the Theorem 1 and Definition 3 for $\alpha$—levels since

\[ \dot{x} = \varphi_t \left( [x(t)]^\alpha \right) \leftrightarrow [x(t)]^\alpha = \left[ \tilde{\varphi}_t (x(t)) \right]^\alpha. \]

**Example 1.** Given the classic initial value problem

\[
\begin{cases}
    x' = -ax \\
    x(0) = x_0,
\end{cases}
\]  

we can verify that its solution is given by $\varphi_t(x_0) = x_0e^{-at}$ and that the origin is an asymptotically stable equilibrium point for $a > 0$.

Supposing that the initial condition $x_0$ is uncertain and modelled by a fuzzy set, then (14) can be rewritten in the following way,

\[
\begin{cases}
    x' = -ax \\
    x(0) \in X_0,
\end{cases}
\]  

where the initial condition $X_0$ belongs to $E^1$ so that $[X_0]^\alpha = [x_0^1, x_0^2]$. Then, since $\varphi_t$ is continuous, by Theorem 1 we have that the $\alpha$—levels of the flow of (15) are given by

\[ [\tilde{\varphi}_t(X_0)]^\alpha = \varphi_t ([X_0]^\alpha) = [x_0^1 e^{-at}, x_0^2 e^{-at}]. \]

By Proposition 2, we have that $\chi_{(0)}$ (here, $\chi_{(0)}$ is the characteristic function of 0) is an equilibrium point of (15), since $\dot{x} = 0$ is an equilibrium point of (14).

In this case, we can see directly that $\dot{x} = 0$ is the sole equilibrium point of (15), because

\[ [x_0^1 e^{-at}, x_0^2 e^{-at}] = [x_0^1, x_0^2], \forall t \geq 0, \text{ if } x_0^1 = x_0^2 = 0. \]

Moreover, $\chi_{(0)}$ is asymptotically stable.

- $\chi_{(0)}$ is **stable**.

Given $\varepsilon > 0$, we must show that $\exists \delta > 0$, such that $D \left( X_0, \chi_{(0)} \right) < \delta$, implies that $D(\tilde{\varphi}_t(X_0), \chi_{(0)} ) < \varepsilon$. Taking, $\delta = \varepsilon$, we have

\[
D(\tilde{\varphi}_t(X_0), \chi_{(0)} ) = \sup_{0 \leq \alpha \leq 1} h ( [\varphi_t(X_0)]^\alpha, [\chi_{(0)}]^\alpha ) \\
= \sup_{0 \leq \alpha \leq 1} h ( \varphi_t([X_0]^\alpha), 0 ) \\
= \sup_{0 \leq \alpha \leq 1} \max \{ e^{-at} (| x_0^1 |, | x_0^2 |) \} \\
= e^{-at} \max \{ | x_0^1 |, | x_0^2 | \} = e^{-at} D \left( X_0, \chi_{(0)} \right) \leq D \left( X_0, \chi_{(0)} \right) < \varepsilon, \forall t \geq 0.
\]

- $\chi_{(0)}$ is stable asymptotically

\[
\lim_{t \to +\infty} D(\tilde{\varphi}_t(X_0), \chi_{(0)} ) = \lim_{t \to +\infty} e^{-at} \max \{ | x_0^1 |, | x_0^2 | \} = 0, \text{ where } a > 0.
\]
The following Lemma is very important for this section, because it characterizes the neighborhood of a fuzzy set which is a characteristic function of some compact set of \( \mathbb{R}^n \).

**Lemma 1.** \(^{11}\) Let \( \chi_X \in \mathcal{F}(\mathbb{R}^n) \) be the characteristic function of the compact set \( X \subset \mathbb{R}^n \). Then

\[
B(\chi_X, r) := \{ X_0 \in \mathcal{F}(\mathbb{R}^n) / D(\chi_X, X_0) < r \}
\]

\[
= \{ X_0 \in \mathcal{F}(\mathbb{R}^n) / [X_0^0] \subset B(X, r) \text{ and } X \subset B([X_0^0], r) \},
\]

where \( B(X, r) = \{ x \in \mathbb{R}^n : d(x, X) < r \} \) with, \( d(x, X) = \inf \{ \| x - x_0 \| : x_0 \in X \} \).

**Corollary 1.** \(^{11}\) If \( X = \{ \bar{x} \} \) is a unit set, then

\[
B(\chi_X, r) := \{ X_0 \in \mathcal{F}(\mathbb{R}^n) / D(\chi_{\{\bar{x}\}}, X_0) < r \}
\]

\[
= \{ X_0 \in \mathcal{F}(\mathbb{R}^n) / [X_0^0] \subset B(\bar{x}, r) \}
\]

and,

\[
D(\chi_{\{\bar{x}\}}, X_0) = \sup_{x_0 \in [X_0^0]} \| x_0 - \bar{x} \|.
\]

**Theorem 2.** Let \( \hat{\varphi}_t : \mathcal{F}(\mathbb{R}^n) \rightarrow \mathcal{F}(\mathbb{R}^n) \) be the Zadeh extension of the flow \( \varphi_t \) associated to the IVP (10) and \( \bar{x} \) an equilibrium point of (10). Then,

- \( \chi_{\{\bar{x}\}} \) is stable for the system (13) if, and only if, \( \bar{x} \) is stable for (10);
- \( \chi_{\{\bar{x}\}} \) is asymptotically stable for the system (13) if, and only if, \( \bar{x} \) is asymptotically stable for (10).

**Proof.**

i) First we will prove that \( \chi_{\{\bar{x}\}} \) is stable, given that \( \bar{x} \) is stable. We should prove that, given \( \varepsilon > 0 \), \( \exists \delta = \delta(\varepsilon) > 0 \), such that for all \( X_0 \in \mathcal{F}(\mathbb{R}^n) \),

\[
D(X_0, \chi_{\{\bar{x}\}}) < \delta \text{ then } D(\hat{\varphi}_t(X_0), \chi_{\{\bar{x}\}}) < \varepsilon, \forall t > 0.
\]

If \( D(X_0, \chi_{\{\bar{x}\}}) < \delta \), from Corollary 1,

\[
\sup_{x \in [X_0^0]} \| x - \bar{x} \| < \delta,
\]

which implies

\[
\| \varphi_t(x) - \bar{x} \| < \varepsilon,
\]

for all \( x \in [X_0^0] \subset B(\bar{x}, \delta) \).

Hence,

\[
\sup_{x \in [X_0^0]} \| \varphi_t(x) - \bar{x} \| < \varepsilon \Leftrightarrow \sup_{y \in \varphi_t([X_0^0])} \| y - \bar{x} \| < \varepsilon, \Leftrightarrow \sup_{y \in \hat{\varphi}_t([X_0^0])} \| y - \bar{x} \| < \varepsilon,
\]

since \( \varphi_t \) is continuous.

Thus, by Corollary 1, \( D(\hat{\varphi}_t(X_0), \chi_{\{\bar{x}\}}) < \varepsilon, \forall t > 0 \).

In order to show the stability of \( \bar{x} \), given the stability of \( \chi_{\{\bar{x}\}} \), it is enough to see that

\[
\| x - \bar{x} \| = D(\chi_{\{\bar{x}\}}, \chi_{\{\bar{x}\}})
\]

and
\[ \hat{\varphi}_t \chi_{(x_\varepsilon)} = x_{(\varphi_0(x_\varepsilon))}. \]

(ii) If \( \bar{x} \) is asymptotically stable, then \( \bar{x} \) is stable and, by item (i), we have that \( \chi_{(x)} \) is stable. According to Lemma 1, we should show that, given \( \varepsilon > 0 \), \( \exists t_0 > 0 \) such that if \( t > t_0 \), then

\[ \varphi_t ([X_0]^0) \subset B(\bar{x}, \varepsilon). \]

By hypothesis we have that \( \bar{x} \) is asymptotically stable. From this it follows that \( \exists \tau > 0 \), such that

\[ \lim_{t \to \infty} \| \varphi_t(x_0) - \bar{x} \| = 0 \quad \text{for} \quad \| x_0 - \bar{x} \| < r, \]

Suppose that \( D \left( X_0, \chi_{(x_\varepsilon)} \right) < r \). By Corollary 1 it is enough to prove that for each \( \varepsilon > 0 \) there is \( t_0 > 0 \) such that if \( t > t_0 \), then \( \varphi_t ([X_0]^0) \subset B(\bar{x}, \varepsilon) \).

Given \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( \| x - \bar{x} \| < \delta \) then \( \| \varphi_0(x) - \bar{x} \| < \varepsilon \) for all \( t \geq 0 \), by the stability of the point \( \bar{x} \).

Now, for each point \( x \in [X_0]^0 \), there is a natural number \( t_0 \) such that \( \varphi_{t_0}(x) \in B(\bar{x}, \delta) \), since the equilibrium point \( \bar{x} \) attracts each point in \( B(\bar{x}, r) \).

Therefore, using the continuity of \( \varphi_{t_0} \), we can choose an open neighborhood \( V_{x_0} \) of \( x_0 \) such that

\[ \varphi_{t_0}(V_{x_0}) \subset B(\bar{x}, r). \] (16)

By doing this for every point of the compact set \([X_0]^0\) we have an open cover of \([X_0]^0\) by the \( V_{x_0} \).

From this cover we select a finite subcover \( \{ V_{x_01}, ..., V_{x_0k} \} \) corresponding to points \( x_01, ..., x_0k \) in \([X_0]^0\). Moreover, \( \varphi_{t_0+t}(V_{x_0}) \subset B(\bar{x}, \varepsilon), \forall t \geq 0 \),

because

\[ \varphi_{t_0+t}(V_{x_0}) = \varphi_t(\varphi_{t_0}(V_{x_0})) \subset B(\varphi_t(\bar{x}), \varepsilon) = B(\bar{x}, \varepsilon). \]

Then, if we take \( t_0 = \max \{ t_{01}, ..., t_{0k} \} \), we easily see that, for each \( x_0i \in [X_0]^0 \) and \( t > 0 \),

\[ \varphi_{t_0+t}(V_{x_0i}) \subset B(\bar{x}, \varepsilon), \]

since \( x_0i \) belongs the one of the \( V_{x_0i} \). This proves that \( \varphi_{t_0+t}([X_0]^0) \subset B(\bar{x}, \varepsilon) \).

The converse follows immediately like in item (i).

The corollary below characterizes the stability of fuzzy equilibrium points of the type \( \chi_{(x)} \) for (13) through the sign of the eigenvalues associated to a deterministic equilibrium \( \bar{x} \) of (10).

**Corollary 2.** Let \( \bar{x} \) be an equilibrium point of (10) and \( f: \mathbb{R}^n \to \mathbb{R}^n \) with \( f'(\bar{x}) = 0 \) and \( \lambda_i \) the eigenvalues associated to the \( \bar{x} \). Then
If $\Re(\lambda_i) < 0$ for all $i$, then $\chi_{(\varepsilon)}$ is asymptotically stable for (13);

If $\Re(\lambda_i) > 0$, for some $i$, then $\chi_{(\varepsilon)}$ is unstable for (13). Here, $\Re(\lambda_i)$ is the real part of the eigenvalue.

Proof.

It follows from the fact that, according to Theorem 2, if $\bar{x}$ is an equilibrium asymptotically stable or unstable equilibrium point for (10), then $\chi_\varepsilon$ will be also an equilibrium point of the same type for (13).

Again we recall some definitions from Hale\(^1\).

A subset $A$ is said to attract a subset $C \subset X$ under a family of applications $T_t : X \to X$ if

$$\rho(T_t(C), A) \xrightarrow{t \to \infty} 0,$$

where $\rho(T_t(C), A) = \sup_{y \in T_t(C)} \inf_{x \in A} \|x - y\|$.

The subset $A$ is called a local attractor if $A$ is compact invariant and there exists a bounded neighborhood $B$ of $A$ such that $A$ attracts $B$.

**Theorem 3.** Let $\hat{\varphi}_t : F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ be the Zadeh’s extension of the continuous function $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$. If $\bar{X}$ is an equilibrium point of (13) and,

$$\lim_{t \to \infty} D(\hat{\varphi}_t(X), \bar{X}) = 0, \text{ for } D(X, \bar{X}) < r,$$

then the $\alpha$–levels $[\bar{X}]^\alpha$ attract the $\alpha$–levels $[X_0]^\alpha$ through the $\varphi_t$.

**Proof.**

If $D(X, \bar{X}) < r$ and $X \in F(\mathbb{R}^n)$, then by Lemma 1 we have that $X \in B(\bar{X}, r)$.

Therefore, given $\varepsilon > 0$, $\exists \ t_0 = t_0(\varepsilon)$, such that $D(\hat{\varphi}_t(X), \bar{X}) < \varepsilon, \forall t \geq t_0$. This implies

$$h([\hat{\varphi}_t(X)]^\alpha, [\bar{X}]^\alpha) < \varepsilon,$$

or rather,

$$h(\varphi_t([X]^\alpha), [\bar{X}]^\alpha) < \varepsilon,$$

$\forall \alpha \in [0, 1]$.

Thus, since

$$h(\varphi_t([X]^\alpha), [\bar{X}]^\alpha) = \inf \{s : \varphi_t([X]^\alpha) \subset B([\bar{X}]^\alpha, s)/[\bar{X}]^\alpha \subset B(\varphi_t([X]^\alpha), s)\},$$

we can conclude that $\varphi_t([X]^\alpha) \subset B([\bar{X}]^\alpha, \varepsilon), \alpha \in [0, 1]$, that is, the levels of $[\bar{X}]^\alpha$ attract the levels of $[X]^\alpha$ through the $\varphi_t$.

**Corollary 3.** Let $\varphi_t$ and $\hat{\varphi}_t$ be as in Theorem (3). Then:

(i) If $X \subset \mathbb{R}^n$ is compact and invariant under $\varphi_t$ with $[\bar{X}]^0 \subset B(\bar{X}, \delta)$ and $\bar{X} \subset B([\bar{X}]^1, r)$, then $X = \chi_X$ (that is, the fixed points are isolated);
Mizukoshi, M. T., Barros, L. C. de, Bassanezi, R.C.

(ii) If $\overline{X}^\alpha \subset B([\overline{X}]^1, r)$ and $\overline{X}^0 \subset B([\overline{X}]^\alpha, r)$, for some $\alpha \in [0, 1]$, then $[\overline{X}]^0 = [\overline{X}]^1$.

(iii) If $[\overline{X}]^0 \subset B([\overline{X}]^1, r)$, then $[\overline{X}]^0 = [\overline{X}]^1$.

Proof.

(i) First we note that $X \subset \mathbb{R}^n$ is compact and invariant under the $\varphi_t$ (see Hale for more details), then by definition $\varphi_t(X) = X$. Therefore, for $\alpha \in [0, 1]$ $[\widehat{\varphi}_t(\chi_X)]^\alpha = \varphi_t(\chi_X) = \varphi_t(X) = [\chi_X]^\alpha,$ or rather, $\chi_X$ is an equilibrium point of $\widehat{\varphi}_t$.

Now, by Lemma 1, we have that $[\overline{X}]^0 \subset B(X, r)$ and $X \subset B([\overline{X}]^1, r)$, which is equivalent to $D(X, \chi_X) < r$. Therefore, from the hypothesis that $\lim_{t \to \infty} D(\widehat{\varphi}_t((X), \chi_X) = 0,$

or,

$$\lim_{t \to \infty} D(\chi_X, \widehat{X}) = 0,$$

we have $\chi_X = \widehat{X}$.

(ii) Immediate consequence of (i) for $X = [\overline{X}]^\alpha$.

(iii) It follows immediately from (ii).

From Lemma 1 and Corollary 3 it follows that the only equilibrium point of $\widehat{\varphi}_t$ which is asymptotically stable, with $h([\overline{X}]^0, [\overline{X}]^1) < r$, is the characteristic function of some compact set $X \subset \mathbb{R}^n$.

The FIVP with the field given by Zadeh’s extension of the deterministic field may have other equilibrium points besides those that are foreseen by Proposition 2. The following example illustrates this fact as also as the use of the results obtained in Section 4.

Example 2. Let us consider the deterministic Verhulst model

$$\begin{cases} x' = ax(1 - x) \\ x(0) = x_0, \end{cases}$$

whose solution is given by

$$\varphi_t(x_0) = \frac{x_0}{x_0 - (x_0 - 1)e^{-at}}.$$ 

The equilibrium points of (17) are 0 and 1. The first one is unstable while the latter is asymptotically stable, since $f'(0) = a > 0$ and $f'(1) = -a < 0$, where $f(x) = ax(1 - x)$.

Now, associated to IVP (17) we can have three types of FIVP:

(i) Considering the initial condition to be fuzzy, that is, $x_0 \in X_0, X_0 \in \mathcal{F}(\mathbb{R})$;
(ii) Considering just $a$ as a fuzzy set;
(iii) Considering that both, the parameter and the initial condition are fuzzy.
In what follows study each case, separately.

Case (i) Let us suppose that only the initial condition is fuzzy, then we have the FIVP
\[
\begin{align*}
x' &= ax(1 - x) \\
x(0) &= \bar{X}_0,
\end{align*}
\]
where \(X_0 \in \mathcal{F}(\mathbb{R})\) with its \(\alpha\)-levels given by \([X_0]_\alpha = [x_{\alpha 01}, x_{\alpha 02}]\), \(\alpha \in [0, 1]\).

Since \(\varphi_t\), the flow associated to (17), is a continuous function and increasing for \(x_0 \geq 0, i = 1, 2\), then the \(\alpha\)-levels of the flow of (18) to \(\alpha \in [0, 1]\) are given by
\[
\hat{\varphi}_t([X_0])_\alpha = \varphi_t([X_0]_\alpha) = [\varphi_t(x_{\alpha 01}), \varphi_t(x_{\alpha 02})] = \left[\begin{array}{c}
x_{\alpha 01} - (x_{\alpha 01} - 1)e^{-at} \\
x_{\alpha 02} - (x_{\alpha 02} - 1)e^{-at}
\end{array}\right].
\]

The equilibrium points of (18) are obtained from
\[
\hat{\varphi}_t(X_0) = [X_0]^\alpha,
\]
for \(\alpha \in [0, 1]\), or by solving the following system of equations
\[
\begin{align*}
x_{\alpha 01} - (x_{\alpha 01} - 1)e^{-at} &= x_0^\alpha \\
x_{\alpha 02} - (x_{\alpha 02} - 1)e^{-at} &= x_0^\alpha,
\end{align*}
\]
\(\forall t \geq 0\).

From (19) it follows that \(x_{\alpha 01} = x_{\alpha 02} = 0\) or \(x_{\alpha 01} = x_{\alpha 02} = 1\). Thus, by Proposition 2, \(\chi_{\{0\}}\) and \(\chi_{\{1\}}\) are equilibrium points of (18).

From Theorem 2 it follows that \(\chi_{\{0\}}\) is unstable and \(\chi_{\{1\}}\) is asymptotically stable. Figure 1 shows the stability of \(\chi_{\{1\}}\) near the stable equilibrium of the deterministic model.

![Fig. 1. Dynamic of \(\hat{\varphi}_t\) around \(\chi_{\{1\}}\).](image-url)
a) $\chi_{[0,1]}$ is an equilibrium point.

By Theorem 1 it follows that

$$\hat{\phi}_t(\chi_{[0,1]}) = \phi_t([0,1]) = [\phi_t(0), \phi_t(1)] = [0, 1] = \chi_{[0,1]}^\alpha, \forall \alpha \in [0, 1].$$

b) $\chi_{[0,1]}$ is unstable.

In order to prove this we shall use Lemma 1. Considering the initial condition $X_0 = \chi_{[\delta,1]}$, where $0 \leq \delta \leq 1$, we have

$$D(X_0, \chi_{[0,1]}) = \sup_{0 \leq \alpha \leq 1} H([\delta, 1], [0, 1]) = || \delta \|.$$ 

On the other hand, we have

$$[\hat{\phi}_t(X_0)]^\alpha = [\phi_t([\delta, 1])] = [\phi_t(\delta), \phi_t(1)] = [\phi_t(\delta), 1],$$

since $\phi_t$ is increasing and $\phi_t(1) = 1$, thus

$$\lim_{t \to +\infty} D(\hat{\phi}_t(X_0), \chi_{[0,1]}) = \lim_{t \to +\infty} \sup_{0 \leq \alpha \leq 1} H([\phi_t(\delta), 1], [0, 1]) = \lim_{t \to +\infty} \frac{\delta}{\delta - (\delta - 1)e^{-at}} = 1.$$ 

Therefore, $[0, 1] \not\subset B(\phi_t([X_0]^\alpha), \epsilon)$ for $t \in \mathbb{R}$ sufficiently large.

It follows from Theorem 3 and Lemma 1 that $\chi_{[0,1]}$ is an unstable equilibrium point for (18).

Case (ii) In this case first we make the change of variables $y = (y_1, y_2) = (x, a)$ in (17) to obtain

$$\begin{cases}
  y'_1 = y_1 y_2 (1 - y_1) \\
  y'_2 = 0 \\
  y_0 = (x_0, a_0),
\end{cases}$$

where the flow associated to (20) is given by

$$\psi_t(y_0) = (\phi_t(x_0), a) = \left(\frac{x_0}{x_0 - (x_0 - 1)e^{-at}}, a_0\right).$$

Supposing that just the parameter $a$ is replaced by a fuzzy set $A_0$ with $supp A_0 \subset (0, +\infty)$ we have that (20) can be rewritten in the following way

$$\begin{cases}
  y'_1 = y_1 y_2 (1 - y_1) \\
  y'_2 = 0 \\
  y_0 \in (x_0, A_0),
\end{cases}$$

where $A_0 \in \mathcal{F}(\mathbb{R})$ with its $\alpha$-levels given by $[A_0]^\alpha = [a_0^\alpha, a_2^\alpha]$.

In this case, for $\alpha \in [0, 1]$, we have

$$[\hat{\psi}_t(Y_0)]^\alpha = \psi_t([Y_0]^\alpha) = \psi_t([x_0] \times [A_0]^\alpha) = \psi_t([x_0] \times [a_0^\alpha, a_2^\alpha])$$

$$= \left\{ \begin{array}{ll}
  \left[ \frac{x_0}{x_0 - (x_0 - 1)e^{-a_1^\alpha t}}, \frac{x_0}{x_0 - (x_0 - 1)e^{-a_2^\alpha t}} \right], & \text{if } x_0 < 1 \\
  \left[ \frac{x_0}{x_0 - (x_0 - 1)e^{-a_2^\alpha t}}, \frac{x_0}{x_0 - (x_0 - 1)e^{-a_1^\alpha t}} \right], & \text{if } x_0 > 1,
\end{array} \right.$$
because $\psi_t$ is increasing, with respect to the variable $a$, for $x_0 < 1$, and decreasing for $x_0 > 1$.

Here, the equilibrium points are obtained from

$$\psi_t([Y_0]^\alpha) = [Y_0]^\alpha, \alpha \in [0,1]$$

or,

$$(\varphi_t(x_0), [a_1^\alpha, a_2^\alpha]) = (x_0, [a_1^\alpha, a_2^\alpha])$$

Thus, we obtain $x_0 = 0$ or $x_0 = 1$.

Therefore, from Proposition 2 the only equilibrium points are $\chi_{(0)}$ and $\chi_{(1)}$ which by Theorem 2, are respectively unstable and stable. Figure 2 shows the stability of $\chi_{(1)}$, where the parameter is a fuzzy number.

![Fig. 2. Dynamic of $\tilde{\varphi}_t$ around $\chi_{(1)}$, with a fuzzy.](image)

**Case (iii)** Let us suppose that both, the initial condition and the parameter are fuzzy, then (20) can be rewritten as

$$\begin{cases}
y_1' = y_1 y_2 (1 - y_1) \\
y_2' = 0 \\
y_0 \in (X_0, A_0),
\end{cases}$$

(22)

where $X_0 \in \mathcal{F}(\mathbb{R})$ and $A_0 \in \mathcal{F}(\mathbb{R})$ are fuzzy numbers with its $\alpha$–levels given by $[A_0]^\alpha = [a_1^\alpha, a_2^\alpha]$ and $[X_0]^\alpha = [x_0^{a_1}, x_0^{a_2}]$, respectively. Therefore, for $\alpha \in [0,1]$, we have

$$[\tilde{\psi}_t(Y_0)]^\alpha = (\varphi_t([X_0]^\alpha, [A_0]^\alpha))$$

We note that

$$[\tilde{\psi}_t(Y_0)]^\alpha = [Y_0]^\alpha$$
is equivalent to

\[
(\varphi_t([X_0], [A_0]) = ([X_0], [A_0])
\]

Thus, the equilibrium points are obtained from

\[
\varphi_t([x_0^0, x_0^1]) = [x_0^0, x_0^1], \forall t
\]

or,

\[
\left[ \frac{x_0^0}{x_0^0 - (x_0^0 - 1)e^{-at}}, \frac{x_0^1}{x_0^1 - (x_0^1 - 1)e^{-at}} \right] = [x_0^0, x_0^1].
\]

Then,

\[
x_0^0 = x_0^1 = 0 \text{ or } x_0^1 = x_0^1 = 1.
\]

Therefore, it follows from Proposition 2 that the equilibrium point are \(\chi^0, \chi^1, \chi^0\). The stability for each point can be studied in the same way as in case (ii).

5. Conclusion

In this paper we explored the fuzzification methods proposed by Mizukoshi et. al\(^2\) for fuzzy differential equations obtained from deterministic equations, where some parameter and/or initial condition are uncertain, and given by fuzzy subsets. More recently, such equations appear on a natural way in a variety of areas. In engineering Oberguggenberger et. al\(^12\) have made use of such equations. They have also been widely used in biomathematics, as one can see in Barros and Bassanezi\(^13\), Barros et. al\(^14,15,16\), Jafelice et. al\(^17\) and Krivan\(^18\).

This paper has suggested a methodology to obtain solutions for fuzzy differential equations through Zadeh’s extension principle applied to the deterministic solution. For autonomous systems, Mizukoshi et. al\(^2\) proved that solutions produced by this methodology coincide with those proposed by Hullermeier (see also Diamond\(^19\)) for fuzzy differential inclusions. The main advantage of using the methodology applied here is its simplicity. Its use does not require any knowledge of multivalued analysis as in the case of the differential inclusions proposed by Hullermeier.

Finally, we could have opted to traditional fuzzy differential equations, which use the Hukuhara derivative\(^20\). However, such approach suffers from a noticeable shortcoming, whereas in this case the class of fuzzy solutions consists of divergent process\(^21,22\) and this fact makes it difficult to study the stability. From the theoretical point of view, this is a great advantage for our method. The stability concept is easily introduced as an extension of the classic theory for differential equations, as we have shown in Section 4 and illustrated in Example 2.
References