RELIABILITY OF SOME AGEING NUCLEAR POWER PLANT SYSTEMS: A SIMPLE STOCHASTIC MODEL

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ABSTRACT

The random number of failure-related events in certain repairable ageing systems, like certain nuclear power plant components, during a given time interval, may be often modelled by a compound Poisson distribution. One of these is the Polya-Aeppli distribution. The derivation of a stationary Polya-Aeppli distribution as a limiting distribution of rare events for stationary Bernouilli trials with first order Markov dependence is considered. But if the parameters of the Polya-Aeppli distribution are suitable time functions, we could expect that the resulting distribution would allow us to take into account the distribution of failure-related events in an ageing system. Assuming that a critical number of damages produce an emergent failure, the abovementioned results can be applied in a reliability analysis. It is natural to ask under what conditions a Polya-Aeppli distribution could be a limiting distribution for non-homogeneous Bernouilli trials with first order Markov dependence. In this paper this problem is analyzed and possible applications of the obtained results to ageing or deteriorating nuclear power plant components are considered. The two traditional ways of modelling repairable systems in reliability theory: the “as bad as old” concept, that assumes that the replaced component is exactly under the same conditions as was the aged component before failure, and the “as good as new” concept, that assumes that the new component is under the same conditions of the replaced component when it was new, are briefly discussed in relation with the findings of the present work.

1. INTRODUCTION

The ultimate (but always elusive) goal of any nuclear engineer engaged in design, maintenance or operation activities related with nuclear power plants (NPP) would be to achieve a downtime as near zero as possible. A careful assessment of vulnerable areas of plant operation to allow the prevention of sequences of damage events that could result in the failure of some significant component is usually successfully done. Nevertheless, in some cases this is not feasible. A reliability approach must be applied to determine both the backup hardware needed in stock, and the average frequency and duration of repair operations. In these cases quality assurance in design, production and operation allows the elimination of systematic or repeatable patterns of failure. An indication of achievement of a suitable optimization in these processes is the random nature of the events that cause failures and that the failures themselves are rare events. So, in this case the theory of stochastic processes is the appropriate framework for mathematical modelling. Within this framework the failure processes may be considered as chains of rare events or equivalently as point processes. For
each stochastic process in continuous time with a discrete state space, there is a point process where the points are the random times of entry to a pre-assigned state or sets of states. In reliability theory what matters is the entrance in a failure state or the exit from it.

The purpose of the present paper is to discuss some reliability aspects that could be of interest in the design and maintenance of NPP. The random number of damage or failure-related events $N(t)$ during the interval $(0, t)$ in certain repairable ageing systems may be composed, from time to time, by clumps of almost simultaneous rare events. If they are considered as truly simultaneous, it is possible to expect to be able to describe them by compound Poisson distributions (also known as “contagious” distributions) [1] defined in $(0, t)$. It is conceivable that this same random number could be described also in discrete time, using random chains of rare events (failure-related) that are not independent from a probabilistic standpoint: we consider random damages or failures in such conditions that the occurrence of one of them modifies the probability of occurrence of the next one. Under suitable conditions, it could be expected that the probability distribution of the number of occurrences $N_n$ of a certain rare event in a great number $n$ of trials in a chain approaches a probability distribution of the compound Poisson type in continuous time, if the time interval $\Delta t$ between trials and the number of trials verify $n \cdot \Delta t = t$ while $n$ is made to increase. The study of this kind of convergence of discrete distributions of failure-related events defined on chains, towards discrete distributions of failure-related events defined in continuous time, allows the interpretation of compound Poisson distributions and their related stochastic processes by models of more elementary nature in which well defined and meaningful mechanisms of random interaction may be introduced. In fact $\Delta t$ may be related with cyclical processes in the subsystems of the NPP. In certain reliability problems $\Delta t$ can be connected with the periods of mechanical vibrations induced in solid components of pumps, pipes, heat exchangers, fuel assemblies and structural parts of the nuclear isle. In other situations it could have a connection with the slower periods of pressure and temperature oscillations. Thus, the parameters of the distributions of failure-related events may be connected with the basic conditional probabilities of the chains. The following discussion is aimed to give a concrete example of the abovementioned method. It will be focussed in the Polya-Aeppli distribution of probabilities (also known as the Poisson-Geometric distribution), its relation with random processes (both in the stationary and non-stationary cases), and its relations with Markov chains of rare events.

If the repair time is negligible when compared with the time scale of the failure process, two approaches are often used in reliability theory in NPP: the non-stationary Poisson Process and the Renewal Process. As stressed by Saldanha, Frutuoso e Melo, and Noriega [2], the first approach may be too pessimistic, since it assumes that the system after repair is as bad as it was before repair and the second approach too optimistic, since it assumes that the system, after repair is as good as new. To get a balance between these traditional approaches, a more realistic modelling of failure and repair processes in an ageing system must be done. As noticed in [2], the explicit use of the complete intensity function allows a more realistic modelling of failure and repair processes in an ageing system, between the extremes of the “as bad as old” and the “as good as new” concepts. The theoretical approach proposed in the present work could be considered as a complement to the mathematical framework commonly applied in NPP reliability analysis. It is intended as a suggestion to further explore the feasibility of applying the approach in suitable NPP reliability problems.
2. POLYA-AEPPLI DISTRIBUTIONS, MARKOV CHAINS AND RELIABILITY

2.1. The Polya-Aeppli distribution as limit of the distribution of rare events in a succession of Bernoulli trials with first order Markov dependence

We first consider a succession of independent Bernoulli trials. Let $p$ be the probability of $E$ in a single trial, and $q$ the probability of $E'$ (the complementary event, $p + q = 1$) also in a single trial, and $N_n$ the random number of events $E$ in $n$ consecutive trials. If the mean value $n \cdot p$ of $N_n$ tends to a constant $\mu$ when $n \to \infty$, while the number $n$ of trials increases without bounds, then it is well known that the probability $P_n(k) = P(N_n = k), (k = 0,1,2,...,n)$ tends to a Poisson distribution. This is possible if $p = p(n) = \frac{1}{n} \cdot (\mu + \epsilon_n), \text{ with } \epsilon_n \to 0$ when $n$ increases, so that $E$ becomes increasingly rare. The probability generating function (pgf) $g_n(z) = \sum_{k=0}^{n} P_n(k) \cdot z^k$ of the random variable $N_n$ is $g_n(z) = (q + p \cdot z)^n = \left(1 + \frac{\mu + \epsilon_n}{n}, (z-1)\right)^n$.

When $n$ increases, $g_n(z)$ approaches $g(z) = e^{\mu(z-1)}$, the pgf of a Poisson distribution. A fairly general theorem of probability theory [3] allows us to conclude that if a sequence of pgf $g_n(z)$ converges to a pgf $g(z)$, the corresponding sequence $P_n(k)$ converges to $P(k)$, the distribution of probabilities of the random variable characterized by $g(z)$. Then, in fact, $P_n(k)$ converges to a Poisson distribution.

Now, let us consider a succession of Bernoulli trials with first order Markov dependence (a Markov chain with two elements in its state space). In this case the probability of occurrence of $E$ in the trial number $n$ depends of the result of the trial number $n-1$. Either directly as was done by Cernuschi and Saleme [4], or indirectly as was done by Bernstein [5], it is possible to calculate the probability generating function (pgf) $g_n(z) = \sum_{k=0}^{n} P_n(k) \cdot z^k$ corresponding to the random number of occurrences $N_n$ of $E$ in $n$ consecutive trials along the chain. The following formula is obtained ([4], [5])

$$g_n(z) = \frac{(b(z)-\lambda_2(z)) \cdot (\lambda_1(z))^n - (b(z)-\lambda_1(z)) \cdot (\lambda_2(z))^n}{\lambda_1(z)-\lambda_2(z)} \tag{1}$$

The following equations are verified:

$$\lambda_1(z) = \frac{c(z) + \sqrt{(c(z))^2 - 4 \cdot a(z)}}{2} \tag{2a}$$

$$\lambda_2(z) = \frac{c(z) - \sqrt{(c(z))^2 - 4 \cdot a(z)}}{2} \tag{2b}$$

$$a(z) = (p'-p) \cdot z \quad \text{(2c)}$$

$$b(z) = 1 - p \cdot (1-z) \quad \text{(2d)}$$

$$c(z) = 1 - p + p' \cdot z \quad \text{(2e)}$$

Here $p$ is the probability of the event $E$ in a trial, if in the previous trial $E'$ occurred, and $p'$ is the probability of $E$ if in the previous trial $E$ occurred.

The mean number of occurrences of $E$ in $n$ trials, $\mu_n$ is obtained from formula (1) for the pgf, and the relation $\mu_n = g_n'(1)$ ($g_n'(1)$ is the derivative of $g_n(z)$ taken for $z = 1$).

Thus we obtain:

$$\mu_n = \frac{n \cdot p}{1 - (p'-p)} \cdot \frac{p' \cdot (p'-p)}{1 + p - p'} \cdot \left(1 - (p'-p)^e\right) \tag{3}$$

We can increase $n$ without bound while maintaining $\mu_n = \mu_\infty$ constant, if the probabilities $p$ and $p'$ verify:

$$p = p(n) = \frac{1}{n} \cdot (\mu + \epsilon_n) \quad \text{(4a)}$$

$$p' = p'(n) = p_\infty + \epsilon_n' \quad \text{(4b)}$$
Both \( \epsilon_n \) and \( \epsilon'_n \) tend to zero while \( n \) tends to \( \infty \). Substituting (4a) and (4b) in formulae (2a) to (2e), from formula (1) it follows, after several fairly lengthy calculations ([6]), that \( g_n(z) \) is asymptotically equivalent to \( (\lambda_i(z))^n \) and both approach to:

\[
g(z) = e^{\mu \left( \frac{1 - p(z)}{1 - p_\infty} \right) z} \]  \hspace{1cm} (6)

Equation (6) may be restated as a compound Poisson distribution:

\[
g(z) = e^{\mu \left( \frac{1 - p(z)}{1 - p_\infty} \right) z} \]  \hspace{1cm} (7)

The distribution that corresponds to (7) is known as the Poisson-Geometric or Polya-Aeppli distribution. (If \( p_\infty = 0 \), (7) reduces to the pgf of Poisson distribution). Applying the same theorem used above in the case of independent Bernoulli trials, we conclude that \( P_n(k) \), for Bernoulli trials of an increasingly rare event \( E \) with strong first order Markov dependence converges to a Polya-Aeppli distribution.

### 2.2. Ageing systems, failure related rare events, and critical numbers to failure

If the parameter \( \mu = \lambda \cdot t \), being \( \lambda \) a constant, and if \( p_\infty \) is constant, then the distribution of the random number \( N(t) \) of failure-related events in \((0,t)\) corresponds to a stationary process: a non ageing system. But if \( \mu = \int_0^t A(u) \cdot du \) for a certain \( A(t) \) (possibly increasing with time) and \( p_\infty = p_\infty(t) \) (not necessarily increasing with time) the Polya-Aeppli distribution is non-stationary and could be used to describe an ageing system.

If the critical number of failure-related events above which a failure is produced during \((0,t)\) is \( n_c \), the reliability \( R(t) \) would be given by:

\[
R(t) = \sum_{k=0}^{k=n} P(N(t) = k) \]  \hspace{1cm} (8)

It is possible to give a closed form analytical formulae for the probabilities \( P(N = k) = P(k) \) for the Polya-Aeppli distribution, but it is somewhat cumbersome to calculate directly with them ([7], [8]). If \( n_c \) is not too small (usually greater than 3 or 4) it is better to apply the method of steepest descent to estimate the probabilities \( P(k) \) for \( k > n_c \), as was done by Cernuschi and Castagnetto in their generalization of the distributions of Greenwood-Yule and Lüders [9]. In the case of the Polya-Aeppli distribution we obtain a fairly good approximation, beginning with the complex variable formula

\[
P(k) = \frac{1}{2\pi i} \oint \frac{g(z)}{z^{k+1}} \, dz \] (where the integral is taken on any simple closed contour that encloses the origin of the complex plane and excludes the singularity \( z = \frac{1}{p_\infty} \) of the function \( g(z) \) given by equations(6) or (7)). The result of this procedure may be found in reference [10].

Then the reliability may be estimated from:

\[
R(t) = 1 - \sum_{k=n_c+1}^{k=\infty} P(N(t) = k) \]  \hspace{1cm} (9)

### 2.3. Some remarks about non-stationary Polya-Aeppli distributions, non-homogeneous Markov chains and reliability in ageing systems

It is natural to ask about a connection between non-homogeneous chains and non-stationary Polya-Aeppli distributions in the framework of reliability problems for ageing systems. Since the research of R. von Mises it is known that a sequence of independent Bernoulli trials with varying probabilities from trial to trial, gives a distribution for \( N_n \) that approaches to a Poisson distribution of parameter \( \mu \) if the sum of the series of the probabilities of
occurrence of the event $E$ (that is the mean number of occurrences) $E[N_n] = \sum_{j=1}^{\infty} p_j(n)$ converges to $\mu$ and if $\max_{1 \leq j \leq n} \{p_j\}$ tends to zero when the number $n$ of trials tends to $\infty$[11]. So asymptotically $E$ is a rare event in every trial. A particularly interesting case, that may be generalized to Markov chains, is obtained if we assume a causative factor $c(n)>1$ such that $p_{j+1}(n)=c(n) \cdot p_j(n)$ for every $j$. Now $E[N_n] = \left( \frac{c(n)^n - 1}{c(n)-1} \right) p_1(n)$ so if this expression must tend to a constant, $c(n)$ must tend to 1 from above and $p_1(n)$ must tend to zero as $\frac{1}{n}$.

The simplest model of a non-homogeneous sequence of Bernouilli trials with first order Markov dependence was constructed using the causative matrix concept by Harary, Lipstein and Styan [12]. Let us suppose that $p'_m = P(E_m / E_{m-1})$ is the probability of the $E$ in the $m$ trial, given that $E$ occurred in the $m-1$ trial, and $q'_m = P(E'_m / E_{m-1})$, $p_m = P(E_m / E'_m - 1)$, $q_m = P(E'_m / E'_m - 1)$ the other conditional probabilities. With these conditional probabilities the probability transition matrix $M_m$ is constructed ($m_{11} = p'_m$, $m_{12} = p_m$, $m_{21} = q'_m$, $m_{22} = q_m$). Then if the chain has a constant causative matrix $C$, $M_{m+1} = CM_m$ for every $m = 1,2,3,...$. So, if $c_{ij}$ represent the elements of the causative matrix, for every $m$ the following scalar equations apply:

$$
P'_{m+1} = c_{11} \cdot p'_m + c_{12} \cdot q'_m \quad q'_{m+1} = c_{21} \cdot p'_m + c_{22} \cdot q'_m
$$

$$
P_{m+1} = c_{11} \cdot p_m + c_{12} \cdot q_m \quad q_{m+1} = c_{11} \cdot p_m + c_{12} \cdot q_m
$$

(9)

The causative matrix is pseudo-stochastic in the sense that the off diagonal elements may be negative, but always $c_{11} + c_{12} = 1$, $c_{21} + c_{22} = 1$.

In dealing with non-stationary chains it is advisable to work with vector probability generating functions that describe the distribution of the random number $N_m$ of occurrences of $E$ in a finite number of trials. So, let us introduce the (column) vector pgf $\hat{g}_m(z)$ with elements $g'_m(z)$ and $g''_m(z)$. The partial pgf $g'_m(z)$ describes the distribution of probabilities of $E$ in $m$ trials given that $E$ occurs in trial number $m$, while $g''_m(z)$ does the same thing when $E$ does not occur in trial number $m$. Then:

$$
g'_{m+1}(z) = z \cdot p'_m \cdot g'_m(z) + z \cdot p_m \cdot g''_m(z) \quad g''_{m+1}(z) = q'_m \cdot g'_m(z) + q_m \cdot g''_m(z)
$$

(10)

In matrix terms: $\hat{g}_{m+1}(z) = M_m(z) \hat{g}_m(z)$ From (9) and (10) this matrix equation is obtained: $M_{m+1}(z) = C(z) M_m(z)$. Being $C(z)$ a matrix whose first row is the first row of $C$ multiplied by $z$ and whose second row is identical to the second row of $C$. From these equations the pgf for $n$ consecutive trials $\hat{g}_n(z)$ may be obtained in terms of powers of $C(z)$ and $M_1$.

The spectral properties of the matrices $M_m(z) = C^{m-1}(z) M_1(z)$ were already studied for the special case $z=1$ by Le Maire and Gauffrey [13]. A possible extension of the results could be applied to discuss the asymptotic behavior of $\hat{g}_n(z)$ when both $n \to \infty$ and the mean value of $N_n$ converges. However, for our purpose here we give a heuristic argument closely related with the derivation of Polya-Aeppli distribution developed in 2.1. So, let us consider $n$ trials in a non-stationary chain with transition probabilities from the trial number $m$ to the trial $m+1$ given by $p'_m(n), q'_m(n), p_m(n), q_m(n)$ ($1 \leq m \leq n$) and the elements of the causative matrix are also functions of $n$: $c_{11}(n), c_{12}(n), c_{21}(n), c_{22}(n)$.

If $E$ is increasingly rare as $n \to \infty$, so that $q_m(n) \to 1$ (and as a consequence $p_m(n) \to 0$), but the conditioning related to $E$ is so strong that $p'_m(n) \to p_\emptyset$ (and as a consequence $q'_m(n) \to 1 - p_\emptyset$), then from equations (9) it follows that the causative matrix $C(n)$ tends to the unit matrix. By a continuity argument we could expect that also for the case of a non-homogeneous sequence...
of Bernouilli trials with first order Markov dependence and a causative matrix connecting the matrices of transition probabilities, the distribution of probabilities of the random number of occurrences of $E$ in $n$ trials converges to a Polya-Aeppli distribution.

3. CONCLUSIONS

(1) A connection between Markov chains and compound Poisson distributions, with focus in continuous random processes related with the Polya-Aeppli distribution that might be of interest from the standpoint of the mathematical modeling of some reliability problems posed by random mechanical vibrations and other oscillatory processes in NPP, was studied.

(2) The approach developed in this paper may be considered of the conservative “as bad as old” type if the parameters of the asymptotic non-stationary Polya-Aeppli distribution describes an ageing system due to the cumulative effect of progressive damage processes.

(3) Further research could be done to complete some of these findings and to extend the suggested approach to construct less conservative models from the reliability standpoint.

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