ON THE MINIMAL POLYNOMIAL AND AUTOMORPHISM GROUP OF A GRAPH

by

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This thesis discusses the use of the characteristic polynomial and minimal polynomial of the adjacency matrix of a graph to characterize its automorphism group. We first consider the reducibility of the minimal polynomial of a graph and see how this reflects the properties of the graph and its automorphism group. Then we study the relationship between the number of orbits of a subgroup of the automorphism group of a graph and the factorization of its characteristic polynomial. Finally we present an algorithm to determine the automorphism partitioning of a graph using its characteristic polynomial. Most of the results can also be extended to directed graphs and to graphs with parallel edges and loops.
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I. INTRODUCTION

In this thesis we will examine the relationship between the automorphism group of a graph $G$ and the minimal and characteristic polynomials of $G$.

In Chapter I we will present definitions and results that are related to the problems to be studied throughout this thesis. In Chapter II, we will consider the problem of using the minimal polynomial of the adjacency matrix to characterize the graph and its automorphism group. Chapter III is devoted to examining the structure of a graph and its automorphism group in terms of various properties of the adjacency matrix. Chapter IV deals with extensions and applications of results obtained in the previous chapters. Algorithmic considerations will also be included.

1.1 Basic Definitions and Notations

Some basic graph theoretic concepts will be given in this section. Notation introduced in the definitions will be used consistently throughout this thesis.

Definition 1.1

A graph $G$ of order $n$ is an ordered pair $G = (V(G), E(G))$ consisting of a nonempty finite set $V(G)$, whose elements are called vertices (points), and a set $E(G)$, whose elements are called edges, such that each $e \in E(G)$ is identified with an unordered pair $(u, v)$ of
distinct vertices \( u, v \in V(G) \). We say that the edge 
\( e = (u, v) \) joins the vertices \( u \) and \( v \). \( V(G) \) will be called 
the \textit{vertex set} and \( E(G) \) the \textit{edge set}. The number of 
elements \( |V(G)| \) in \( V(G) \) will be denoted by \( n \).

In general, we will use \( v_1, v_2, \ldots, v_n \) to represent 
the \( n \) vertices in \( V(G) \); sometimes only the subscripts will 
be used when no confusion will arise.

If an ordered pair \( (u, v) \) is used to identify an edge 
\( e \in E(G) \), then \( G \) is called a \textit{digraph} and \( e = (u, v) \) a \textit{line} 
(or \textit{arc}) from \( u \) to \( v \).

\textbf{Definition 1.2}

The term \textit{graph} is sometimes used in a more general 
sense, i.e. "loops" and "parallel edges" are allowed. More 
precisely a \textit{general graph} is a triple \( G = (V(G), E(G), \gamma_G) \) 
where \( V(G) \) is a nonempty set of \textit{vertices}, \( E(G) \) (disjoint 
from \( V(G) \)) is a nonempty set of \textit{edges}, and \( \gamma_G \) is an 
\textit{incidence function} that associates with each edge of \( G \) an 
unordered pair of vertices of \( G \).

An edge \( e = (u, v) \) is called a \textit{loop} if \( u = v \). Two 
edges \( e_1 \) and \( e_2 \) are said to be \textit{parallel} if 
\( \gamma_G(e_1) = \gamma_G(e_2) \). Thus a graph in the sense of 
Definition 1.1 is a general graph with no loops and no 
parallel edges. We will sometimes use the adjective 
"simple" to distinguish graphs from general graphs.
Definition 1.3

Two vertices $u, v \in V(G)$ are said to be adjacent, denoted by $u \text{ adj } v$, if $(u, v) \in E(G)$. The degree of a vertex $v$, denoted by $\deg v$, is defined as the number of vertices adjacent to it; and a graph is said to be $k$-regular (or regular of degree $k$) if every vertex has degree $k$.

From the adjacency relation we can define the adjacency function $\vartheta$ on $V(G) \times V(G)$ to $\{0, 1\}$ as follows:

$$\vartheta(u, v) = \begin{cases} 
1 & \text{if } u \text{ adj } v, \\
0 & \text{otherwise.}
\end{cases}$$

Definition 1.4

For any $u, w \in V(G)$, a path of length $p$ from $u$ to $w$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_p$ such that $v_0 = u$, $v_p = w$ and $v_i \text{ adj } v_i$ for $i = 1, 2, \ldots, p$. A path from $u$ to $u$ will be called a cycle.

Definition 1.5

A graph $G$ is said to be connected if for any pair of vertices $u, v \in V(G)$ there exists a path from $u$ to $v$.

Definition 1.6

Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then the matrix

$$A(G) = [a_{ij}]$$
where \( a_{ij} = \mathcal{E}(v_i, v_j) \) is called the \textit{adjacency matrix} of \( G \).

**Definition 1.7**

The \textit{characteristic (minimal) polynomial of a graph} \( G \) with adjacency matrix \( A(G) \) is the characteristic (minimal) polynomial of \( \lambda(G) \), and will be denoted by \( f(G) \) (\( p(G) \) for the minimal polynomial).

The \textit{characteristic roots (or eigenvalues) of a graph} are taken to be the characteristic roots of \( A(G) \).

**Definition 1.8**

The \textit{index} of a graph \( G \), denoted by \( r = r(G) \), is defined to be the largest characteristic root of \( G \).

**Definition 1.9**

Two graphs \( G_1 = (V(G_1), E(G_1)) \) and \( G_2 = (V(G_2), E(G_2)) \) are said to be \textit{isomorphic}, denoted by \( G_1 \cong G_2 \), if there exists a bijection \( \gamma \), called an \textit{isomorphism}, from \( V(G_1) \) to \( V(G_2) \) such that
\[
(u_1, u_2) \in E(G_1) \text{ iff } (\gamma(u_1), \gamma(u_2)) \in E(G_2).
\]

**Definition 1.10** (cf. [16, p. 161])

An \textit{automorphism} of a graph \( G \) is a permutation of the vertex set \( V(G) \) which preserves adjacency.

Thus an \textit{automorphism} of a graph \( G \) is an isomorphism of \( G \) onto itself.
We note that the set of all automorphisms of a graph G forms a group under the usual operation of permutation multiplication. This group of permutations is the automorphism group \( \Gamma(G) \) of G and each subgroup of \( \Gamma(G) \) will be called a group of automorphisms.

**Definition 1.11**

A graph \( G_1 = (V(G_1), E(G_1)) \) is a subgraph of \( G = (V(G), E(G)) \) if \( V(G_1) \subseteq V(G) \) and \( E(G_1) \subseteq E(G) \). \( G_1 \) is an induced subgraph if \( e = (u, v) \in E(G) \) such that \( u, v \in V(G_1) \) implies \( e \in E(G_1) \).

For any subset \( S \subseteq V(G) \), the induced subgraph with vertex set \( V(G) \setminus S \) will be denoted by \( G - S \). When \( S \) is the singleton set \( \{s\} \) the notation \( G - s \) will be employed.

**Definition 1.12**

A subgraph \( G_1 \) of a graph G is called a component of G if it is a maximal connected subgraph of G.

**Definition 1.13**

Let \( G = (V(G), E(G)) \) be a graph. Then its complement, denoted by \( \bar{G} \), is the graph \( \bar{G} = (V(G), \bar{E}(G)) \) where \( e = (u, v) \in \bar{E}(G) \) whenever \( u \neq v \) and \( e \notin E(G) \).
Definition 1.14

The simple graph of order $n$ in which every two distinct vertices are adjacent is called the \textit{complete graph} on $n$ vertices. It will be denoted by $K_n$. Its complement $\overline{K_n}$, in which all vertices are non-adjacent, is called the \textit{void graph} (or \textit{empty graph}) on $n$ vertices.

Definition 1.15

A graph $G$ is \textit{bipartite} if the vertices of $G$ can be partitioned into two sets $V_1$ and $V_2$ such that $v_1$ is adjacent to $v_2$ implies $[v_1 \in V_1 \text{ and } v_2 \in V_2]$ or $[v_1 \in V_2 \text{ and } v_2 \in V_1]$.

Definition 1.16

A \textit{cycle} $C_n$ on $n$ vertices is a connected, 2-regular graph.

Definition 1.17

A \textit{subdivision} of an edge $(u, v)$ refers to the replacement of an edge $(u, v)$ by two edges $(u, w)$, $(v, w)$ where $w$ is a new vertex in the graph.

Definition 1.18

Two graphs $G_1$, $G_2$ are said to be \textit{homeomorphic} if there exists $G_3$ such that $G_1$ and $G_2$ can be obtained from $G_3$ by a sequence of subdivisions of edges.
Definition 1.19

The **chromatic number** $\chi(G)$ of a graph $G$ is defined as the minimum number of colors that is required to color the vertices of $G$ so that no two adjacent vertices have the same color.

Definition 1.20 (See [2, p.42])

A **vertex cut** of a graph $G$ is a subset $V'$ of $V(G)$ such that $G - V'$ is disconnected. A **$k$-vertex cut** is a vertex cut of $k$ elements. If $G$ has at least one pair of distinct nonadjacent vertices, the **connectivity** $\kappa(G)$ of $G$ is the minimum $k$ for which $G$ has a $k$-vertex cut; otherwise, we define $\kappa(G)$ to be $|V(G)| - 1$.

The following notation will also be used consistently throughout this thesis:

$(b_1, b_2, ..., b_n)'$ will be used to denote an $n$-vector with $b_1, b_2, ..., b_n$ as the coordinates.

$<\sigma_1, \sigma_2, ..., \sigma_m>$ will denote the group of permutations generated by the permutations $\sigma_1, \sigma_2, ..., \sigma_m$.

$e$ will be used to denote the identity permutation and $\{e\}$ the trivial permutation group containing only the identity permutation.

For any group $H$, $|H|$ will be used to denote the order of $H$.

$D_n$ will be used to denote the dihedral group on $n$
elements and $S_n$ the symmetric group on $n$ elements.

For any matrix $B$, $f(B)$ will be used to denote the characteristic polynomial of $B$ and $p(B)$ the minimal polynomial of $B$.

$I_n$ will be used to denote the identity matrix of order $n$. When the order is clear from the context, the subscript will be omitted.

1.2 Basic Properties of the Automorphism Group

In this section we discuss basic properties of the automorphism group of a graph. The following three well-known results are obvious from the relevant definitions.

Theorem 1.1

Let $\sigma$ be a permutation on the vertices of a graph $G$ and $P_\sigma$ its corresponding permutation matrix. Then

$\sigma$ is an automorphism of $G$ iff $P_\sigma A = A P_\sigma$,

where $A = A(G)$.

Theorem 1.2

$\Gamma(G) = \Gamma(\overline{G})$.

Theorem 1.3

$\Gamma(K_n) = \Gamma(\overline{K_n}) = S_n$

where $S_n$ is the symmetric group on $n$ elements.
Definition 1.21

Two vertices $x, y \in V(G)$ of a graph $G$ are said to be **similar**, denoted by $x \sim y$, if there exists at least one automorphism of $G$ mapping $x$ to $y$.

Definition 1.22

The **automorphism partitioning** of a graph $G$ is a partitioning of the vertices of $G$ such that $x$ and $y$ are in the same cell if and only if $x \sim y$.

Hence the automorphism partitioning of $G$ gives precisely the orbits of $\Gamma(G)$.

Definition 1.23

A **fixed-point** of a graph $G$ is a vertex fixed by all the automorphisms of $G$.

Definition 1.24

A graph is called an **identity graph** if no two distinct vertices $x$ and $y$ are similar.

Definition 1.25

A graph is **point-symmetric** (or transitive on the vertices) if every two vertices $x$ and $y$ are similar.

Thus for a point-symmetric graph $G$ the automorphism partitioning is $V(G)$. Similarly we say that a graph $G$ is **line-symmetric** (or transitive on the edges) if, for every
pair of edges $p$ and $q$, there exists at least one automorphism of $G$ mapping the endpoints of $p$ to the endpoints of $q$.

**Theorem 1.4**

If $u$, $v$ are two similar vertices of a graph $G$, then $G - u \cong G - v$.

We will now consider the following operations on graphs and groups (see [16, p.21 and p.163]):

Let $G_1$ and $G_2$ be graphs with disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$, respectively. The union $G = G_1 \cup G_2$ has vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The join $G_1 \star G_2$ consists of $G_1 \cup G_2$ and all edges joining $V_1$ and $V_2$.

For any connected graph $G$, we write $nG$ for the graph with $n$ components each isomorphic with $G$.

The product $G_1 \times G_2$ has vertex set $V = V_1 \times V_2$. The edge set $E$ of $G_1 \times G_2$ is defined as follows:

Consider any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then $u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $|u_i - v_i| = 1$ for some $i$ or $u_2 = v_2$ and $u_1 = v_1$.

Let $H$ be a permutation group of degree $d$ acting on the set $X = \{x_1, x_2, \ldots, x_d\}$, and let $K$ be another permutation group of degree $e$ acting on the set $Y = \{y_1, y_2, \ldots, y_e\}$. The sum $H + K$ is a permutation
group which acts on the disjoint union $X \cup Y$ and whose elements are all the ordered pairs of permutations $\alpha$ in $H$ and $\beta$ in $K$, written $\alpha + \beta$. Any element $z$ of $X \cup Y$ is permuted by $\alpha + \beta$ according to the rule:

$$(\alpha + \beta)(z) = \begin{cases} \alpha z, & z \in X \\ \beta z, & z \in Y. \end{cases}$$

The cartesian product $H \times K$ of $H$ and $K$ is a permutation group which acts on the set $X \times Y$ and whose permutations are all the ordered pairs, written $\alpha \times \beta$, of permutations $\alpha$ in $H$ and $\beta$ in $K$. The element $(x, y)$ of $X \times Y$ is permuted by $\alpha \times \beta$ as follows:

$$(\alpha \times \beta)(x, y) = (\alpha x, \beta y).$$

The wreath product $H[K]$ of $H$ around $K$ also acts on the set $X \times Y$. For each $\alpha \in H$ and any sequence $(\beta_1, \beta_2, \ldots, \beta_d)$ of $d$ permutations in $K$, there is a unique permutation in $H[K]$ written $(\alpha; \beta_1, \beta_2, \ldots, \beta_d)$ such that for $(x_i, y_j)$ in $X \times Y$:

$$(\alpha; \beta_1, \beta_2, \ldots, \beta_d)(x_i, y_j) = (\alpha x_i, \beta_j y_j).$$

The following are results concerning the relationship between operations on graphs and operations on their automorphism groups.

**Theorem 1.5** [18]

If $G_1$, $G_2$, ..., $G_n$ are pairwise non-isomorphic, connected graphs and

$$G = \sum_{k=1}^{n} m_k G_k,$$

then
\[ \Gamma(G) = \sum_{k=1}^{n} S_{\pi_k} [\Gamma(G_k)]. \]

**Corollary 1.1** ([16, p.166])

\[ \Gamma(G_1 + G_2) \cong \Gamma(G_1) + \Gamma(G_2) \]

if and only if no component of \( G_1 \) is isomorphic with a component of \( G_2 \).

**Theorem 1.6** ([32]; [16, p.166])

\[ \Gamma(G_1 \circ G_2) \cong \Gamma(G_1) \times \Gamma(G_2) \]

if and only if \( G_1 \) and \( G_2 \) are relatively prime (see [16, p.166] or [32] for a definition of prime graphs).

1.3 **Motivation**

From Theorem 1.1, we see that the construction of the automorphism group of a graph \( G \) is equivalent to finding all permutation matrices that commute with \( \Lambda(G) \). The solution to this problem obviously depends on the structure of \( \Lambda(G) \). The minimal and characteristic polynomials of \( \Lambda(G) \) reflect, to some extent, properties of \( \Lambda(G) \). Hence we expect to obtain some properties of the automorphism group of a graph by studying the properties of its minimal and characteristic polynomials.

The problem of determining the automorphism group of a graph is closely related to the graph isomorphism problem. In fact, it is known that the automorphism
The partitioning problem is algorithmically equivalent to the graph isomorphism problem. That means, given an algorithm that can solve the automorphism partitioning problem, we can use it to solve the graph isomorphism problem and vice versa. More detailed treatment of this connection will be considered in Section 4.3.

Applications can be found in retrieving information which is stored graphically. Chemical compounds, for example, can be represented by graphs corresponding to their structural formulas, and we may desire to identify all the chemical compounds that have a given radical or substructure. This problem involves finding subgraphs of the graphs representing the chemical compounds which are isomorphic to a given graph.

Sussenguth[39] describes an algorithm for matching chemical structures. The algorithm uses various properties of the vertices of the two graphs to partition the vertex sets. In chemical structures, these properties include atom and bond types. Actually these properties can be represented by means of labels on vertices and weights on edges. Sussenguth's algorithm can also be used to check whether a chemical compound contains a certain substructure. We note that the general problem of identifying isomorphic graphs is considerably more difficult than matching chemical structures.
1.4 Statement of Problems

The following two problems will be studied:

(i) characterization of the automorphism group of a graph from properties of its adjacency matrix.

(ii) construction of the automorphism groups of graphs whose adjacency matrices share some common properties.

We shall concern ourselves mainly with finite simple graphs (as defined in Section 1.1). Various extensions will also be considered in Chapter IV.

1.5 Literature Review

In this section, we are going to review briefly results that are related to the problems mentioned in Section 1.4.

The problem of the existence of graphs with a given group was posed by König[21]. In 1938 Frucht[11] proved that for any finite group H there exist infinitely many graphs with automorphism groups abstractly isomorphic to H. Later the same author[10] extended this result to connected regular graphs of degree three. Then in 1957, Sabidussi[33] generalized this result to graphs with certain given properties. The following is the main result of Sabidussi's generalization.
Theorem 1.7

Given a finite group $H$ of order $> 1$ and an integer $j, 1 \leq j \leq 4$, there exist infinitely many non-homeomorphic connected fixed-point-free graphs $G$ such that (i) the automorphism group of $G$ is isomorphic to $H$, and (ii) $G$ has property $P_j$, where the properties $P_j$ ($j = 1, 2, 3, 4$) are as follows:

$P_1$: The connectivity of $G$ is $k$, where $k$ is an integer $\geq 1$.

$P_2$: The chromatic number of $G$ is $k$, where $k$ is an integer $\geq 2$.

$P_3$: $G$ is regular of degree $k$, where $k$ is an integer $\geq 3$.

$P_4$: $G$ is spanned by a graph $\tilde{Y}$ homeomorphic to a given connected graph $Y$.

The other problem is to construct the automorphism groups of certain particular classes of graphs. Usually the construction of the automorphism groups depends on the knowledge of the structure of graphs in that class.

Frucht, Graver and Watkins [12] considered the problem of characterizing the automorphism groups of the generalized Petersen graphs which are defined as follows:

For integers $n$ and $k$ with $2 \leq 2k < n$, the generalized Petersen graph $G(n, k)$ has been defined to have vertex-set

$V(G(n, k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$

and edge set $E(G(n, k))$ consisting of all edges of the
form \((u_i, u_{i+1}), (u_i, v_i), (v_i, v_{i+k})\), where \(i\) is an integer and all subscripts are to be read modulo \(n\).

However classes of graphs with common algebraic properties can also be considered.

Mowshowitz\([24]\) describes an algorithm for the construction of the automorphism groups of graphs with non-derogatory adjacency matrices (i.e. those with identical minimal and characteristic polynomials). If \(G\) is a graph of this type with \(n\) vertices then Mowshowitz's algorithm will construct the automorphism group \(\Gamma(G)\) in at most \(2^{\lceil \frac{n}{2} \rceil}\) steps.\(^*\)

The difficulty in constructing the automorphism group of a graph increases rapidly as the number of vertices gets larger. Hence it will be desirable if we can decompose the graph into smaller component graphs and construct the automorphism group of the graph from those of the component graphs.

Harary and Palmer \([18]\) studied the automorphism groups of product graphs in relation to the automorphism groups of the component graphs.

Another approach in characterizing the automorphism group of a graph is to find its automorphism partitioning.

\(^*\) \([p]\) denotes as usual the greatest integer \(\leq p\).
Weichsel[40] considered a partition $P$ of the vertices of the graph $G$ which is related to its automorphism group. The partition $P$ is called a star partition of $G$ and is defined as follows:

**Definition 1.26**

$P$ is a **star partition** of $G$ if it satisfies the following:

(i) If $A \in P$, then all elements of $A$ are of the same degree.

(ii) Let $a \in A \in P$ and $b \in B \in P$ with $a$ adjacent to $b$. Then for each $a' \in A$ there is $b' \in B$ such that $a'$ is adjacent to $b'$.

The following theorem is given in [40]:

**Theorem 1.8**

Let $G$ be a graph whose only star partition is the trivial partition (i.e., the partition whose elements are singleton sets). Then $G$ is an identity graph.

An algorithm for finding a star partition is also described in [40]. It is worthwhile to note that the automorphism partition of a graph $G$ is a star partition of $G$ and the algorithm produces only the coarsest of all star partitions. Also the algorithm does not yield anything for a regular graph since in this case the set of all vertices of the graph forms the coarsest star partition.
The route that we are going to take is via the characteristic polynomial and minimal polynomial of the adjacency matrix of the graph. The eigenvalues of a graph are the roots of the characteristic polynomial and the totality of all eigenvalues is called the spectrum of the graph. A list of the eigenvalues and characteristic polynomials of connected graphs with not more than six vertices is given in Tables I and II of the Appendix. The following theorems are results concerning the relations between the eigenvalues of a graph and its automorphism group.

**Theorem 1.9 [36]**

If the automorphism group of a connected graph $G$ is transitive on the edges, then there is but one positive simple eigenvalue of $G$ and it is the largest eigenvalue.

If $G$ is bipartite then $r(G)$ and $-r(G)$ are the only two non-zero simple eigenvalues of $G$.

**Theorem 1.10 [27]**

Let $G$ be a graph of order $n$ with adjacency matrix $A = A(G)$, and automorphism group $\Gamma = \Gamma(G)$ (regarded as a group of permutation matrices). If $A$ has $n$ distinct eigenvalues, then every $P$ in $\Gamma$ has order two (which, of course, implies that $\Gamma$ is Abelian).

The following theorem is an extension of the previous
Let $G$ be a finite graph with $n$ vertices (directed or undirected, with or without loops), $A = A(G)$ be its adjacency matrix, and $\Gamma(G)$ be its automorphism group. If the eigenvalues of $A$ (in the complex number field) are distinct, then $\Gamma(G)$ is Abelian.

It is well known that the characteristic polynomial of a graph does not characterize a graph completely. Graphs with the same characteristic polynomial are called cospectral graphs. Some examples of cospectral graphs and digraphs are given by Harary, King, Mowshowitz and Read in [17]. Also it can be easily seen from the following example that cospectral graphs do not necessarily have isomorphic automorphism groups.
\[ \Gamma(G_1) \cong D_4 \quad \Gamma(G_2) \cong S_4 \]

characteristic polynomial of \( G_1 \)

\[ = \text{characteristic polynomial of } G_2 \]

\[ = x^5 - 4x^3 = x^3(x - 2)(x + 2) \]

From the previous example we see that using only information from the characteristic polynomial of a graph does not enable us to determine the automorphism group. In the following chapters we are going to study how the properties of the minimal polynomial and characteristic polynomial of a graph reflect those of the automorphism group.
II. MINIMAL POLYNOMIAL OF THE ADJACENCY MATRIX

In this chapter, we will study the relationship between the automorphism group and the minimal polynomial of a graph.

We will use \( f(G; x) \) and \( p(G; x) \) to denote the characteristic polynomial and minimal polynomial of a graph \( G \) in the indeterminate \( x \). When the dependence on the indeterminate is clear from the context, we will employ the usual notation \( f(G) \) and \( p(G) \).

The following theorem gives a characterization of the minimal polynomial of any graph.

**Theorem 2.1**

The minimal polynomial of any graph \( G \) is a product of distinct irreducible polynomials over the integers, i.e. the multiplicity of any factor in the expression for the minimal polynomial is one.

**Proof**

Let \( A \) be the adjacency matrix of \( G \). Since \( A \) is symmetric the minimal polynomial of \( A \) consists of distinct irreducible factors over the real number field. Hence it is also a product of distinct irreducible factors over the integers.

From the above theorem we see that the minimal polynomial of a graph can be determined by the characteristic polynomial. For, if we have computed the
characteristic polynomial of a graph and decomposed it into irreducible factors then the minimal polynomial is simply the product of the distinct irreducible factors. Hence cospectral graphs also have the same minimal polynomial.

The following theorem by Mowshowitz [24] gives a sufficient condition for a graph to be an identity graph.

**Theorem 2.2**

If the characteristic polynomial of a graph is irreducible over the integers, then the automorphism group of the graph is trivial.

In the next section, we shall study graphs that have irreducible minimal polynomials.

2.1 **Irreducible Minimal Polynomial**

The adjacency matrix of a graph consists of 0, 1 entries and hence is a non-negative matrix. Thus results on non-negative matrices can be applied to adjacency matrices of graphs. The following are concepts and results in the theory of non-negative matrices that are relevant in this section.
An n x n matrix $A$ (n \geq 2) is said to be irreducible if no permutation matrix $P$ exists such that

$$PAP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where $A_{11}, A_{22}$ are square matrices.

**Theorem 2.3** (Frobenius–Perron theorem [8])

Let $A$ be an irreducible non-negative matrix. Then $A$ has a characteristic root $r > 0$ such that

(i) there is associated to $r$ a positive eigenvector $x$;

(ii) if $\lambda$ is any characteristic root of $A$, $|\lambda| \leq r$;

(iii) $r$ increases when any element of $A$ increases;

(iv) $r$ is a simple root.

Let $p$ be the minimal polynomial of a graph and assume $p$ is irreducible of degree $m$. If $f$ is the characteristic polynomial and has degree $n$, then $m|n$ and

$$f = p^k$$

where

$$k = n/m.$$

**Theorem 2.4**

If $G$ is a graph with $f$ and $p$ described above as its characteristic and minimal polynomials respectively, then $G$ is the union of $k$ cospectral connected identity graphs with characteristic and minimal polynomials = $p$. 
Proof (By Frobenius-Perron Theorem)

If $G$ is connected, then the adjacency matrix $A$ is obviously irreducible. Therefore the maximal characteristic root $r$ of $f$ is simple; but every characteristic root of $f$ is of multiplicity $k$. Hence $G$ is connected only if $k = 1$.

Let $G_i$ be a component of $G$. Then since $f(G_i) | f$, we have $f(G_i) = p^h$ for some $h$ such that $1 \leq h \leq k$. Since $G_i$ is connected, the maximal characteristic root $r_i$ of $f(G_i)$ is simple, so $h = 1$. Therefore every component of $G$ has characteristic polynomial $p$, which implies that the components of $G$ are cospectral.

By Theorem 2.2, if $p$ is irreducible the components are identity graphs. Since $f$ is the product of the characteristic polynomials of the components, the number of components is $k$.

Before we study the automorphism groups of graphs with irreducible minimal polynomials, we shall first prove the following simple lemma.

**Lemma 2.1**

If $G_1$ is an identity graph and $G_2$ is isomorphic to $G_1$, there is only one isomorphism from $G_1$ to $G_2$.

**Proof**

Let $\sigma_1, \sigma_2$ be two distinct isomorphisms from $G_1$ to
Take any \((u, v) \in V(G_1) \times V(G_2)\) where \(V(G_1), V(G_2)\) denote the vertex sets of \(G_1\) and \(G_2\) respectively. Let \(E(G_1), E(G_2)\) denote the edge sets of \(G_1\) and \(G_2\) respectively. Then

\((u, v) \in E(G_2)\) \iff \((\sigma_2^{-1}\sigma_1 u, \sigma_2^{-1}\sigma_1 v) \in E(G_1)\).

So \(\sigma_2^{-1}\sigma_1\) is an automorphism of \(G_2\), and \(\sigma_2^{-1}\sigma_1 \neq e\) contradicts the fact that \(G_1\) is an identity graph. The lemma then follows.

**Corollary 2.1**

If \(G_1, G_2, \ldots, G_k\) are isomorphic identity graphs, then

\[\Gamma(G) \equiv S_k, \text{ the symmetric group on } k \text{ elements and} \]

\[|\Gamma(G)| = k! \]

where \(G = G_1 \cup G_2 \cup \ldots \cup G_k\).

**Theorem 2.5**

Let the minimal polynomial \(p\) of a graph \(G\) be irreducible, and

\[f = p^k\] be the characteristic polynomial of the graph
where k is a positive integer. Then*

\[ \Gamma(G) = \sum_{i=1}^{r} S_{k_i} \]

and \(|\Gamma(G)| = k_1! k_2! \ldots k_r!\),

for some \(k_1, k_2, \ldots, k_r\) such that

\[ k = k_1 + k_2 + \ldots + k_r. \]

**Proof**

By Theorem 2.4, G contains k components. Now let us collect the k components into groups of mutually isomorphic graphs and let us assume that there are r such groups with \(k_1, k_2, \ldots, k_r\) connected graphs in each group.

Consider the set of automorphisms that fix all vertices except those of graphs in the \(i\)-th group. This set is isomorphic to \(S_{k_i}\). Therefore

\[ \Gamma(G) = S_{k_1} + S_{k_2} + \ldots + S_{k_r} \]

and

\[ |\Gamma(G)| = k_1! k_2! \ldots k_r! \]

with \(k_1 + k_2 + \ldots + k_r = k\).

We note that Theorem 2.5 assumes that there exist at least \(r\) cospectral connected identity graphs. When all the

* See [16, p.163] or Section 1.2 for definitions of operations on permutation groups.
components are isomorphic, we have $r$ equal to 1. The following theorems are proved in [25] and [22].

**Theorem 2.6** [25]

Given any positive integer $k$, there exists an integer $n$ such that there are at least $k$ non-isomorphic connected regular graphs with $n$ vertices all having the same characteristic polynomial.

**Theorem 2.7** [22]

For any positive integer $k$ there are $k$ cospectral graphs with given automorphism group.

Thus we are led to conjecture that there exist non-isomorphic, cospectral, connected identity graphs whose common characteristic polynomial is irreducible.

### 2.2 Reducible Minimal Polynomial

Since the diagonal entries of the adjacency matrix of a graph are always equal to zero, we see immediately that if the minimal polynomial of a graph is linear then it must be the monic polynomial $x$ and this is true if and only if the graph is a void graph.

Next we consider the case when the minimal polynomial is quadratic.
Theorem 2.8

Let \( p = x^2 + ax + b \) be the minimal polynomial of a graph \( G \) with \( n \) vertices. Then \( b \leq 0 \) and \( G \) consists of \( n/(1 - b) \) copies of complete graphs of order \( 1 - b \) and 
\[
p = (x + 1)(x + b).
\]

Proof

If \( A \) is the adjacency matrix of \( G \), we have
\[
p = x^2 + ax + b \quad \text{and} \quad A^2 + aA + bI = 0.
\]
Let \( B = [b_{ij}] = A^2 \); then

(i) \( b_{ii} \) = degree of vertex \( i \),

(ii) \( b_{ij} \) = number of vertices adjacent to both \( i \) and \( j \) \((i \neq j)\).

Since \( B = -aA - bI \) and \( a_{ii} = 0 \), (i) implies \( G \) is regular of degree \( -b \) and so \( b \leq 0 \).

Let \( E(G) \) be the edge set of \( G \). If \( (i, j) \notin E(G) \), \( i \neq j \), then the number of vertices adjacent to both \( i \) and \( j \) equals 0; and if \( (i, j) \in E(G) \), then the number of vertices adjacent to both \( i \) and \( j \) equals \( -a \).

Now take any two vertices \( i \) and \( j \) \((i \neq j)\) such that \((i, j) \notin E(G)\). Let
\[
S_i = \{k | (i, k) \in E(G)\}, \quad S_j = \{k | (j, k) \in E(G)\}.
\]
Clearly
\[
S_i \cap S_j = \emptyset.
\]
Take any two vertices \( k, l \in S_i \).

\((k, l) \notin E(G)\) implies that the number of vertices adjacent to both \( k \) and \( l \) is 0, but \( i \) is adjacent to both \( k \)
and 1, so we must have

\((k, 1) \in E(G)\).

Therefore every pair of vertices in \(S_i\) is adjacent. Also every vertex in \(S_i\) is adjacent to \(i\); hence \(S_i \cup \{i\}\) is a complete graph.

Since the number of vertices adjacent to both \(k\) and \(l\) is \(-a\), the order of \(S_i \cup \{i\}\) is \(-a + 2\).

Similarly for \(S_i \cup \{j\}\).

So \(G\) consists of \(n/(2 - a)\) copies of complete graphs of order \(2 - a\); but \(G\) is regular of degree \(-b\), so

\[-b = -a + 1.\]

Therefore \(G\) consists of \(n/(1 - b)\) copies of complete graphs of order \(1 - b\).

Now \(-b = -a + 1\), so that \(a = b + 1\).

Therefore \(p = x^2 + ax + b = x^2 + (b + 1)x + b = (x + 1)(x + b)\).

**Corollary 2.2**

If \(p = (x + 1)(x + b)\) is the minimal polynomial of a graph \(G\) with \(n\) vertices, then*

\[\Gamma(G) = S_{-b}^n S_{1-b}^{1-b}\]

and \(|\Gamma(G)| = (n/(1 - b))!(1 - b)!^n/(1 - b)\).

* See [16, p. 164] or Section 1.2 for a definition of the wreath product \(A[B]\) of \(A\) around \(B\).
Now let us consider some operations on graphs (as defined in Section 1.2) and their effects on the characteristic polynomials of the component graphs.

**Definition 2.2**

A graph $G$ is said to be integral (or, equivalently, to have integral spectrum) if all the eigenvalues of $G$ are integers.

It follows from Definition 2.2 that the minimal and characteristic polynomials of an integral graph consist only of linear factors.

The following theorem on complements of regular graphs is due to Sachs [34].

**Theorem 2.9**

Let $G$ be a regular graph of degree $r$ and with $n$ vertices, then

$$f(\overline{G}; x) = (-1)^n \frac{x - n + r + 1}{x + r + 1} f(G; -x - 1).$$

**Corollary 2.3** [19]

If a regular graph $G$ is integral then so is $\overline{G}$.

The following results are proved in [35] and [19].
Theorem 2.10 [35]

Let \( f(G; x) = \prod (x - \lambda_i) \) and \( f(H; x) = \prod (x - \mu_j) \) be the characteristic polynomials of two graphs \( G \) and \( H \).

Then*

\[
\begin{align*}
    f(G \times H; x) &= \prod \prod (x - \lambda_i - \mu_j) \\
    f(G \wedge H; x) &= \prod \prod (x - \lambda_i \mu_j) \\
    f(G \ast H; x) &= \prod \prod (x - \lambda_i \mu_j - \lambda_i - \mu_j) .
\end{align*}
\]

Corollary 2.4 [19]

If \( G_1 \) and \( G_2 \) are integral, then so are \( G_1 \times G_2 \), \( G_1 \wedge G_2 \) and \( G_1 \ast G_2 \).

Theorem 2.11 [35]

If \( G_1 \) and \( G_2 \) are regular graphs of degrees \( r_1 \) and \( r_2 \) and with \( n_1 \) and \( n_2 \) vertices respectively, then

\[
    f(G_1 + G_2; x) = (x^2 - (r_1 + r_2) x + r_1 r_2 - n_1 n_2) \frac{f(G_1) f(G_2)}{(x - r_1)(x - r_2)} .
\]

Corollary 2.5 [19]

The join \( G_1 \ast G_2 \) of two regular graphs is integral if and only if both \( G_1 \) and \( G_2 \) are integral and

\[(r_1 - r_2)^2 + 4n_1 n_2 \text{ is a perfect square.}\]

* See [35] and [16] for definitions of the operations on graphs quoted in this theorem.
Next, we are going to consider families of graphs with linear factors in their minimal polynomials.

(i) **Complete graphs and void graphs**

Let $n$ be $\geq 2$. Then

$$f(K_n; x) = (x - n + 1)(x + 1)^{n-1},$$

so that

$$p(K_n; x) = (x + 1)(x - n + 1).$$

The complement $\overline{K}_n$ of $K_n$ is the void graph on $n$ vertices.

$$f(\overline{K}_n; x) = x^n$$

and

$$p(\overline{K}_n; x) = x.$$

Therefore all factors of the minimal polynomials of $K_n$ and $\overline{K}_n$ are linear for $n = 2, 3, \ldots$. Also we know that only void graphs can have linear minimal polynomials, and connected graphs with quadratic minimal polynomials are complete.

Since $\Gamma(G) = \Gamma(G)$ and $\Gamma(K_n) = S_n$, we have

$$\Gamma(K_n) = \Gamma(\overline{K}_n) = S_n,$$

and $|\Gamma(K_n)| = |\Gamma(\overline{K}_n)| = n!$.

(ii) **Complete bipartite graphs**

Let $K_{m,n}$ denote the complete bipartite graph (see [16, p.17]) on $V_1$ and $V_2$ where $|V_1| = m$, $|V_2| = n$.

Schwenk[35] has shown that

$$f(K_{m,n}; x) = (x^2 - mn)x^{m+n-2}$$

which implies

$$p(K_{m,n}; x) = x(x^2 - mn)$$

for $m + n > 2$.

Therefore $K_{m,n}$ is integral if and only if $mn$ is a
perfect square. Also if \( m = n \), \( K_{m,m} \) and \( \overline{K}_{m,m} \) are both integral.

\[
\overline{K}_{m,m} = 2K_m,
\]

\[
f(\overline{K}_{m,m} ; x) = (x - m + 1)^2 (x + 1)^{2m-2},
\]

and for \( m > 1 \)

\[
p(\overline{K}_{m,m} ; x) = (x - m + 1)(x + 1).
\]

Since \( K_{m,n} = \overline{K}_m + \overline{K}_n \), the above results are actually particular cases of Corollary 2.5.

If \( m \neq n \), then by Corollary 1.1,

\[
\Gamma(K_{m,n}) \equiv \Gamma(\overline{K}_m) + \Gamma(\overline{K}_n) = S_m + S_n.
\]

Therefore

\[
|\Gamma(K_{m,n})| = m!n!.
\]

If \( m = n \),

\[
\overline{K}_{m,m} = 2K_m \text{ implies } \Gamma(\overline{K}_{m,m}) = S_2[S_m].
\]

Hence

\[
\Gamma(K_{m,m}) = \Gamma(\overline{K}_{m,m}) = S_2[S_m]
\]

and

\[
|\Gamma(K_{m,m})| = |\Gamma(\overline{K}_{m,m})| = 2(m!)^2.
\]

When \( m = 1 \), \( n \geq 2 \), the resulting graphs are called claws (or stars). The star \( K_{1,n} \) is integral if and only if \( n \) is a perfect square, in which case we have

\[
f(K_{1,n}; x) = x^{n-1}(x^2 - n),
\]

\[
p(K_{1,n}; x) = x(x^2 - n),
\]

\[
\Gamma(K_{1,n}) \equiv S_n,
\]

and

\[
|\Gamma(K_{1,n})| = n!.
\]

(iii) Cubes

Let \( Q_n \) denote the \( n \)-dimensional cube. Then
It has been proved in [35] that

\[ f(Q_n; x) = \prod_{i=0}^{n} (x - n + 2i) \quad \text{and} \]

\[ p(Q_n; x) = \prod_{i=0}^{n} (x - n + 2i). \]

Thus \( Q_n \) is integral for all \( n \).

Since \( Q_n \) is regular of degree \( n \) for all \( n \) we also have that \( \overline{Q}_n \) is integral for all \( n \).

By Theorem 2.9, we have

\[ f(Q_n; x) = \frac{x - 2^n + n + 1}{x + n + 1} \prod_{i=0}^{n} (-x - 1 - n + 2i) \]

\[ = (x - 2^n + n + 1) \prod_{i=1}^{n} (x + 1 + n - 2i). \]

For \( n = 1 \), \( p(\overline{Q}_1; x) = x \);

\( n = 2 \), \( p(\overline{Q}_2; x) = (x - 1) (x + 1) \), and

\( n > 2 \),

\[ p(\overline{Q}_n; x) = (x - 2^n + n + 1) \prod_{i=1}^{n} (x + 1 + n - 2i). \]

The following theorem has been proved by Palmer [29] (for a definition of the exponentiation group, see [16, p. 177]).

**Theorem 2.12**

If \( G \) is connected and prime (with respect to the product defined in Section 1.2), then
\[ \Gamma(G^n) = \left[ \Gamma(G) \right]^S_n. \]

**Corollary 2.6**

\[ \Gamma(Q_n) = \Gamma(\bar{Q}_n) = [S_2]^S_n \]

and \[ |\Gamma(Q_n)| = |\Gamma(\bar{Q}_n)| = n!2^n. \]

**Proof**

This follows immediately from the fact that \( K_2 \) is prime and

\[ |B|^A| = |A| \cdot |B|^d \]

where \( d \) is the size of the set on which the permutation group \( A \) acts.

When \( n = 2 \),

\[ Q_2 = K_2 \times K_2 = C_4. \]

Therefore

\[ f(C_4; x) = \prod_{i=0}^{2} (x - 2 + 2i) \]

\[ = x^2(x - 2)(x + 2), \]

and \[ p(C_4; x) = x(x - 2)(x + 2). \]

Also \[ \Gamma(C_4) = \Gamma(\bar{C}_4) = [S_2]^S_2 \cong D_4, \] the dihedral group on four elements, and hence

\[ |\Gamma(C_4)| = |\Gamma(\bar{C}_4)| = 8. \]

(iv) **Trees of diameter 3**

If \( T \) is a tree with diameter 3, it is easy to see that \( T \)'s center is a pair of adjacent vertices. Moreover, \( T \) is completely characterized by the degrees of the two
center vertices.

The following theorem concerning the coefficients of the characteristic polynomial of an arbitrary tree is proved by Mowshowitz[25].

**Theorem 2.13**

Let $T$ be a tree of order $n$, then for $0 \leq k \leq n$, the $k$-th coefficient $a_k$ of

$$f(T) = \sum_{i=0}^{n} (-1)^i a_i x^{n-i}$$

is given by

$$a_k = \begin{cases} (-1)^r h_r(T), & \text{if } k = 2r \text{ for some } r \geq 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $h_r(T) = \text{number of collections of } r \text{ nonadjacent edges in } T$.

**Corollary 2.7**

If $T$ is a tree of diameter 3, and $m$, $n$ are the degrees of the center vertices then

$$f(T; x) = x^m + x^n - 4(x^4 - (m + n - 1)x^2 + (m - 1)(n - 1)).$$

**Proof**

$h_1(T) = \text{number of edges in } T = m + n - 1$.

Obviously,

$h_2(T) = (m - 1)(n - 1)$

and $h_r(T) = 0$

for all $r \geq 3$.

Therefore
Let \( f(T; x) = x^{m+n} - (m + n - 1)x^{m+n-2} + (m - 1)(n - 1)x^{m+n-4} \)
\[ = x^{m+n-4}(x^4 - (m + n - 1)x^2 + (m - 1)(n - 1)). \]

![Diagram of a tree with diameter 3](image)

**Figure 2.1: A Tree of Diameter 3**

**Corollary 2.8**

If \( T \) is a tree of diameter 3, and \( m, n \) are the degrees of the center vertices, then \( T \) is integral if and only if the following are satisfied:

(i) \((m + n - 1)^2 - 4(m - 1)(n - 1) = k^2\)

for some integer \( k \), and

(ii) if (i) holds for some integer \( k \) then

\((m + n - 1) + k\)/2 and \((m + n - 1) - k\)/2

are perfect squares.

**Proof**

Obvious from the characteristic polynomial of \( T \).

In particular, if \( m = n \) we have the following theorem.
Theorem 2.14

If \( T \) is a tree of diameter 3, and the degrees of the center vertices are both equal to \( m \), then \( T \) is integral if and only if

\[ m = p^2 + p + 1 \]

for some positive integer \( p \).

Proof

If \( m = n = p^2 + p + 1 \) for some positive integer \( p \), then

\[
\begin{align*}
(m + n - 1)^2 - 4(m - 1)(n - 1) &= (2p^2 + 2p + 1)^2 - 4(p^2 + p)^2 \\
&= 4p^4 + 4p^2 + 1 + 8p^3 + 4p^2 + 4p - 4p^4 - 8p^3 - 4p^2 \\
&= 4p^2 + 4p + 1 = (2p + 1)^2.
\end{align*}
\]

Let \( k = 2p + 1 \). Then

\[
\begin{align*}
((m + n - 1) + k)/2 &= (2p^2 + 2p + 1 + 2p + 1)/2 \\
&= (2p^2 + 4p + 2)/2 = p^2 + 2p + 1 = (p + 1)^2, \\
and
((m + n - 1) - k)/2 &= (2p^2 + 2p + 1 - 2p - 1)/2 \\
&= p^2, \text{ which implies } T \text{ is integral.}
\end{align*}
\]

Now suppose \( T \) is integral. Then

\[
\begin{align*}
(m + n - 1)^2 - 4(m - 1)(n - 1) &= (2m - 1)^2 - 4(m - 1)^2 \\
&= 4m^2 - 4m + 1 - 4m^2 + 8m - 4 \\
&= 4m - 3.
\end{align*}
\]

If \( 4m - 3 = k^2 \) for some positive integer \( k \), then

\[
\begin{align*}
((m + n - 1) + k)/2 \text{ and } ((m + n - 1) - k)/2 \text{ must be perfect squares.}
\end{align*}
\]
Let
\[(m + n - 1) + k)/2 = ((2m - 1) + k)/2 = q^2 \]
\[((m + n - 1) - k)/2 = ((2m - 1) - k)/2 = p^2 \]
for some positive integers p and q. Obviously we have \(q > p\).

Then \(p^2 + q^2 = 2m - 1\),
and \(p^2q^2 = ((2m - 1)^2 - 4m + 3)/4\)
\[= (4m^2 - 4m + 1 - 4m + 3)/4\]
\[= (4m^2 - 8m + 4)/4 = m^2 - 2m + 1 = (m - 1)^2.\]
Therefore \(m - 1 = pg\)
and \(p^2 + q^2 = 2(pg + 1) - 1 = 2pq + 1\), which implies
\((p - q)^2 = 1\), so that \(q = p + 1\).
Hence \(m = pg + 1 = p(p + 1) + 1 = p^2 + p + 1\).

Integral trees of diameter 3 having unequal degrees for the center vertices are more difficult to find. A list of all integral trees of order \(\leq 500\) is given in Table III of the Appendix. From this we see that the smallest integral tree of diameter 3 with unequal degrees for the center vertices is one with \(m = 51\) and \(n = 99\).

The automorphism group of a tree with diameter 3 can be determined quite easily. If \(T\) is a tree of diameter 3 having center vertices of degrees \(m\) and \(n\) respectively, then
\[\Gamma(T) \equiv S_m \cdot S_n,\]
\[|\Gamma(T)| = (m - 1)! (n - 1)! \text{ for } m \neq n;\]
and $|\Gamma(T)| \equiv S_2[S_{m-1}]$

$|\Gamma(T)| = 2((m - 1)!)^2$, for $m = n$.

(v) **Subdivision graphs of stars $K_{l,n}$**

Let $G_n$ denote the subdivision graph of $K_{l,n}$.

![Subdivision Graph of $K_{l,4}$]

We are going to use the following theorem by Schwenk[35] to compute $f(G_n; x)$.

**Theorem 2.15**

Let $v$ be a vertex of a graph $G$ and let $C(v)$ be the collection of cycles containing $v$. Then

$$f(G) = xf(G - v) - \sum_{u \text{ adj } v} f(G - v - u)$$

$$- 2 \sum_{Z \in C(v)} f(G - V(Z))$$

where $V(Z)$ is the set of vertices contained in $Z$. 
Corollary 2.9

\[ f(G_n; x) = x(x^2 - n - 1)(x - 1)^{n-1}(x + 1)^{n-1}. \]

Proof

Let \( v \) be the vertex of degree \( n \) in \( G_n \). Then \( G_n - v \) consists of \( n \) copies of \( K_2 \). Therefore

\[ f(G_n - v; x) = (x - 1)^n (x + 1)^n. \]

For any \( u \) adjacent to \( v \), \( G_n - u - v \) consists of an isolated vertex and \( n - 1 \) copies of \( K_2 \).

Hence \( f(G_n - v - u; x) = x(x - 1)^{n-1}(x + 1)^{n-1}. \)

Since there are \( n \) vertices adjacent to \( v \),

\[ \sum_{u \text{ adj } v} f(G_n - v - u; x) = nx(x - 1)^{n-1}(x + 1)^{n-1}. \]

Since \( G_n \) is a tree, \( C(v) = \emptyset \).

So we have

\[ f(G_n; x) = x(x - 1)^n (x + 1)^n - nx(x - 1)^{n-1}(x + 1)^{n-1} \]

\[ = x(x - 1)^{n-1}(x + 1)^{n-1}((x - 1)(x + 1) - n) \]

\[ = x(x - 1)^{n-1}(x + 1)^{n-1}(x^2 - n - 1). \]

Thus \( G_n \) is integral if and only if \( n + 1 \) is a perfect square, i.e. when \( n = p^2 - 1 \) for some positive integer \( p \).

It is obvious that

\[ \Gamma(G_n) \equiv S_n \text{ and } |\Gamma(G_n)| = n!. \]

Obviously, there are other integral graphs which do not fall into any of the above mentioned families. Some of them are composite graphs of integral graphs such as the one shown in Figure 2.3.
Some other integral graphs which have not yet been classified are shown in Figure 2.4 (see also [19]).

(i) \quad (ii)

\begin{align*}
\mathcal{C}_6 & \quad G \\
 f(\mathcal{C}_6; x) & = (x - 1)^2 (x + 1)^2 (x - 2) (x + 2) \\
 f(G; x) & = x(x + 1)(x - 1)^2 (x + 2)^2 (x - 3) 
\end{align*}

We note that \( \mathcal{C}_6 = \Pi(3, 2, 1) \), the symmetric block design with parameters \( v = 3, k = 2 \) and \( \lambda = 1 \).

Obviously many graphs that have both linear and
non-linear factors can be found in families of graphs described in (ii), (iv) and (v) above.

Next we are going to state a theorem by Schwenk[35] and use it to compute the characteristic polynomials of a family of graphs.

**Theorem 2.16**

Let \( uv \) denote a line of \( G \) joining the vertices \( u \) and \( v \) and let \( C(uv) \) be the set of all cycles containing this line. The characteristic polynomial of \( G \) satisfies

\[
f(G) = f(G - uv) - f(G - u - v) - 2 \sum_{Z \in C(uv)} f(G - V(Z))\]

where \( V(Z) \) is the set of vertices contained in \( Z \).

We shall use this theorem to compute the characteristic polynomial of \( K_n - uv \) where \( uv \) is the line in \( K_n \) (\( n \geq 3 \)) joining any two vertices \( u \) and \( v \).

By Theorem 2.16, we have

\[
f(K_n - uv) = f(K_n) + f(K_n - u - v)
+ 2 \sum_{Z \in C(uv)} f(K_n - V(Z))
= (x - n + 1) (x + 1)^{n-1}
+ (x - n + 3) (x + 1)^{n-3}
+ 2 \sum_{Z \in C(uv)} f(K_n - V(Z)).
\]

Let \( Z \) be a cycle of length \( k + 2 \) containing \( uv \). Then
\[ f(K_n - \nu(\mathcal{Z})) = (x - n + k + 3)(x + 1)^{n-k-3}. \]

Obviously there are \( n-2 \) such cycles; therefore
\[ f(K_n - \nu) = (x - n + 1)(x + 1)^{n-1} + (x - n + 3)(x + 1)^{n-3} \]
\[
+ 2 \sum_{k=1}^{n-2} \binom{n-2}{k} (x - n + k + 3)(x + 1)^{n-k-3}.
\]

Using induction we can show that this expression is equivalent to
\[ x(x + 1)^{n-3}(x^2 - (n - 3)x - 2(n - 2)). \]

So we have
\[ f(K_n - \nu) = x(x + 1)^{n-3}(x^2 - (n - 3)x - 2(n - 2)). \]
III. ADJACENCY MATRIX AND AUTOMORPHISM GROUP

Two main topics will be studied in this chapter. First we will look at graph properties that correspond to spectral properties of the adjacency matrix. Then the relationship between the characteristic polynomial of the adjacency matrix and the automorphism group will be examined.

3.1 Spectrum of a Graph

As defined in Section 1.5, the spectrum of a graph is the set of all eigenvalues (with repetitions) of the adjacency matrix; the index of a graph G, denoted by r, is the largest value in its spectrum. If G is connected, then by the Frobenius–Perron theorem, r is a simple root of the characteristic polynomial of A.

In this section, we will consider graph properties that can be characterized by the properties of its spectrum. The following theorem is a simple result for graphs with small index.

Theorem 3.1

If \( r < 2 \), then every component of G is a tree.

Proof

If \( G_1 \) is a component of G which is not a tree then \( G_1 \) contains a cycle, say \( v_1, v_2, \ldots, v_p, v_1 \).

Let \( \mathbf{b} \) be the eigenvector for \( A(G_1) \) corresponding to
the index of $G_1$, such that $b_1, b_2, \ldots, b_p$ are the entries in $b$ corresponding to $v_1, v_2, \ldots, v_p$, respectively.

By the Frobenius-Perron Theorem, all entries of $b$ are positive, and since (i) $v_1$ is adjacent to $v_t, v_p$, (ii) $v_p$ is adjacent to $v_t, v_{p-1}$, and (iii) for $i = 2, 3, \ldots, p - 1$, $v_i$ is adjacent to $v_{i-1}, v_{i+1}$, we have

$$r, b_1 \geq b_2 + b_p$$
$$r, b_p \geq b_1 + b_{p-1}$$

and $r, b_i \geq b_{i-1} + b_{i+1}$ for $i = 2, 3, \ldots, p - 1$.

Hence

$$r, \sum_{i=1}^{p} b_i \geq 2 \sum_{i=1}^{p} b_i$$

which implies $r, \geq 2$; but $r, \leq r$ implies $r \geq 2$.

So for $r < 2$ every component of $G$ must be a tree.

Smith[36] proved a stronger result as stated in the following theorem.

**Theorem 3.2**

The index of a graph is $< 2$ if and only if each component of $G$ is a proper subgraph of one of the graphs in Figure 3.1.
The following theorem is given in [5].

**Theorem 3.3**

Let $r$ be the index of a finite, connected graph $G$ with $n$ vertices, and let $q_{\text{min}}$, $\bar{q}$ and $q_{\text{max}}$, respectively, be the minimum, average, and maximum degree of the vertices. Then the following inequalities hold:

(i) $q_{\text{min}} \leq \bar{q} \leq r \leq q_{\text{max}}$,

(ii) $2\cos \frac{\pi}{n + 1} \leq r \leq n - 1$.

We will now present some other inequalities on the
index of a graph and characterize those graphs which attain equalities.

**Theorem 3.4**

If \( k \) is the maximum degree* of a connected graph \( G \), then \( r^2 \geq k \geq r \).

**Proof**

From Theorem 3.3, we have \( k \geq r \).

Now let \( A(G) \) be the adjacency matrix of \( G \) with vertex set \( \{v_1, v_2, \ldots, v_n\} \) in which \( v_i \) is of degree \( k \). Also let \((a, b_2, b_3, \ldots, b_n)\) be the eigenvector of \( A(G) \) associated with \( r \).

Then \( ra = \sum_{i \in K} b_i \)

where \( K = \{i \mid v_i \text{ is adjacent to } v_1\} \).

Therefore \( |K| = k \).

Now for each \( i \in K \),

\( rb_i = a + \sum_{j \in K_i} b_j \)

where \( K_i = \{j \mid v_j \text{ is adjacent to } v_i, j \neq 1\} \).

Hence

\[ r \sum_{i \in K} b_i = \sum_{i \in K} a + \sum_{i \in K} \sum_{j \in K_i} b_j. \]

By the Frobenius-Perron Theorem, \( a > 0 \) and \( b_j > 0 \) for

* The maximum degree of a graph \( G \) will also be denoted by \( k \) in Corollaries 3.1 and 3.2.
j = 2, 3, ..., n.

Therefore

\[ r^2 a \geq ka \]

and \( r^2 \geq k \).

**Corollary 3.1**

G is a connected graph with \( r^2 = k \) if and only if G is a claw.

**Proof**

If G is a claw with n vertices, then

\[ k = n - 1 \] and \( f(G; x) = x^{n-2}(x^2 - n + 1) \). This implies

\[ r = \sqrt{n - 1}, \] and thus

\[ r^2 = k. \]

Now assume that \( r^2 = k \). Then we have

\[ r^2 a = ka + \sum_{i \in K} \sum_{j \in K_i} b_{ij}, \] which implies

\[ \sum_{i \in K} \sum_{j \in K_i} b_{ij} = 0. \] From this it follows that if \( K_i \neq \emptyset \)

\[ b_{ij} = 0 \] for all \( j \in K_i, i \in K \).

Since G is connected \( b_{ij} > 0 \) for all \( j = 2, 3, ..., n \),

so that \( K_i = \emptyset \) for all \( i \in K \), and \( v_i \) is adjacent only to \( v_i \) for all \( i \in K \).

Therefore the induced subgraph with vertex set \( \{v_i\} \cup \{v_i \mid i \in K\} \) is a claw.

* See p.33 for the definition of a claw.
Now deg \( v_i = k \), deg \( v_i = 1 \), for all \( i \in K \) implies that no other vertex in \( G \) is connected to any of the vertices in \( \{v_i\} \cup \{v_i | i \in K\} \); but this contradicts the fact that \( G \) is connected. Hence \( V(G) = \{v_i\} \cup \{v_i | i \in K\} \), and \( G \) is a claw.

**Corollary 3.2**

\( G \) is a connected graph with \( r = k \) if and only if \( G \) is regular of degree \( r \).

**Proof**

If \( G \) is regular of degree \( r \), then obviously \( k = r \).

Now assume \( r = k \),

\[ d_j = \text{degree of vertex } v_j \text{ with } d_i = k, \text{ and } A \text{ is the adjacency matrix of } G \text{ with vertex set } \{v_1, v_2, \ldots, v_n\}. \]

Also let \((b_1, b_2, \ldots, b_n)'\) be an eigenvector of \( A \) associated with \( r \). Then we have

\[
A(b_1, b_2, \ldots, b_n)' = r(b_1, b_2, \ldots, b_n)'
\]

i.e. \( a_{11} b_1 + a_{12} b_2 + \ldots + a_{1n} b_n = rb_1 \),

\( a_{21} b_1 + a_{22} b_2 + \ldots + a_{2n} b_n = rb_2 \)

\( \ldots \)

\( a_{n1} b_1 + a_{n2} b_2 + \ldots + a_{nn} b_n = rb_n \).

Rewriting we have
\[(a_1 - r)b_1 + a_{12}b_2 + \ldots + a_{1n}b_n = 0\]
\[a_{21}b_1 + (a_{22} - r)b_2 + \ldots + a_{2n}b_n = 0\]
\[\ldots\]
\[a_{n1}b_1 + a_{n2}b_2 + \ldots + (a_{nn} - r)b_n = 0.\]

Clearly,
\[\left(\sum_{i=1}^{n} a_{i1}\right) - r\right) b_1 + \left(\sum_{i=1}^{n} a_{i2}\right) - r\right) b_2 + \ldots + \left(\sum_{i=1}^{n} a_{in}\right) - r\right) b_n = 0.\]

This implies
\[(k - r)b_1 + (d_2 - r)b_2 + \ldots + (d_n - r)b_n = 0.\]

Now \(r = k\) implies \(r \geq d_i\) for all \(i = 1, 2, \ldots, n\).

Since \(b_i > 0\) for all \(i = 1, 2, \ldots, n\),
\(d_i = r\) for all \(i = 1, 2, \ldots, n\).

In Theorem 3.3, we know that the index \(r\) of a connected graph \(G\) satisfies the following inequality:
\[2\cos \frac{\pi}{n+1} \leq r \leq n - 1.\]

By Theorem 3.1, the lower bound can be raised to 2 if \(G\) is not a tree. Hence, for any connected graph which is not a tree we have
\[2 \leq r \leq n - 1.\]

Obviously for any graph with \(n\) vertices, the maximum possible degree is \(n - 1.\). So if \(r = n - 1\), then by
Corollary 3.2, G is regular of degree \( n - 1 \), i.e. \( G = K_n \).

Now for any graph \( G \) which is not the complete graph, \( G \) must be a subgraph of \( K_n - uv \) for some \( uv \) such that \( uv \notin E(G) \).

By the Probenius-Perron Theorem, we know that the index of \( G \) is less than or equal to the index of \( K_n - uv \), which is equal to

\[
\frac{(n - 3 + \sqrt{(n - 3)^2 + 8(n - 2)})}{2} \quad (\text{see Section 2.2}).
\]

So we have the following inequality for the index \( r \) of a graph \( G \) which is not the complete graph:

\[
r \leq \frac{(n - 3 + \sqrt{(n - 3)^2 + 8(n - 2)})}{2}.
\]

The spectrum of a graph contains the index and hence should provide more information on the structure of the graph.

The following theorem, as stated in [7], indicates how a regular graph can be recognized by its spectrum.

**Theorem 3.5**

Let \( \{ \lambda_1 = r, \lambda_2, \lambda_3, \ldots, \lambda_n \} \) be the spectrum of a graph \( G \), where \( r \) denotes the index of \( G \). Then \( G \) is regular if and only if

\[
\frac{1}{n} \sum_{i=1}^{n} \lambda_i^2 = r.
\]

More results on the characterization of a graph by means of its spectrum can be found in [5], [7] and [41].

The foregoing results on characterization of a graph
by means of its spectrum obviously depend on the computation of the characteristic roots of the graph. Usually some knowledge of the structure of a graph will help in computing its characteristic polynomial, and thus its spectrum. In [5], the symmetrical properties of a graph have been used to simplify the calculation of the characteristic polynomial. We will show one method in which this can be done.

Let \( G \) be a graph with vertex set \( \{u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_m; v_{m+1}, v_{m+2}, \ldots, v_{2m}\} \) such that the sets \( V_1 = \{v_1, v_2, \ldots, v_m\} \) and \( V_2 = \{v_{m+1}, v_{m+2}, \ldots, v_{2m}\} \) are symmetrical about \( U = \{u_1, u_2, \ldots, u_p\} \). That is to say, for \( i, j = 1, 2, \ldots, m \),

\[
(v_i, v_j) \in E(G) \iff (v_{m+i}, v_{m+j}) \in E(G),
\]

\[
(v_i, v_{m+j}) \in E(G) \iff (v_{m+i}, v_j) \in E(G),
\]

and for any vertex \( u \in U \),

\[
(v_i, u) \in E(G) \iff (v_{m+i}, u) \in E(G).
\]

Hence we can partition the adjacency matrix \( A(G) \) as follows:

\[
A = \begin{bmatrix}
C & R & R \\
R' & P & Q \\
R' & Q & P
\end{bmatrix}
\]

where \( C = [c_{ij}] \), \( P = [p_{ij}] \), and \( Q = [q_{ij}] \).
\[ R = [r_{ij}] \]

and \( c_{ij} = \begin{cases} 1 & \text{if } u_i \text{ is adjacent to } u_j, \\ 0 & \text{otherwise}, \end{cases} \quad i, j = 1, 2, \ldots, p; \]

\[ p_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise}, \end{cases} \quad i, j = 1, 2, \ldots, m; \]

\[ q_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_{m+i}, \\ 0 & \text{otherwise}, \end{cases} \quad i, j = 1, 2, \ldots, m; \]

\[ r_{ij} = \begin{cases} 1 & \text{if } u_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, m. \]

Let \( I \) be the identity matrix of order \( 2m \times p \). Then

\[
\det(xI - A) = \det \begin{bmatrix} xI_p - C & -R & -R \\ -R' & xI_m - P & -Q \\ -R' & -Q & xI_m - P' \end{bmatrix}
\]

\[
= \det \begin{bmatrix} xI_p - C & -2R & -R \\ -R' & xI_m - P - Q & -Q \\ -R' & xI_m - P - Q & xI_m - P \end{bmatrix}
\]

\[
= \det \begin{bmatrix} xI_p - C & -2R & -R \\ -R' & xI_m - P - Q & -Q \\ 0 & 0 & xI_m - P + Q \end{bmatrix}
\]

\[
= \det(xI_m - P + Q) \det \begin{bmatrix} xI_p - C & -2R \\ -R' & xI_m - P - Q \end{bmatrix}.
\]

Hence the calculation of the characteristic polynomial of \( A \) is reduced to calculating the
characteristic polynomials of two (three if $R = 0$) smaller matrices. The following example will illustrate this method more clearly.

**Example 3.1**

Consider the graph $G$ shown in Figure 3.2.

![Figure 3.2: A Graph with Symmetry](image)

Here $U = \{5\}$, $V = \{1, 4\}$, $V = \{2, 3\}$; so

$C = [0]$,

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$Q = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$,

$R = [1 \\ 1]$, 

$\det(xI_2 - P + Q) = \det\begin{bmatrix} x & 0 \\ 0 & x + 1 \end{bmatrix} = x(x + 1)$,

$\det\begin{bmatrix} xI_2 - C & -2R \\ -R & xI_2 - P - Q \end{bmatrix}$
\[
\begin{bmatrix}
x & -2 & -2 \\
-1 & x & -2 \\
-1 & -2 & x - 1
\end{bmatrix}
\]

\[
= \det \begin{bmatrix}
x & -2 \\
-1 & x \\
-1 & -2
\end{bmatrix}
\]

\[= x^3 - x^2 - 8x + 2.\]

Therefore
\[f(G; x) = x(x + 1)(x^3 - x^2 - 8x + 6)
\]
\[= x(x + 1)^2 (x^2 - 2x - 6).\]

3.2 Characteristic Polynomial and Automorphism Group of a Graph

In this section the relationship between the characteristic polynomial and the automorphism group of a graph will be studied in detail.

The following theorem is proved by Mowshowitz [24].

**Theorem 3.6**

Let D be a digraph with adjacency matrix A, automorphism group \( \Gamma \), and characteristic polynomial \( f \) of degree \( n \); and let \( k \) be the number of orbits of \( \Gamma \). Then there exists a polynomial of degree \( k \) dividing \( f \).

For simplicity sake, we will restrict ourselves to undirected graphs; the generalization to digraphs will be discussed in the next chapter. Before we prove the next theorem we will need some lemmas.
Lemma 3.1

If $H$ is a group of automorphisms of a graph $G$, and if $R, S$ ($R$ may be equal to $S$) are any two orbits of $H$, then for $v_i, v_j \in R$,

the number of vertices in $S$ adjacent to $v_i$

$= \text{the number of vertices in } S \text{ adjacent to } v_j$.

Proof

If $v_i, v_j \in R$, then there exists $\sigma \in H$ such that $\sigma(v_i) = v_j$.

Take any $w \in S$ such that $(v_i, w) \in E(G)$. Then $(v_i, w) \in E(G)$ iff $(v_j, \sigma(w)) \in E(G)$,

and since $\sigma(w) \in S$, we have

number of vertices in $S$ adjacent to $v_i$

$= \text{number of vertices in } S \text{ adjacent to } v_j$.

Now let $H$ be a group of automorphisms of a graph $G$. We define the orbit-matrix and orbit-digraph as follows:

Definition 3.1

Let $P_1, P_2, \ldots, P_k$ be the orbits of $H$. For each $P_i$ take $v_i \in P_i$ to be its representative. The matrix

$M(H) = [m_{ij}]$

with $m_{ij} = \text{number of vertices in } P_j \text{ adjacent to } v_i$, $i, j = 1, 2, \ldots, k$, is called the orbit-matrix of $H$ (with
respect to \( G \)\). The weighted digraph with vertex set \( \{v_1, v_2, \ldots, v_k\} \) and arc weights \( (v_i, v_j) = m_{ij} \), is called the \textit{orbit-digraph} of \( H \).

Obviously the orbit-matrix and orbit-digraph of any group of automorphisms of a graph are independent of the choice of representatives. We will make this definition clear by the following example.

\textbf{Example 3.2}

Let \( G = K_{1,4} \), the claw on five vertices with centre \( \{1\} \), and \( H = \langle (25), (34) \rangle \). Obviously \( H \) is a group of automorphisms of \( G \). The orbits of \( H \) are \( \{1\} \), \( \{2, 5\} \) and \( \{3, 4\} \). Let 1, 2 and 3 be the representatives. Then

\[
M(H) = \begin{bmatrix}
0 & 2 & 2 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

and the corresponding orbit-digraph is shown in Figure 3.3.

\* The phrase in parentheses will be omitted whenever the term is clear from the context.
Now if \( H_1 \) is a group of automorphisms of a graph \( G \), \( H_2 \) a subgroup of \( H_1 \), and \( B_1, B_2, \ldots, B_k \) are the orbits of \( H_1 \) with sizes \( b_1, b_2, \ldots, b_k \), respectively, then \( H_2 \) has orbits \( A_{i1}, A_{i2}, \ldots, A_{i_{p_1}} \):

\[
A_{i1}, A_{i2}, \ldots, A_{i_{p_1}};
\]

\[
A_{i2}, A_{i2}, \ldots, A_{i_{p_2}};
\]

\[
\ldots
\]

\[
A_{i_k}, A_{i_k}, \ldots, A_{i_{p_k}}
\]

with sizes \( a_{i1}, a_{i2}, \ldots, a_{i_{p_1}} \):

\[
a_{i1}, a_{i2}, \ldots, a_{i_{p_1}};
\]

\[
a_{i2}, a_{i2}, \ldots, a_{i_{p_2}};
\]

\[
\ldots
\]

\[
a_{i_k}, a_{i_k}, \ldots, a_{i_{p_k}}
\]

respectively.

The \( A_{ij} \) and \( a_{ij} \) satisfy the relations.
Let $T = [t_{ij}]$ be the $k \times k$ matrix defined by
\[ t_{ij} = \text{number of vertices in } B_i \text{ adjacent to a vertex in } B_j, \]
i, j = 1, 2, \ldots, k.

Clearly $T = M(H_i)$, the orbit–matrix of $H_i$.

Let $S$ be the matrix partitioned into $k^2$ submatrices, where the $ij$–th submatrix
\[ S_{ij} = [s_{lm}^{ij}], \]
and $s_{lm}^{ij} = \text{number of vertices in } A_{jm} \text{ adjacent to a vertex in } A_{im}$,
i, j = 1, 2, \ldots, k, \quad l = 1, 2, \ldots, p_i; \quad m = 1, 2, \ldots, p_j.

$S = M(H_2)$ is the orbit–matrix of $H_2$.

Now we have the following lemma.

**Lemma 3.2**

The characteristic polynomial of $T$ divides that of $S$, i.e.
\[ \det(xI - T) | \det(xI - S). \]

**Proof**

Since $A_{im} \subseteq B_i$ and \[ \bigcup_{m=1}^{p_i} A_{jm} = B_j \]
i, j = 1, 2, \ldots, k,
and the $A_{jm}$'s are mutually disjoint, we have
\[ \sum_{m=1}^{p_j} s_{lm}^{ij} = t_{ij} \quad i, j = 1, 2, \ldots, k. \]

Now let $(z_1, z_2, \ldots, z_k)'$ be an eigenvector of $T$.
corresponding to the eigenvalue \( \lambda \). Then we have

\[
\sum_{j=1}^{k} t_{ij} z_j = \lambda z_i \quad \text{for all } i = 1, 2, \ldots, k.
\]

Consider the expression

\[
S(z_1, z_2, \ldots, z_i; z_1, z_2, \ldots, z_i; \ldots; z_k, z_k, \ldots, z_k)'.
\]

The \((p_1 + 1)\)-th term, where \(1 < 1 < p_1\), is

\[
\sum_{j=1}^{k} \left( \sum_{m=1}^{p_j} s_{k_m} \right) z_j = \sum_{j=1}^{k} t_{ij} z_j = \lambda z_i.
\]

Hence \( \lambda \) is also an eigenvalue of \( S \), from which it follows that

\[
\det(xI - T) | \det(xI - S).
\]

The next theorem follows readily from Lemma 3.2.

**Theorem 3.7**

Let \( H_1, H_2, \ldots, H_q \) be groups of automorphisms of a graph \( G \) with \( H_1 = \{e\} \), \( H_q = \Gamma(G) \) and \( H_i \) a subgroup of \( H_{i+1} \) for all \( i = 1, 2, \ldots, q - 1 \). If \( T_i = M(H_i) \), the orbit-matrix of \( H_i \), then we have

\[
f(T_q') | f(T_{q-1}'), f(T_{q-1}') | f(T_{q-2}'), \ldots, f(T_2) | f(T_1) = f(G).
\]

**Corollary 3.3**

If \( H \) is a group of automorphisms of a graph \( G \) and if \( H \) has \( k \) orbits, then there exists a polynomial of degree \( k \) that divides \( f(G) \).
The converse of Corollary 3.3 is, however, not true as shown by the following example.

Example 3.3

Consider the path on five vertices as shown in Figure 3.4. The characteristic polynomial is \( x(x - 1)(x + 1)(x^2 - 3) \) and the only nontrivial subgroup of the automorphism group is \( H = \{e, (15)(24)\} \), the automorphism group itself.

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

Figure 3.4: The Path on Five Vertices

\( H \) has three orbits \{1, 5\}, \{2, 4\} and \{3\}. The orbit-matrix of \( H \) with representatives 1, 2 and 3 is

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 2 & 0 \\
\end{bmatrix}
\]

The characteristic polynomial of this matrix is \( x(x^2 - 3) \). Hence by Lemma 3.2 or Theorem 3.7 there does not exist a group of automorphisms which has \( x, x - 1, x + 1, x^2 - 3, x(x - 1), x(x + 1), (x - 1)(x^2 - 3), (x - 1)(x + 1) \) or \( (x + 1)(x^2 - 3) \) as the characteristic polynomial of its orbit-matrix.

The following theorem is given by Yap in [42].
Theorem 3.8

The characteristic polynomial of the adjacency matrix $A$ of a weighted digraph $D$ with $n$ vertices whose automorphism group $\Gamma$ is non-trivial is the product of the characteristic polynomials of the following two matrices.

$$B = [b_{ij}], \quad C = [c_{ij}]$$

defined by:

1. $b_{ij} = \sum_{v_k \in P_i} a_{kj} \quad i, j = 1, 2, \ldots, k,$

$v_i$ is the representative of the $i$-th orbit, $P_i$,

$i = 1, 2, \ldots, k,$

$k = \text{number of orbits of } \Gamma.$

2. $c_{ij} = a_{k+i,k+j} - a_{k+i,j} \quad \text{if } v_{k+j} \in P_i$

$i, j = 1, 2, \ldots, n - k.$

Yap called the two factors the characteristic polynomial of the symmetric part of $A$ and the characteristic polynomial of the complementary part of $A$, respectively. It happens that the characteristic polynomial of the symmetric part of $A$ equals the characteristic polynomial of the orbit-matrix of $\Gamma(D)$.

We will now define a matrix $C(H)$ which is analogous to the complementary part of $A$.

Let $G$ be a graph and $H$ a nontrivial subgroup of the automorphism group of $G$; also let $P_1, P_2, \ldots, P_k$ be the orbits of $H$, and $v_1, v_2, \ldots, v_k$ the representatives used in computing $\mathbb{N}(H)$. 
Let $P_i = \{v_{i1}, v_{i2}, \ldots, v_{ip_i}\}$

$P_2 = \{v_{z1}, v_{z2}, \ldots, v_{zp_2}\}$

\[\vdots\]

$P_k = \{v_{k1}, v_{k2}, \ldots, v_{kp_k}\}$

where $p_i = |P_i|$. Let $B = \{i \mid p_i > 1\}$ and $|B| = b$.

Now we assume (relabeling indices if necessary) that $p_i > 1$ for $1 \leq i \leq b$

and $p_i = 1$ for $b < i \leq k$.

Also let $\mathcal{E}(u, v)$ be the adjacency function of $G$. Then the orbit-complement-matrix and orbit-complement-digraph of $H$ with respect to $G$ will be defined as follows.

**Definition 3.2**

Let $C(H)$ be the partitioned matrix with $b^2$ submatrices such that the $ij$-th submatrix

$C_{ij} = [c_{im}^{ij}]$

where $c_{im}^{ij} = \mathcal{E}(v_{i1}, v_{j1}, \ldots, v_{ip_i, v_{j1}, \ldots, v_{j(p_j - 1)}}$)

1 = 1, 2, \ldots, p_i - 1;

$m = 1, 2, \ldots, p_i - 1$; and

$i, j \in B$.

Then $C(H)$ is called the orbit-complement-matrix of $H$, and the weighted digraph with vertex set
and weights given by \( C(H) \) is called the **orbit-complement-digraph** of \( H \).

Then we have the following theorem (cf. [42]).

**Theorem 3.9**

Let \( H \) be a non-trivial subgroup of the automorphism group of a graph \( G \) with \( n \) vertices and \( D_1, D_2 \) the orbit-digraph and orbit-complement-digraph of \( H \) respectively. Then

\[
f(G) = f(D_1) f(D_2).\]

**Proof**

The proof is essentially the same as the proof of Theorem 2 in [42].

Let \( P_1, P_2, \ldots, P_k \) be the orbits of \( H \) and \( v_1, v_2, \ldots, v_k \) their representatives. Then if \( A \) is the adjacency matrix of \( G \) with vertex set \( \{v_1, v_2, \ldots, v_k; v_{k+1}, v_{k+2}, \ldots, v_n\} \), we have
\[
f(G) = \begin{bmatrix} x - a_{11} & -a_{12} & \cdots & -a_{1k} & -a_{1,k+1} & \cdots & -a_{1,n} \\ -a_{21} & x - a_{22} & \cdots & -a_{2k} & -a_{2,k+1} & \cdots & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{k,1} & -a_{k,2} & \cdots & x - a_{kk} & -a_{k,k+1} & \cdots & -a_{k,n} \\ -a_{k+1,1} & -a_{k+1,2} & \cdots & -a_{k+1,k} & x - a_{k+1,k+1} & \cdots & -a_{k+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \cdots & -a_{n,k} & -a_{n,k+1} & \cdots & x - a_{nn} \end{bmatrix}
\]

where \( a_{ij} \) is the \( ij \)-th element of \( A \), \( i, j = 1, 2, \ldots, n \).

If we add to column \( j \), \( j \leq k \), all columns \( 1 \geq k + 1 \), such that \( v_k \in P_j \), then we can write \( f(G) \) as follows:

\[
f(G) = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}
\]

where

\[
P = \begin{bmatrix} x - b_{11} & -b_{12} & \cdots & -b_{1k} \\ -b_{21} & x - b_{22} & \cdots & -b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{k1} & -b_{k2} & \cdots & x - b_{kk} \end{bmatrix},
\]

\[
b_{ij} = \sum_{v_k \in P_j} a_{ij} \quad i, j = 1, 2, \ldots, k.
\]
\[ Q = \begin{bmatrix}
-a_{1,k+1} & -a_{1,k+2} & \cdots & -a_{1,n} \\
-a_{2,k+1} & -a_{2,k+2} & \cdots & -a_{2,n} \\
& & \ddots & \\
-a_{k,k+1} & -a_{k,k+2} & \cdots & -a_{k,n}
\end{bmatrix} \]

\[ R = [r_i] \text{ where } r_i \text{ is the } i\text{-th row of } R \text{ and } \]
\[ r_i = (0, 0, \ldots, 0, x, 0, 0, \ldots, 0) \uparrow \text{ t-th position} \]
\[- (b_{t1}, b_{t2}, \ldots, b_{tk}) \text{ if } v_{k+i} \in P_t. \]
(Note that the entries in \( r_i \) are the same as those of the t-th row in \( P \) because)
\[ \sum_{v_k \in P_t} a_{k+i, \ell} = b_{tj}, \quad j = 1, 2, \ldots, k, \]
whenever \( v_{k+i} \in P_t. \}

\[ S = \begin{bmatrix}
x - a_{k+1,k+1} & -a_{k+1,k+2} & \cdots & -a_{k+1,n} \\
-a_{k+2,k+1} & x - a_{k+2,k+2} & \cdots & -a_{k+2,n} \\
& & \ddots & \\
-a_{n,k+1} & -a_{n,k+2} & \cdots & x - a_{nn}
\end{bmatrix} \]

Now for the i-th row in \([R \ S] \), if \( r_i \) contains the indeterminate \( x \) in the t-th position, then subtract from it the t-th row of \([P \ Q]\).

Then
\[ f(G) = \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \]

with
\[ T = xI_{n-k} - W \]

where \( W = [w_{ij}] \)

and \( w_{ij} = a_{k+i,k+j} - a_{t,k+j} \) if \( v_{k+i} \in P_t \),

\[ i, j = 1, 2, \ldots, n - k. \]

Obviously \( \det P = f(D_1) \) and \( \det W = f(D_2) \) and

\[ f(G) = (\det P) \times (\det W), \]

which implies \( f(G) = f(D_1)f(D_2) \).

Therefore, if we know a nontrivial subgroup of the automorphism group of a graph then the computation of its characteristic polynomial will be reduced to computing the characteristic polynomials of two smaller weighted digraphs. We will illustrate this by the following example.

**Example 3.4**

Consider the graph \( G \) shown in Figure 3.2. Obviously \( \{e, (12)(34)\} \) is a group of automorphisms and the corresponding orbit-matrix and orbit-complement-matrix with 1, 3 and 5 as the representatives of the orbits are given by:

\[
\begin{bmatrix}
0 & 2 & 1 \\
2 & 1 & 1 \\
2 & 2 & 0
\end{bmatrix}
\]

and
The characteristic polynomials are $x^3 - x^2 - 8x - 6$ and $x(x + 1)$ respectively. Hence

$$f(G) = x(x + 1)(x^3 - x^2 - 8x - 6) = x(x + 1)^2(x^2 - 2x - 6).$$

Note that the method used in Section 3.1 is a special case of Theorem 3.9.

### 3.3 Graphs With Labels

We will now study graphs whose vertices have labels attached to them. The following is a precise definition of graphs with labels.

**Definition 3.3**

A **graph with labels** is an ordered triple

$$G = (V(G), E(G), l_G)$$

where $V(G)$, $E(G)$ are the usual vertex set and edge set respectively, and $l_G$ is the labeling function defined on $V(G)$ to a set of labels, say $\{1, 2, \ldots, p\}$ where $p \geq 2$ is the number of distinct labels.

If $l_G(u) \neq l_G(v)$ for every pair of distinct vertices $u, v \in V(G)$, then we call $G$ a **labeled graph**.

The notion of automorphisms of graphs can be extended
to graphs with labels as follows.

**Definition 3.4**

$\sigma$ is an **automorphism** of $G = (V(G), E(G), l_G)$ if for all $u, v \in V(G)$,

$(u, v) \in E(G)$ iff $(\sigma(u), \sigma(v)) \in E(G)$

and $l_G(u) = l_G(\sigma(u))$.

Obviously the automorphisms of $G$ also form a group.

The usual characteristic polynomial of a graph $G$ is defined as $\det(xI - A(G))$. In order to take into account the labels of the vertices, we will also label the corresponding indeterminates.

**Definition 3.5**

Let $G = (V(G), E(G), l_G)$ with label set $\{1, 2, \ldots, p\}$ and vertex set $V(G) = \{v_1, v_2, \ldots, v_{k_1}, v_{k_1+1}, \ldots, v_{k_1+k_2}, \ldots, v_{k_1+k_2+\ldots+k_{p-1}+1}, \ldots, v_{k_1+k_2+\ldots+k_{p-1}+2}, \ldots, v_n\}$ such that $l_G(v_1) = l_G(v_2) = \ldots = l_G(v_{k_1}) = 1$,

$l_G(v_{k_1+1}) = l_G(v_{k_1+2}) = \ldots = l_G(v_{k_1+k_2}) = 2$,

\ldots

$l_G(v_{k_1+k_2+\ldots+k_{p-1}+1}) = l_G(v_{k_1+k_2+\ldots+k_{p-1}+2}) = \ldots$
The characteristic polynomial of $G$, denoted by $f(G)$, is the determinant of

$$
\begin{array}{cccc}
  x_1 & k_1 \text{ terms} & & \\
  x_1 & x_2 & k_2 \text{ terms} & \\
  x_1 & x_2 & x_3 & \ddots & & & -A(G) \\
  & & & & k_p \text{ terms} & \\
  & & & & x_p & \\
  & & & & & x_p
\end{array}
$$

We note that the characteristic polynomial of a graph with labels is a polynomial in $p$ indeterminates where $p$ is the number of distinct labels.

Theorem 3.9 can obviously be extended to graphs with labels as follows:

**Theorem 3.10**

Let $H$ be a non-trivial group of automorphisms of $G = (V(G), E(G), l_G)$ and $D_1$, $D_2$ the orbit-digraph and orbit-complement-digraph of $H$ respectively. Also let $v_1, v_2, \ldots, v_n$ be the vertices of $G$ such that $v_1, v_2, \ldots, v_k$ are the representatives of the orbits $P_1, P_2, \ldots, P_k$ of $H$ with
\( l_G(v_i) = l_i \quad i = 1, 2, \ldots, n. \)

Then \( f(G) = f(D_1)f(D_2), \)

where \( f(D_i) = \) the determinant of

\[
\begin{bmatrix}
x_{k_1} & 0 & & \\
& x_{k_2} & & \\
& & \ddots & \\
& & & x_{k_n}
\end{bmatrix} - M(H)
\]

and \( f(D_2) = \) the determinant of

\[
\begin{bmatrix}
x_{k+1} & 0 & & \\
& x_{k+2} & & \\
& & \ddots & \\
& & & x_{n}
\end{bmatrix} - C(H).
\]

Now we have the following result which is analogous to Corollary 3.3.

**Corollary 3.4**

If \( H \) is a group of automorphisms of a graph \( G = (V(G), E(G), l_G) \) with \( p \) types of labels, and if \( H \) has \( k_i \) orbits whose elements have label \( i \), then there exists a polynomial of degree \( k_i \) in \( x_i \) (for all \( i = 1, 2, \ldots, p \)) that divides \( f(G) \).

A list of graphs with two types of labels and at most four vertices is given in Table IV of the Appendix.
IV. EXTENSIONS AND APPLICATIONS

Various extensions and applications will be considered in this chapter. In particular, we will extend the concepts of adjacency matrices and automorphisms to digraphs and non-simple graphs.

4.1 Digraphs

As defined in Section 1.1, a digraph of order \( n \) is an ordered pair \( D = (V(D), E(D)) \) consisting of a nonempty set of vertices \( V(D) \) with \( |V(D)| = n \) and a set of lines \( E(D) \) such that each \( e \in E(D) \) is identified with an ordered pair \( (u, v) \) of distinct vertices \( u, v \in V(D) \).

Certain important concepts of a digraph will be given in the following definitions.

**Definition 4.1**

Let \( D \) be a digraph with vertex set \( V(D) = \{v_1, v_2, \ldots, v_n\} \). The matrix

\[
A(D) = [a_{ij}]
\]

where

\[
a_{ij} = \begin{cases} 
1 & \text{if there is a line in } E(D) \text{ from } v_i \text{ to } v_j, \\
0 & \text{otherwise},
\end{cases}
\]

is called the **adjacency matrix** of \( G \).

The characteristic polynomial \( f(D) \), and the minimal polynomial \( p(D) \) of \( D \) are given by \( f(A(D)) \) and \( p(A(D)) \),
respectively.

**Definition 4.2**

σ is an **automorphism** of a digraph D if σ is a permutation on the vertex set V(D) such that

\[
\sigma(u_i) = v_i \quad \text{implies} \quad \left\{ \begin{array}{ll}
(u_i, u_2) \text{ and } (v_i, v_2) \text{ are both in } E(D) \\
\sigma(u_1) = v_2 \end{array} \right.
\]

The main difference between the adjacency matrix of a digraph and that of a graph is the fact that the latter is symmetric. If, in a digraph D,

\[(u, v) \in E(D) \text{ iff } (v, u) \in E(D), \]

then A(D) will be identical to A(G) where G is the graph having the same vertex set and

\[(u, v) \in E(G) \text{ iff } (u, v) \in E(D). \]

So we can regard graphs simply as symmetrical digraphs.

Now, since the adjacency matrix of a digraph is nonnegative, all the foregoing results on adjacency matrices of graphs hold equally well for digraphs if the assumption that the adjacency matrix is symmetric is not required.

We will now explicitly state some of the results as extended to digraphs.
Theorem 4.1

Let \( H_1, H_2, \ldots, H_q \) be groups of automorphisms of a digraph \( D \) with \( H_1 = \{e\}, H_q = \Gamma(D) \) and \( H_i \) a subgroup of \( H_{i+1} \) for all \( i = 1, 2, \ldots, q - 1 \). If \( T_i = M(H_i) \) then

\[
\Gamma(D) = \bigcap_{i=1}^{q} T_i = \bigcap_{i=1}^{q} f(T_i) = f(T_q) = \bigcap_{i=1}^{q} f(T_q) = \bigcap_{i=1}^{q} f(T_{q-1}) = f(T_{q-1}) \bigcap f(T_{q-2}) = \cdots = f(T_2) \bigcap f(T_1) = f(D).
\]

Corollary 4.1

If \( f(D) \) is irreducible over the integers, then \( \Gamma(D) \) is trivial.

By virtue of Corollary 4.1 we expect to have a result analogous to Theorem 2.4. We will need the following concepts for a digraph.

Definition 4.3 (cf. [16, p. 198])

A path of length \( p \) from \( u \) to \( w \) in a digraph \( D \) is a sequence of distinct vertices \( v_0 = u, v_1, v_2, \ldots, v_{p-1}, v_p = w \) such that \((v_{i-1}, v_i) \in E(D)\) for all \( i = 1, 2, \ldots, p \). A vertex \( v \) is said to be reachable from \( u \) if there is a path from \( u \) to \( v \). A digraph is said to be strongly connected if every two vertices are mutually reachable. A strongly connected component of \( D \) is a maximal strongly connected subdigraph of \( D \).

We have the following analogue of Theorem 2.4.
Theorem 4.2

If $D$ is a digraph with an irreducible minimal polynomial, and characteristic polynomial $f = p^k$, then $D$ has $k$ strongly connected components which are cospectral identity digraphs with characteristic and minimal polynomials $p$.

The following is an example of a digraph which illustrates Corollary 4.1 and Theorem 4.2.

Example 4.1

The digraph $D$ shown in Figure 4.1 is the union of two digraphs which are converses of one another, i.e. one can be obtained from the other by reversing the directions of all the lines. Obviously digraphs that are converse to each other have the same characteristic polynomial because their adjacency matrices are transposes of one another.

![Figure 4.1](image)

**Figure 4.1**

An Identity Digraph

with Irreducible Characteristic Polynomial

and its Converse
Now \( f(D) = (x^4 - 2x^2 - x - 1)^2 \)

and \( p(D) = x^4 - 2x^2 - x - 1 \).

The two strongly connected components of \( D \) are identity digraphs because both of them have \( p(D) \) as the characteristic polynomial, and this polynomial is irreducible.

As introduced in Section 3.2, a weighted digraph is simply a digraph with weights attached to its lines. The adjacency matrix \( A(D) \) of a weighted digraph \( D \) with vertex set \( \{v_1, v_2, \ldots, v_n\} \) is given by

\[
A(D) = [a_{ij}]
\]

where

\[
a_{ij} = \begin{cases} 
\text{the weight on the line from } v_i \text{ to } v_j \text{ if } & \\
0 & \text{if there is no line from } v_i \text{ to } v_j. 
\end{cases}
\]

The edge set of a weighted digraph will be the set of lines which have nonzero weights.

### 4.2 Non-simple Graphs

If we allow loops and multiple edges in the edge set of a graph, then the graph will be a non-simple graph as defined in Section 1.1.

**Definition 4.4**

The adjacency function \( \Psi_G \) of a non-simple graph \( G = (V(G), E(G), \Psi_G) \) is defined as follows:
for \( u, v \in V(G) \)

\[ \psi_G(u, v) = \text{number of edges in } E(G) \text{ joining } u \text{ and } v. \]

Then the adjacency matrix of a non-simple graph will be defined as in Definition 1.6 with the above adjacency function. Note that the adjacency matrix of a non-simple graph is also nonnegative and symmetric; hence most of the results in Chapter II and Chapter III apply to non-simple graphs as well. Also we note that a non-simple graph can be viewed as a symmetric weighted digraph in which the weight on the line from \( u \) to \( v \) is \( \psi_G(u, v) \).

4.3 The Graph Isomorphism Problem

The relationship between the automorphism partitioning problem and graph isomorphism problem will be discussed in this section.

First, we assume that we have an algorithm to find the automorphism partitioning of a graph; then we will show how to use the algorithm to solve graph isomorphism problems.

Let \( G_1 \) and \( G_2 \) be two arbitrarily chosen graphs. Since components of \( G_1 \) must correspond to components of \( G_2 \) whenever \( G_1 \) and \( G_2 \) are isomorphic, it suffices to consider the case when \( G_1 \) and \( G_2 \) are both connected.

Let \( G, U G_2 = G \). Then \( G, \cong G_2 \) if and only if each orbit of \( \Gamma(G) \) contains elements of \( V(G_1) \) and \( V(G_2) \).

Now assume that we have an algorithm to solve the
The graph isomorphism problem. For any two vertices \( x, y \) of a graph \( G \), form the graph \( G_x \) with two types of labels by assigning \( x \) label 1 and the other vertices of \( G \) label 2. \( G_y \) is formed similarly. Then \( x \sim y \) iff \( G_x \cong G_y \).

The orbit containing \( x \) will be the set of all vertices similar to \( x \). The other orbits will be constructed similarly. Hence we can obtain the automorphism partitioning of a graph by means of an algorithm for testing whether two graphs are isomorphic.

Also note that when two graphs are isomorphic then their orbit-digraphs, defined in Section 3.2, will also be isomorphic. Hence comparing the orbit-digraphs will act as a filter to screen out non-isomorphic graphs when testing whether two given graphs are isomorphic.

### 4.4 Algorithmic Considerations

We will now describe an algorithm for constructing the automorphism partitioning of a graph.

The following is an outline of the algorithm. We will assume that the graph \( G \) being considered is a finite simple graph. Extensions to deal with other graphs can be done by slight modifications.

**Step 1:** Find the characteristic polynomial \( f(G) \) in factorised form.

**Step 2:** Let \( f(G) \) have \( k_i \) factors of degree \( i \). Note that
\[ \sum_{i} i \cdot k_i = n, \text{ where } n \text{ is the order of } G. \]

Find the set

\[ NP = \{ \sum_{i} i \cdot \delta_i : \delta_i \text{ are integers}, 0 \leq \delta_i \leq k_i \} \setminus \{0, n\}, \]

i.e. the set of the degrees of all possible nontrivial factorizations of \( f(G) \).

Let \( P \) be the trivial partitioning, \( H = \{e\} \).

**Step 3:** Let \( p \) be the maximum value in \( NP \). Merge cells in \( P \) so as to make the number of cells equal to \( p \). Call the new partitioning \( P' \).

**Step 4:** Check the following condition:

(\#) For every pair of distinct vertices \( u, v \) in the same cell \( C_1 \),
the number of vertices in a cell \( C_2 \) that are adjacent to \( u \)
= the number of vertices in \( C_2 \) adjacent to \( v \).

Here we may have \( C_2 = C_1 \).

If (\#) holds, go to Step 6.

**Step 5:** If all partitionings with \( p \) cells obtained from \( P \) by merging cells have been checked, go to Step 7.
If not, replace \( P' \) by another partitioning with \( p \) cells obtained from \( P \) by merging cells. Go to Step 4.

**Step 6:** Search for an automorphism \( \sigma \) of \( G \) such that \( \langle H, \sigma \rangle \) will have \( P' \) as the set of orbits. If there exists such a \( \sigma \), set \( H \) to \( \langle H, \sigma \rangle \), set \( NP \) to \( NP \setminus \{p\} \), set
P to P', and go to Step 3.
If not, go to Step 5.

**Step 7:** If p = minimum value in NP, stop with P the automorphism partitioning of G.
If not, set NP to NP\{p\}, and go to Step 3.

We will illustrate the algorithm by the following example.

**Example 4.2**
Consider the graph G as shown in Figure 3.2.

The algorithm, when applied to G, will run as follows:

**Step 1:** \( f(G) = x(x + 1)^2(x^2 - 2x - 6) \).
**Step 2:** \( k_1 = 3, k_2 = 1, \)
\( NP = \{1, 2, 3, 4\}, \)
\( P = \{1\}, \{2\}, \{3\}, \{4\}, \{5\}; \)
\( H = \{e\}. \)
**Step 3:** \( p = 4, \)
\( P' = \{1, 2\}, \{3\}, \{4\} \) is a partitioning obtained from P by merging cells and the number of cells in \( P' \) is 4.
**Step 4:** 1 and 2 are adjacent to 3, 4 and 5; 1 and 2 are not adjacent, hence the condition (*) is satisfied.
**Step 6:** The only possible candidate is (12) and this is
indeed an automorphism; hence
\[ H = \langle (12) \rangle, \]
\[ NP = \{1, 2, 3\}, \]
\[ P = \{1, 2\}, \{3\}, \{4\}, \{5\}. \]

**Step 3:** \( p = 3 \),
\[ P' = \{1, 2, 3\}, \{4\}, \{5\} \] is a partitioning with three cells obtained from \( P \) by merging cells.

**Step 4:** In \( \{1, 2, 3\} \), 3 is adjacent to 1 and 2 but 2 is adjacent to 3 only; hence (*) is violated.

**Step 5:** \( P' = \{1, 2, 4\}, \{3\}, \{5\} \) is another partitioning with three cells obtained from \( P \) by merging cells.

**Step 4:** In \( \{1, 2, 4\} \), 4 is adjacent to 1 and 2 but 2 is adjacent to 4 only, hence (*) is violated.

**Step 5:** \( P' = \{1, 2, 5\}, \{3\}, \{4\} \) is another candidate partitioning.

**Step 4:** (*) is violated when comparing vertices 2 and 5 in the cell \( \{1, 2, 5\} \).

**Step 5:** \( P' = \{1, 2\}, \{3, 4\}, \{5\} \) is another candidate partitioning.

**Step 4:** 1 and 2 are both adjacent to 3, 4 and 5.
3 and 4 are both adjacent to 1, 2 and 5.
Hence (*) is satisfied.

**Step 6:** \((34)\) is an automorphism that satisfies the requirements; hence
\[ H = \langle (12), (34) \rangle, \]
\[ NP = \{1, 2\}. \]
\[ P = \{1, 2\}, \{3, 4\}, \{5\}. \]

**Step 3:** \( p = 2 \),

\[ P' = \{1, 2, 3, 4\}, \{5\} \] is a candidate partitioning.

**Step 4:** (*) is violated when comparing vertices 2 and 4 in cell \( \{1, 2, 3, 4\} \).

**Step 5:** \( P' = \{1, 2, 5\}, \{3, 4\} \) is another candidate partitioning.

**Step 4:** (*) is violated when comparing vertices 2 and 5 in cell \( \{1, 2, 5\} \).

**Step 5:** \( P' = \{1, 2\}, \{3, 4, 5\} \) is another candidate partitioning.

**Step 4:** 1 and 2 are both adjacent to 3, 4 and 5.

3, 4 and 5 are all adjacent to 1 and 2.

Hence (*) is satisfied.

**Step 6:** (354) is an automorphism that satisfies the requirements; hence

\[ H = \langle (12), (34), (354) \rangle, \]

\[ NP = \{1\}, \]

\[ P = \{1, 2\}, \{3, 4, 5\}. \]

**Step 3:** \( p = 1 \),

\[ P' = \{1, 2, 3, 4, 5\} \] is the only candidate partitioning.

**Step 4:** (*) is violated when comparing 2 and 5 in cell \( \{1, 2, 3, 4, 5\} \).

**Step 5:** The only partitioning with one cell obtained from merging cells in P has been checked.
\textbf{Step 7:} \( p = 1 = \text{minimum value in } \mathbb{NP}, \text{ hence} \)

\[ P = \{1, 2\}, \{3, 4, 5\} \text{ is the automorphism partitioning of } G. \]

We will now consider the number of operations required for the algorithm. As noted in [20, p. 560], the characteristic polynomial of a matrix can be found in \( O(n^3) \) operations. The factorization of the characteristic polynomial presents some problems. However, we note that the factorization of a polynomial over \( \mathbb{GF}(p) \), where \( p \) is a prime, can be done in \( O(n^3(\log p)^2 + n^2(\log p)^3) \) operations. These factorizations will then give some idea of the possible factorization of the polynomial over the integers. This process does not yield a polynomial bounded algorithm for complete factorization of a polynomial over the integers. However, since we are only interested in the degrees of possible factorizations, we may stop this process when the number of primes considered is too large.

The most time-consuming operations are Step 4 and Step 6. The number of partitionings that have to be considered in Step 4 depends on the factorizations of the characteristic polynomial and the structure of the automorphism group. The worst case occurs when the automorphism group of a graph \( G \) does not contain any nontrivial subgroups and the characteristic polynomial of \( G \) is completely reducible into linear factors. In this
case we may have to check over all partitionings that refine the automorphism partitioning and then search for a generator of the automorphism group. The search for an automorphism in Step 6 can be done by first making an assignment which is impossible in the previous admissible partitioning. This will reduce the number of operations required to find an automorphism.

A list of the automorphism partitionings of simple connected graphs with not more than six vertices is given in Table V of the Appendix.

4.5 Conclusion

We have established some connections between the automorphism group of a graph and its characteristic polynomial. We have seen how the reducibility of the minimal polynomial of a graph reflects the properties of the graph and its automorphism group. We have also studied the relationship between the number of orbits of a subgroup of the automorphism group of a graph and the factorization of its characteristic polynomial. Although the results have been proved for simple graphs, extensions to other graphs are immediate.

We have also considered an algorithm to construct the automorphism partitioning of a graph by making use of the characteristic polynomial. The algorithm requires the computation of the characteristic polynomial and its
factorization; hence the efficiency of the algorithm will also depend on these computations.

Also since the factorization of the characteristic polynomial is related to the orbits of the subgroups of the automorphism group, we would expect results on the automorphism group of a graph to assist in the exploration of the properties of the characteristic polynomial, and vice versa.


32. G. Sabidussi, Graph multiplication, Math. Z. 72(1960), 446 - 457.

33. G. Sabidussi, Graphs with given group and given graph theoretical properties, Canad. J. Math. 9(1957), 515 -525.


### APPENDIX

**Table I**

**Spectra of Simple Connected Graphs up to n = 6 Vertices**

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| 3   | ![Graph](image) | $x^5 - 4x^3$  

= $x^3(x - 2)(x + 2)$ |
| 5   | ![Graph](image) | $x^5 - 5x^3 + 5x - 2$  

= $(x - 2)(x^2 + x - 1)^2$ |
| 5   | ![Graph](image) | $x^5 - 5x^3 + 2x$  

= $x(x^4 - 5x^2 + 2)$ |
| 5   | ![Graph](image) | $x^5 - 5x^3 - 2x^2 + 4x + 2$  

= $(x - 1)(x + 1)(x^3 - 4x - 2)$ |
| 5   | ![Graph](image) | $x^5 - 5x^3 - 2x^2 + 3x$  

= $x(x^2 - x - 3)(x^2 + x - 1)$ |
| 5   | ![Graph](image) | $x^5 - 5x^3 - 2x^2 + 2x$  

= $x(x + 1)(x^3 - x^2 - 4x + 2)$ |
| 5   | ![Graph](image) | $x^5 - 6x^3$  

= $x^3(x^2 - 6)$ |
| 5   | ![Graph](image) | $x^5 - 6x^3 - 2x^2 + 4x$  

= $x(x + 2)(x^3 - 2x^2 - 2x + 2)$ |
| 5   | ![Graph](image) | $x^5 - 6x^3 - 4x^2 + 5x + 4$  

= $(x - 1)(x + 1)^2(x^2 - x - 4)$ |
| 5   | ![Graph](image) | $x^5 - 6x^3 - 4x^2 + 3x + 2$  

= $(x + 1)(x^4 - x^3 - 5x^2 + x + 2)$ |
| 5   | ![Graph](image) | $x^5 - 6x^3 - 4x^2 + 2x$  

= $x(x^4 - 6x^2 - 4x + 2)$ |
| 5   | ![Graph](image) | $x^5 - 7x^3 - 4x^2 + 2x$  

= $x(x + 1)(x^3 - x^2 - 6x + 2)$ |
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<td>[x^5 - 7x^3 - 6x^2 + 3x + 2] = ((x^2 + x - 1)(x^3 - x^2 - 5x - 2))</td>
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<td>[x^5 - 7x^3 - 6x^2] = (x^2(x - 3)(x + 1)(x + 2))</td>
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<td>[x^5 - 7x^3 - 8x^2 + 2] = ((x + 1)^2(x^3 - 2x^2 - 4x + 2))</td>
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<td>[x^5 - 8x^3 - 8x^2] = (x^2(x + 2)(x^2 - 2x - 4))</td>
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<td><img src="image5" alt="Graph" /></td>
<td>[x^5 - 8x^3 - 10x^2 - x + 2] = ((x + 1)^2(x^3 - 2x^2 - 5x + 2))</td>
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<td><img src="image6" alt="Graph" /></td>
<td>[x^5 - 9x^3 - 14x^2 - 6x] = (x(x + 1)^2(x^2 - 2x - 6))</td>
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<td><img src="image7" alt="Graph" /></td>
<td>[x^5 - 10x^3 - 20x^2 - 15x - 4] = ((x - 4)(x + 1)^4)</td>
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<td><img src="image8" alt="Graph" /></td>
<td>[x^6 - 5x^4 + 6x^2 - 1] = ((x^3 - x^2 - 2x + 1)(x^3 + x^2 - 2x - 1))</td>
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<td><img src="image9" alt="Graph" /></td>
<td>[x^6 - 5x^4 + 5x^2] = (x^2(x^4 - 5x^2 + 5))</td>
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<td>[x^6 - 5x^4 + 5x^2 - 1] = ((x - 1)(x + 1)(x^4 - 4x^2 + 1))</td>
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<td><img src="image11" alt="Graph" /></td>
<td>[x^6 - 5x^4 + 4x^2] = (x^2(x - 1)(x + 1)(x - 2)(x + 2))</td>
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<td>[x^6 - 5x^4 + 3x^2] = (x^2(x^4 - 5x^2 + 3))</td>
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| 3   | ![Graph](image1) | $x^6 - 5x^4$  
\quad = x^4(x^2 - 5)$ |
| 4   | ![Graph](image2) | $x^6 - 6x^4 + 9x^2 - 4$  
\quad = (x - 1)^2(x + 1)^2(x - 2)(x + 2)$ |
| 5   | ![Graph](image3) | $x^6 - 6x^4 + 8x^2 - 2x - 1$  
\quad = (x - 1)(x^2 + x - 1)(x^3 - 4x - 1)$ |
| 6   | ![Graph](image4) | $x^6 - 6x^4 + 6x^2$  
\quad = x^2(x^4 - 6x^2 + 6)$ |
| 7   | ![Graph](image5) | $x^6 - 6x^4 - 2x^3 + 8x^2 + 4x - 1$  
\quad = (x + 1)^2(x^4 - 2x^3 - 3x^2 + 6x - 1)$ |
| 8   | ![Graph](image6) | $x^6 - 6x^4 + 5x^2$  
\quad = x^2(x - 1)(x + 1)(x^2 - 5)$ |
| 9   | ![Graph](image7) | $x^6 - 6x^4 + 5x^2 - 1$  
\quad = (x^3 - 2x^2 - x + 1)(x^3 + 2x^2 - x - 1)$ |
| 10  | ![Graph](image8) | $x^6 - 6x^4 - 2x^3 + 7x^2 + 4x$  
\quad = x(x + 1)(x^4 - x^3 - 5x^2 + 3x + 4)$ |
| 11  | ![Graph](image9) | $x^6 - 6x^4 + 4x^2$  
\quad = x^2(x^4 - 6x^2 + 4)$ |
| 12  | ![Graph](image10) | $x^6 - 6x^4 - 2x^3 + 7x^2 + 2x - 1$ |
| 13  | ![Graph](image11) | $x^6 - 6x^4 - 2x^3 + 6x^2 + 2x - 1$  
\quad = (x - 1)(x + 1)(x^4 - 5x^2 - 2x + 1)$ |
| 14  | ![Graph](image12) | $x^6 - 6x^4 - 2x^3 + 6x^2 - 1$  
\quad = (x^2 - 2x - 1)(x^2 + x - 1)^2$ |
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|   |   | \(= x^2(x^4 - 6x^2 - 2x + 5)\) \\
|   |   | \(x^6 - 6x^4 - 2x^3 + 3x^2\) \\
|   |   | \(= x^2(x + 1)(x^3 - x^2 - 5x + 3)\) \\
| 4 |  | \(x^6 - 7x^4 + 3x^2\) \\
|   |   | \(= x^2(x^4 - 7x^2 + 3)\) \\
|   |   | \(x^6 - 7x^4 - 2x^3 + 3x^2\) \\
<p>|   |   | (= x^2(x - 1)(x^3 - x^2 - 5x + 3)) |</p>
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<td>$x^2 (x^2 + x - 1)(x^2 - x - 5)$</td>
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<td>$x(x + 2)(x^2 - 2)(x^2 - 2x - 2)$</td>
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<td>$(x^2 + x - 1)(x^4 - x^3 - 6x^2 + 3x + 1)$</td>
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<td>$= x(x + 1)^2 (x^3 - 2x^2 - 5x + 4)$</td>
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<td>![Graph]</td>
<td>$x^6 - 9x^4$</td>
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<td>$= x^4 (x - 3) (x + 3)$</td>
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<td>$x^6 - 9x^4 - 4x^3 + 12x^2$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 9x^4 - 4x^3 + 4x^2$</td>
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<td>$= x^2 (x + 1) (x^3 - x^2 - 8x + 4)$</td>
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<td>$x^6 - 9x^4 - 6x^3 + 6x^2 + 4x$</td>
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<td>$= x(x + 1) (x^4 - x^3 - 8x^2 + 2x + 4)$</td>
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<td>$x^6 - 9x^4 - 8x^3 + 9x^2 + 8x - 1$</td>
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<td>$= (x - 1) (x + 1) (x^4 - 8x^2 - 8x - 1)$</td>
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<td>$x^6 - 9x^4 - 8x^3 + 9x^2 + 6x - 4$</td>
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<td>$= (x^2 - 2x - 4) (x^2 + x - 1)^2$</td>
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<td>$x^6 - 9x^4 - 8x^3 + 8x^2 + 8x$</td>
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<td>$x^6 - 9x^4 - 8x^3 + 6x^2 + 4x$</td>
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<td>$= x(x + 2) (x^4 - 2x^3 - 5x^2 + 2x + 2)$</td>
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<td>$= x(x^5 - 9x^3 - 8x^2 + 5x + 4)$</td>
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<td>$x^6 - 9x^4 - 8x^3 + 4x^2$</td>
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<td>$= x^2 (x + 2) (x^3 - 2x^2 - 5x + 2)$</td>
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<td>$= (x - 1) (x + 1)^2 (x^3 - x^2 - 7x - 3)$</td>
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<td><img src="image1.png" alt="Graph" /></td>
<td>(x^6 - 9x^4 - 8x^3)</td>
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<td>= (x^3(x + 1)(x^2 - x - 8))</td>
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<td>(x^6 - 9x^4 - 10x^3 + 4x^2 + 6x)</td>
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<td></td>
<td>= (x(x + 1)(x^4 - x^3 - 8x^2 - 2x + 6))</td>
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<td>(x^6 - 9x^4 - 10x^3 + 3x^2 + 4x - 1)</td>
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<td></td>
<td>= ((x + 1)(x^5 - x^4 - 8x^3 - 2x^2 + 5x - 1))</td>
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<td><img src="image2.png" alt="Graph" /></td>
<td>(x^6 - 10x^4 - 8x^3 + 9x^2 + 8x)</td>
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<td>= (x(x - 1)(x + 1)^2(x^2 - x - 8))</td>
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<td>(x^6 - 10x^4 - 8x^3 + 9x^2 + 4x - 1)</td>
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<td>= ((x^3 - 2x^2 - 5x + 1)(x^3 + 2x^2 - x - 1))</td>
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<td>(x^6 - 10x^4 - 6x^3 + 3x^2)</td>
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<td>= (x^2(x + 1)(x^3 - x^2 - 9x + 3))</td>
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<td><img src="image3.png" alt="Graph" /></td>
<td>(x^6 - 10x^4 - 10x^3 + 10x^2 + 8x - 5)</td>
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<td>= ((x^2 - 2x - 5)(x^2 + x - 1)^2)</td>
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<td>(x^6 - 10x^4 - 8x^3 + 4x^2)</td>
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<td>= (x^2(x^4 - 10x^2 - 8x + 4))</td>
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<td>(x^6 - 10x^4 - 10x^3 + 8x^2 + 8x)</td>
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<td>= (x(x + 2)(x^4 - 2x^3 - 6x^2 + 2x + 4))</td>
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<td>(x^6 - 10x^4 - 10x^3 + 6x^2 + 6x - 1)</td>
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<td>(x^6 - 10x^4 - 10x^3 + 5x^2 + 4x)</td>
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<td>= (x(x^5 - 10x^3 - 10x^2 + 5x + 4))</td>
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<td>(x^6 - 10x^4 - 12x^3 + 7x^2 + 14x + 4)</td>
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</table>
| 6 | ![Graph](image1) | \[x^6 - 10x^4 - 12x^3 + 5x^2 + 12x + 4\] <br>\[= (x - 1)(x + 2)(x + 1)^2(x^2 - 3x - 2)\] <br>\[x^6 - 10x^4 - 12x^3 + 4x^2 + 6x - 1\] <br>\[= (x + 1)(x^2 + x - 1)(x^3 - 2x^2 - 6x + 1)\] <br>\[x^6 - 10x^4 - 12x^3 - x^2 + 4x\] <br>\[= x(x + 1)^2(x^3 - 2x^2 - 7x + 4)\] <br>\[x^6 - 10x^4 - 14x^3 + 8x + 3\] <br>\[= (x + 1)^2(x^4 - 2x^3 - 7x^2 + 2x + 3)\] <br>\[x^6 - 10x^4 - 14x^3 - x^2 + 4x\] <br>\[= x(x + 1)(x^4 - x^3 - 9x^2 - 5x + 4)\] <br>\[x^6 - 11x^4 - 12x^3 + 5x^2 + 4x\] <br>\[= x(x^2 + x - 1)(x^3 - x^2 - 9x - 4)\] <br>\[x^6 - 11x^4 - 12x^3 + 3x^2 + 4x - 1\] <br>\[= (x + 1)^2(x^2 + 2x - 1)(x^2 - 4x + 1)\] <br>\[x^6 - 11x^4 - 12x^3\] <br>\[= x^3(x^3 - 11x - 12)\] <br>\[x^6 - 11x^4 - 14x^3 + 4x^2 + 8x\] <br>\[= x(x + 1)(x + 2)(x^3 - 3x^2 - 4x + 4)\] <br>\[x^6 - 11x^4 - 14x^3 + 4x\] <br>\[= x(x + 1)^2(x^3 - 2x^2 - 8x + 4)\] <br>\[x^6 - 11x^4 - 16x^3 + 3x^2 + 16x + 7\] <br>\[= (x - 1)(x + 1)^3(x^2 - 2x - 7)\] <br>\[x^6 - 11x^4 - 16x^3 + x^2 + 10x + 3\] <br>\[= (x + 1)(x^5 - x^4 - 10x^3 - 6x^2 + 7x + 3)\]
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<td>$x^6 - 11x^4 - 16x^3 - 2x^2 + 4x$ $= x(x + 1)(x + 2)(x^3 - 3x^2 - 4x + 2)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 11x^4 - 20x^3 - 9x^2 + 4x + 3$ $= (x + 1)^3(x^3 - 3x^2 - 5x + 3)$</td>
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<td>$x^6 - 12x^4 - 16x^3$ $= x^3(x - 4)(x + 2)^2$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 12x^4 - 18x^3 - 3x^2 + 4x$ $= x(x + 1)(x^4 - x^3 - 11x^2 - 7x + 4)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 12x^4 - 20x^3 - 4x^2 + 8x + 3$ $= (x + 1)(x^2 + x - 1)(x^3 - 2x^2 - 8x - 3)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 12x^4 - 20x^3 - 9x^2$ $= x^2(x + 1)^2(x^2 - 2x - 9)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 12x^4 - 22x^3 - 9x^2 + 6x + 4$ $= (x + 1)^3(x^3 - 3x^2 - 6x + 4)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 13x^4 - 24x^3 - 12x^2$ $= x^2(x + 1)(x + 2)(x^2 - 3x - 6)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 13x^4 - 26x^3 - 15x^2 + 2x + 3$ $= (x + 1)^3(x^3 - 3x^2 - 7x + 3)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 14x^4 - 32x^3 - 27x^2 - 8x$ $= x(x + 1)^3(x^2 - 3x - 8)$</td>
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<td><img src="image" alt="Graph" /></td>
<td>$x^6 - 15x^4 - 40x^3 - 45x^2 - 24x - 5$ $= (x - 5)(x + 1)^5$</td>
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Table III: Integral Trees of Diameter 3 and Order ≤ 500

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<td>$x^2(x - 1)(x + 1)(x - 2)(x + 2)$</td>
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<td>$x^{10}(x - 2)(x + 2)(x - 3)(x + 3)$</td>
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<td>$x^{22}(x - 3)(x + 3)(x - 4)(x + 4)$</td>
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<td>$x^{218}(x - 10)(x + 10)(x - 11)(x + 11)$</td>
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<td>$x^{262}(x - 11)(x + 11)(x - 12)(x + 12)$</td>
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<td>$x^{310}(x - 12)(x + 12)(x - 13)(x + 13)$</td>
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<td>$x^{337}(x - 12)(x + 12)(x - 14)(x + 14)$</td>
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<td>183</td>
<td>183</td>
<td>$x^{362}(x - 13)(x + 13)(x - 14)(x + 14)$</td>
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<td>211</td>
<td>$x^{418}(x - 14)(x + 14)(x - 15)(x + 15)$</td>
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<tr>
<td>482</td>
<td>241</td>
<td>241</td>
<td>$x^{478}(x - 15)(x + 15)(x - 16)(x + 16)$</td>
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### Table IV

**Connected Graphs** with 2 types of Labels and their Characteristic Polynomials (for \( n \leq 4 \))

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<th>( n )</th>
<th>Graph</th>
<th>Characteristic Polynomial</th>
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<tbody>
<tr>
<td>2</td>
<td>( x \rightarrow y )</td>
<td>( xy - 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( y \rightarrow x )</td>
<td>( x^2y - y - x )</td>
</tr>
<tr>
<td>3</td>
<td>( x \rightarrow y \rightarrow x )</td>
<td>( x(xy - 2) )</td>
</tr>
<tr>
<td>3</td>
<td>( y \rightarrow x )</td>
<td>( (x + 1)(xy - y - 2) )</td>
</tr>
<tr>
<td>4</td>
<td>( x \rightarrow x \rightarrow y )</td>
<td>( x^3y - 2xy - x^2 + 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( y \rightarrow x )</td>
<td>( x^3y - xy - 2x^2 + 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( x \rightarrow y \rightarrow y )</td>
<td>( x^2y^2 - y^2 - xy - x^2 + 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( y \rightarrow y )</td>
<td>( x^2y^2 - 3xy + 1 )</td>
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* Graphs that can be obtained by interchanging \( x \) and \( y \) are omitted.
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<th>n</th>
<th>Graph</th>
<th>Characteristic Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Graph 1" /></td>
<td>((xy - x - 1)(xy + x - 1))</td>
</tr>
<tr>
<td>2</td>
<td><img src="image2" alt="Graph 2" /></td>
<td>(x(x^2y - 2y - x))</td>
</tr>
<tr>
<td>3</td>
<td><img src="image3" alt="Graph 3" /></td>
<td>(x^2(xy - 3))</td>
</tr>
<tr>
<td>4</td>
<td><img src="image4" alt="Graph 4" /></td>
<td>(y(x^2y - y - 2x))</td>
</tr>
<tr>
<td>5</td>
<td><img src="image5" alt="Graph 5" /></td>
<td>(x(x^2y - 2x - 2y))</td>
</tr>
<tr>
<td>6</td>
<td><img src="image6" alt="Graph 6" /></td>
<td>((xy + x + y)(xy - x - y))</td>
</tr>
<tr>
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<td><img src="image7" alt="Graph 7" /></td>
<td>(xy(xy - 4))</td>
</tr>
<tr>
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<td><img src="image8" alt="Graph 8" /></td>
<td>((x + 1)(x^2y - xy - 2y - x + 1))</td>
</tr>
<tr>
<td>9</td>
<td><img src="image9" alt="Graph 9" /></td>
<td>(x^3y - 2xy - 2x^2 - 2x + 1)</td>
</tr>
<tr>
<td>10</td>
<td><img src="image10" alt="Graph 10" /></td>
<td>((x + 1)(x^2y - xy - 3x + 1))</td>
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<td>11</td>
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<td>(x^2y^2 - y^2 - 3xy - 2xy + 1)</td>
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<td>((y + 1)(x^2y - y - x^2 - 2x + 1))</td>
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<tr>
<td>x</td>
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<td></td>
<td>(x + 1) (x^2 y - xy - 2y - 2x)</td>
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<td>x</td>
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<td>y</td>
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<td>x(x^2 y - 2y - 3x - 4)</td>
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<td>y(x + 1) (xy - y - 4)</td>
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<td>(x + 1)^2 (xy - 2y - 3)</td>
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<tr>
<td>y</td>
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<tr>
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<td>(x + 1) (y + 1) (xy - y - x - 3)</td>
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$x^5 - 5x^3 + 5x$
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<td>$(x-1)(x^4 - 5x^2 - 2x + 1)$</td>
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<td>$x(x - 1)(x^2 - x - 4)$</td>
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