GRÖBNER BASES AND GRADINGS FOR PARTIAL DIFFERENCE IDEALS

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Abstract. In this paper we introduce a working generalization of the theory of Gröbner bases for the algebras of partial difference polynomials with constant coefficients. Such algebras are free objects in the category of commutative algebras endowed with the action by endomorphisms of a monoid isomorphic to $\mathbb{N}^r$. Since they are not Noetherian algebras, we propose a theory for grading them that provides a Noetherian subalgebras filtration. This implies that the variants of the Buchberger algorithm we developed for partial difference ideals terminate in the finitely generated graded case when truncated up to some degree. Moreover, even in the non-graded case, we provide criteria for certifying completeness of eventually finite Gröbner bases when they are computed within sufficiently large bounded degrees. We generalize also the concepts of homogenization and saturation, and related algorithms, to the context of partial difference ideals. The feasibility of the proposed methods is shown by an implementation in Maple and a test set based on the discretization of concrete systems of non-linear partial differential equations.

1. Introduction

An important idea at the intersection of many algebraic theories consists in studying algebraic structures under the action of operators of different nature, typically automorphisms and derivations. Classical roots of this idea can be found clearly in invariant and representation theory, as well as in the study of polynomial identities satisfied by associative algebras. Recently, topics like algebraic statistic [4] or entanglement theory [22] have given new impulse and applications to the research on such themes. Another fundamental source of inspiration is the theory of differential and difference algebras introduced in the pioneeristic work of Ritt [23, 24] and afterwards developed by Kolchin [16], Cohn [6], Levin [21] and many others. From the point of view of computational methods, starting from the algorithms proposed by Ritt himself, a considerable advancement can be recorded in the differential case (see for instance [25]). Much less has be achieved for the algebras of difference polynomials where working algorithms can be found mainly in the linear case [12]. Nevertheless, the interest for such computations is relevant because of applications in the discretization of systems of differential equations like the automatic generation of finite difference schemes or the consistency analysis of finite difference approximations [9, 11, 19]. The present paper contributes to this research trend by concerning the development of effective methods for systems of linear and non-linear difference equations. In a general and systematic way, we

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introduce a Gröbner bases theory for the ideals of the algebra of partial difference polynomials with constant coefficients. Recent contributions in this direction can be found also in [9, 18]. In particular, owing to the notion of letterplace embedding in [18] one shows that the Gröbner bases computations for ideals of the free associative algebra are a subclass of the same computations for ideals of the algebra of ordinary difference polynomials.

The algebra of partial difference polynomials is a free algebra in the class of commutative algebras that are invariant under the action by endomorphisms of a monoid isomorphic to \( \mathbb{N}^r \). They are fundamental structures in the formal theory of partial difference equations where a set of multivariate functions is assumed algebraically independent together with all partial shifts of them. It is relevant then to study the notion of Gröbner basis and the algorithmic methods to compute them, for the difference ideals that are ideals of difference polynomials invariant under the action of the shifts monoid. Based on a suitable definition of monomial orderings that are compatible with shifts action and the construction of large classes of them, the present paper introduces variants of the Buchberger algorithm for partial difference ideals. These procedures take advantage of the monoid symmetry essentially by killing all S-polynomials in an orbit except for a minimal one. Note that the algebras of difference polynomials are not Noetherian since they are commutative polynomial rings in an infinite number of variables and hence termination is not generally guaranteed for the proposed algorithms. With the aim of improving this situation, we define suitable gradings that are compatible with the monoid action and provide filtrations of the algebra of partial difference polynomials with finitely generated subalgebras. We obtain therefore the termination for finitely generated graded difference ideals when computations are performed within some bounded degree. For non-graded ideals but for monomial orderings compatible with such gradings, we prove also criterions able to certify that a Gröbner basis computation performed over a suitable finite set of variables that is within a sufficiently large degree, is a complete one. Finally, the paper generalizes the notion of saturation to difference ideals with respect to the given gradings and provides the algorithms to perform this ideal operation. As a byproduct, one obtains an alternative algorithm to compute Gröbner bases of non-graded difference ideals via homogeneous computations. By means of an implementation in Maple, all these methods are finally experimented on difference ideals obtained by the discretization of systems of non-linear differential equations.

2. Algebra of Partial Difference Polynomials

Fix \( K \) any field and let \( \Sigma \) be a monoid (semigroup with identity) that we denote multiplicatively. Let \( A \) be a commutative \( K \)-algebra and denote \( \text{End}_K(A) \) the monoid of \( K \)-algebra endomorphisms of \( A \). We call \( A \) a \( \Sigma \)-invariant algebra or briefly a \( \Sigma \)-algebra if there is a monoid homomorphism \( \rho : \Sigma \to \text{End}_K(A) \). In this case, we denote \( \sigma \cdot x = \rho(\sigma)(x) \), for all \( \sigma \in \Sigma \) and \( x \in A \). Let \( A, B \) be \( \Sigma \)-algebras and \( \varphi : A \to B \) be a \( K \)-algebra homomorphism. We say that \( \varphi \) is a \( \Sigma \)-algebra homomorphism if \( \varphi(\sigma \cdot x) = \sigma \cdot \varphi(x) \), for all \( \sigma \in \Sigma \) and \( x \in A \). Let \( A \) be a \( \Sigma \)-algebra and let \( I \subset A \) be an ideal. We call \( I \) a \( \Sigma \)-invariant ideal or simply a \( \Sigma \)-ideal if \( \Sigma \cdot I \subset I \). Clearly, all kernels of \( \Sigma \)-algebra homomorphisms are \( \Sigma \)-ideals.

**Definition 2.1.** Let \( A \) be a \( \Sigma \)-algebra and let \( X \subset A \) be a subset. We say that \( A \) is \( \Sigma \)-generated by \( X \) if \( A \) is generated by \( \Sigma \cdot X \) as \( K \)-algebra. In other words, \( A \)
coincides with the smallest $\Sigma$-subalgebra of $A$ containing $X$. In the same way, one defines $\Sigma$-generation for the $\Sigma$-ideals.

In the category of $\Sigma$-invariant algebras one can define free objects. In fact, let $X$ be a set and denote $x(\sigma)$ each element $(x, \sigma)$ of the product set $X(\Sigma) = X \times \Sigma$. Define $P = K[X(\Sigma)]$ the polynomial algebra in the commuting variables $x(\sigma)$. For any element $\sigma \in \Sigma$ consider the $K$-algebra endomorphism $\sigma: P \to P$ such that $x(\tau) \mapsto x(\sigma \tau)$, for all $x(\tau) \in X(\Sigma)$. Then, one has a faithful monoid representation $P : \Sigma \to \text{End}_K(P)$ such that $\rho(\sigma) = \sigma$ ad hence $P$ is a $\Sigma$-algebra. Note that if $\Sigma$ is a left-cancellative monoid then all maps $\rho(\sigma)$ are injective.

**Proposition 2.2.** Let $A$ be a $\Sigma$-algebra and let $f : X \to A$ be any map. Then, there is a unique $\Sigma$-algebra homomorphism $\varphi : P \to A$ such that $\varphi(x(1)) = f(x)$, for all $x \in X$.

**Proof.** It is sufficient to define $\varphi(x(\sigma)) = \sigma \cdot f(x)$, for all $x \in X$ and $\sigma \in \Sigma$. In fact, one has $\varphi(\tau \cdot x(\sigma)) = \varphi(\tau \sigma \cdot f(x)) = \tau \sigma \cdot f(x) = \tau \cdot (\sigma \cdot f(x)) = \tau \cdot \varphi(x(\sigma))$, for any $\tau \in \Sigma$. \hspace{1cm} $\square$

**Definition 2.3.** We call $P = K[X(\Sigma)]$ the free $\Sigma$-algebra generated by $X$. In fact, $P$ is $\Sigma$-generated by the subset $X(1)$.

From now on, we work only with free $\Sigma$-algebras and we assume that $X = \{x_0, x_1, \ldots\}$ is a finite or countable set and $\Sigma$ is a free commutative monoid generated by a finite set, say \{\sigma_1, \ldots, \sigma_r\}. Note that $\Sigma$ is a cancellative monoid isomorphic to $\mathbb{N}^r$ and the monomorphisms $\rho(\sigma) : P \to P$ have infinite order for all $\sigma \neq 1$. For any $x_i(\sigma) \in X(\Sigma)$, we call $i$ and $\sigma$ respectively the index and the weight of the variable $x_i(\sigma)$. If we put $X(\sigma) = \{x_i(\sigma) \mid x_i \in X\}$ one has clearly $P = \bigotimes_{\sigma \in \Sigma} K[X(\sigma)] = \bigotimes_{x_i \in X} K[x_i(\Sigma)]$, where all subalgebras $K[X(\sigma)]$ are isomorphic to $K[X]$ and all subalgebras $K[x_i(\Sigma)]$ to $K[\Sigma]$. Then, the free $\Sigma$-algebra $P = K[X(\Sigma)]$ is called the algebra of partial difference polynomials (with constant coefficients). The motivation for this name is the following. One understands the variables $x_i(1)$ as algebraically independent functions $u_i(t_1, \ldots, t_r)$ in the variables $t_j$ and the maps $\rho(\sigma_k)$ as the shift operators $u_i(t_1, \ldots, t_r) \mapsto u_i(t_1, \ldots, t_k + h, \ldots, t_r)$ where $h$ is a parameter (mesh step). If $\sigma = \prod_i \sigma_i^{a_i}$ then the variables $x_i(\sigma) = \sigma \cdot x_i(1)$ are the (algebraically independent) shifted functions $u_i(t_1 + a_1 h, \ldots, t_r + a_r h) = \sigma \cdot u_i(t_1, \ldots, t_r)$. Then, a $\Sigma$-ideal $I \subset P$ is also called a partial difference ideal. Since $P$ is not a Noetherian ring, note that such ideals have bases or $\Sigma$-bases which are generally infinite. One uses the term ordinary difference when $r = 1$.

In the next sections we generalize the Gröbner basis theory to the free $\Sigma$-algebra $P = K[X(\Sigma)]$. Clearly, one reobtains the classical theory when $\Sigma = \{1\}$ that is $P = K[X]$. The starting point is to define monomial orderings of $P$ which are compatible with the action of the monoid $\Sigma$.

### 3. Monomial $\Sigma$-orderings

Denote by $M = \text{Mon}(P)$ the set of all monomials of $P$. Note that even if the set $X(\Sigma)$ is infinite (in fact countable), one can endow $P$ by monomial orderings. This is an important consequence of the Higman’s Lemma [15] which can be stated in the following way (see for instance [1], Corollary 2.3 and remarks at beginning of page 5175).
Proposition 3.1. Let \( \prec \) be a total ordering on \( M \) such that

1. \( 1 \leq m \) for all \( m \in M \);
2. \( \prec \) is compatible with multiplication on \( M \), that is if \( m \prec n \) then \( tm \prec tn \), for any \( m, n, t \in M \).

Then \( \prec \) is also a well-ordering of \( M \) that is a monomial ordering of \( P \) if and only if the restriction of \( \prec \) to the variables set \( X(\Sigma) \) is a well-ordering.

Clearly, it is easy to assign well-orderings to the set \( X(\Sigma) \) which is in bijective correspondence to \( \mathbb{N}^{r+1} \). Note that the monoid \( \Sigma \) stabilizes the monomials set \( M \) since it stabilizes \( X(\Sigma) \). We introduce then the following notion.

Definition 3.2. Let \( \prec \) be a monomial ordering of \( P \). We call \( \prec \) a (monomial) \( \Sigma \)-ordering of \( P \) if \( \prec \) is compatible with the \( \Sigma \)-action on \( M \), that is \( m \prec n \) implies that \( \sigma \cdot m \prec \sigma \cdot n \) for all \( m, n \in M \) and \( \sigma \in \Sigma \).

A straightforward consequence of this definition is the following result.

Proposition 3.3. Let \( \prec \) be a monomial \( \Sigma \)-ordering of \( P \). Then \( m \preceq \sigma \cdot m \) for all \( m \in M \) and \( \sigma \in \Sigma \).

Proof. By contradiction, assume that there are \( m, \sigma \) such that \( m \succ \sigma \cdot m \). Then, \( \sigma \cdot m \succ \sigma^2 \cdot m \) and by induction one obtains the infinite descending chain \( m \succ \sigma \cdot m \succ \sigma^2 \cdot m \succ \ldots \) which contradicts that \( \prec \) is a well-ordering. \( \square \)

The orderings on the variable set \( X(\Sigma) \) that can be extended to monomial \( \Sigma \)-orderings are as follows.

Definition 3.4. Let \( \prec \) be a well-ordering of \( X(\Sigma) \). We call \( \prec \) a (variable) \( \Sigma \)-ranking of \( P \) if \( \prec \) is compatible with the \( \Sigma \)-action on \( X(\Sigma) \), that is \( u \prec v \) implies that \( \sigma \cdot u \prec \sigma \cdot v \) for all \( u, v \in X(\Sigma) \) and \( \sigma \in \Sigma \).

As for Proposition 3.3 we have that if \( \prec \) is a \( \Sigma \)-ranking then \( u \preceq \sigma \cdot u \) for all \( u \in X(\Sigma) \) and \( \sigma \in \Sigma \). Moreover, if \( X \) is a finite set then condition \( u \preceq \sigma \cdot u \) for all \( u, \sigma \) implies that \( \prec \) is a well-ordering by applying Dickson’s Lemma (or Higman’s Lemma) to \( \Sigma \) which is isomorphic to \( \mathbb{N}^r \). However, note that in this paper the set \( X \) may be also countable.

Owing to the decompositions \( X(\Sigma) = \bigcup_{\sigma \in \Sigma} X(\sigma) = \bigcup_{x_i \in X} x_i(\Sigma) \) of the variable set of the ring \( P \), we can define \( \Sigma \)-rankings of \( P \) in a natural way. Denote by \( Q \) the monoid \( K \)-algebra defined by the free commutative monoid \( \Sigma = \langle \sigma_1, \ldots, \sigma_r \rangle \). In other words, \( Q = K[\sigma_1, \ldots, \sigma_r] \) is the polynomial algebra in the commutative variables \( \sigma_i \). From now on, we assume that \( \Sigma \) is endowed with a monomial ordering \( \prec \) of \( Q \). By abuse, we call \( \prec \) a monomial ordering of \( \Sigma \).

Definition 3.5. Fix \( \prec \) a monomial ordering of \( \Sigma \). For all \( x_i(\sigma), x_j(\tau) \in X(\Sigma) \), we define:

1. \( x_i(\sigma) \prec x_j(\tau) \) if and only if \( \sigma < \tau \) or \( \sigma = \tau \) and \( i < j \). In other words, \( X(\sigma) \prec X(\tau) \) when \( \sigma < \tau \).
2. \( x_i(\sigma) \prec' x_j(\tau) \) if and only if \( i < j \) or \( i = j \) and \( \sigma < \tau \). In other words, \( x_i(\Sigma) \prec' x_j(\Sigma) \) when \( i < j \).

Clearly \( \prec \) and \( \prec' \) are both \( \Sigma \)-rankings of \( P \) that we call respectively weight and index \( \Sigma \)-ranking defined by a monomial ordering of \( \Sigma \).
For all \( x_i \in X \) and \( \sigma \in \Sigma \) denote \( P(\sigma) = K[X(\sigma)] \), \( M(\sigma) = \text{Mon}(P(\sigma)) \) and \( P(x_i) = K[x_i(\Sigma)] \), \( M(x_i) = \text{Mon}(P(x_i)) \). Owing to the decompositions \( P = \bigotimes_{\sigma \in \Sigma} P(\sigma) = \bigotimes_{\sigma \in \Sigma} P(x_i) \), one has that a monomial \( m \in M \) can be uniquely written as \( m = m(\delta_1) \cdots m(\delta_k) = m(x_{i_k}) \cdots m(x_{i_1}) \), where \( m(\delta_p) \in M(\delta_p), m(x_{i_p}) \in M(x_{i_p}) \) and \( \delta_1 > \ldots > \delta_k, i_1 > \ldots > i_k \). By means of such presentations we can define block monomial orderings of \( P \) extending weight and index ranking. Recall that \( \rho : \Sigma \rightarrow \text{End}_K(P) \) is the faithful monoid representation defined by the action of \( \Sigma \) over \( P \). For any \( \sigma \in \Sigma \) one has that the map \( \rho(\sigma) \) defines an isomorphism between the monoids \( M(1), M(\sigma) \) and hence between the algebras \( P(1), P(\sigma) \). In other words, we have \( M(\sigma) = \sigma \cdot M(1), P(\sigma) = \sigma \cdot P(1) \).

**Definition 3.6.** Fix \( \prec \) a monomial ordering of the subalgebra \( P(1) \subset P \) and extend it to all subalgebras \( P(\sigma) (\sigma \in \Sigma) \) by the isomorphisms \( \rho(\sigma) \). In other words, we put \( \sigma \cdot m \prec \sigma \cdot n \) if and only if \( m \prec n \), for any \( m, n \in M(1) \). Then, for all \( m, n \in M, m = m(\delta_1) \cdots m(\delta_k), n = n(\delta_1) \cdots n(\delta_l) \) with \( \delta_1 > \ldots > \delta_k \), we define \( m \prec_w n \) if and only if \( m(\delta_j) = n(\delta_j) \) if \( j < i \) and \( m(\delta_i) < n(\delta_i) \) for some \( 1 \leq i \leq k \). Clearly, the restriction of \( \prec_w \) to the variables of \( P \) is just the weight \( \Sigma \)-ranking.

**Proposition 3.7.** The ordering \( \prec_w \) is a \( \Sigma \)-ordering of \( P \).

**Proof.** Note that if \( m = m(\delta_1) \cdots m(\delta_k) \in M \) with \( m(\sigma_1) \in M(\sigma_1) \) and \( \delta_1 > \ldots > \delta_k \), then \( \sigma \cdot m = m(\sigma \delta_1) \cdots m(\sigma \delta_k) \), where \( m(\sigma \delta_i) \in M(\sigma \delta_i) \) and \( \sigma \delta_1 > \ldots > \sigma \delta_k \) since \( \sigma \prec \sigma \cdot n \) is a monomial ordering of \( \Sigma \). Assume \( m \prec_w n \) that is \( m(\delta_i) = n(\delta_i) \) for \( j < i \) and \( m(\delta_i) < n(\delta_i) \). Clearly \( m(\sigma \delta_j) = n(\sigma \delta_j) \) for \( j < i \) and one has \( m(\delta_i) < n(\delta_i) \) if and only if \( m(1) < n(1) \) if and only if \( m(\sigma \delta_i) < n(\sigma \delta_i) \).

Then, we conclude that \( \sigma \cdot m \prec_w \sigma \cdot n \). \( \square \)

Note that we have also a monoid faithful representation \( \phi : \mathbb{N} \rightarrow \text{End}_K(P) \) such that the endomorphism \( \phi(i) \) is defined as \( x_j(\sigma) \rightarrow x_{i+j} (\sigma) \) for any \( i, j \geq 0 \) and \( \sigma \in \Sigma \). Clearly \( \phi(i) \) induces isomorphism between the monoids \( M(x_0), M(x_i) \) and the algebras \( P(x_0), P(x_i) \). The algebra \( P(x_0) \) can be easily endowed with a \( \Sigma \)-ordering. For instance, since \( P(x_0) = \bigotimes_{\sigma \in \Sigma} K[x_0(\sigma)] \) one can define a lexicographic ordering as in Definition 3.6.

**Definition 3.8.** Fix \( \prec \) a monomial \( \Sigma \)-ordering of the subalgebra \( P(x_0) \subset P \) and extend it to all subalgebras \( P(x_i) (x_i \in X) \) by the isomorphisms \( \phi(i) \). For any \( m, n \in M, m = m(x_{i_k}) \cdots m(x_{i_1}), n = n(x_{i_k}) \cdots n(x_{i_1}) \) with \( i_k > \ldots > i_1 \) we put \( m \prec_i n \) if and only if \( m(x_{i_k}) = n(x_{i_k}) \) if \( q < p \) and \( m(x_{i_p}) < n(x_{i_p}) \) for some \( 1 \leq p \leq k \). Note that the restriction of \( \prec_i \) to the variables of \( P \) is the index \( \Sigma \)-ranking.

**Proposition 3.9.** The ordering \( \prec_i \) is a \( \Sigma \)-ordering of \( P \).

**Proof.** Note that if \( m = m(x_{i_k}) \cdots m(x_{i_1}) \in M \) with \( m(x_{i_p}) \in M(x_{i_p}) \) and \( i_k > \ldots > i_1 \) then \( \sigma \cdot m = m'(x_{i_k}) \cdots m'(x_{i_1}) \) where \( m'(x_{i_p}) = \sigma \cdot m(x_{i_p}) \in M(x_{i_p}) \).

Suppose \( m \prec_i n \) that is \( m(x_{i_k}) = n(x_{i_k}) \) if \( q < p \) and \( m(x_{i_p}) < n(x_{i_p}) \). We have clearly that \( m'(x_{i_k}) = n'(x_{i_k}) \). Moreover, since \( \prec \) is a \( \Sigma \)-ordering of \( P(x_0) \) and therefore of \( P(x_{i_p}) \), one has also \( m'(x_{i_p}) \prec n'(x_{i_p}) \) that is \( \sigma \cdot m \prec_i \sigma \cdot n \). \( \square \)

We call the above monomial \( \Sigma \)-orderings \( \prec_w, \prec_i \) of \( P \) respectively weight \( \Sigma \)-ordering defined by a monomial ordering of \( P(1) \) and index \( \Sigma \)-ordering of \( P \) defined by a monomial \( \Sigma \)-ordering of \( P(x_0) \). Clearly, both these orderings depend also on a monomial ordering of \( \Sigma \). Note that index \( \Sigma \)-orderings are suitable for generation
of finite difference schemes for partial differential equations [10] [11]. The weight \( \Sigma \)-
orderings are instead compatible with the gradings of the \( \Sigma \)-algebra \( P \) we introduce
in Section 5. For this reason they are suitable for obtaining complete Gröbner bases
from partial computations.

4. Gröbner \( \Sigma \)-bases

From now on, we consider \( P \) endowed with a monomial \( \Sigma \)-ordering \( \prec \). Let
\( f = \sum_i c_i m_i \in P \) with \( m_i \in M, c_i \in K, c_i \neq 0 \). We denote as usual \( \text{lm}(f) = m_k = \max_\prec \{ m_i \} \), \( \text{lc}(f) = c_k \) and \( \text{lt}(f) = \text{lc}(f) \text{lm}(f) \). If \( G \subset P \) we put \( \text{lm}(G) = \{ \text{lm}(f) \mid f \in G, f \neq 0 \} \) and we denote as \( \text{LM}(G) \) the ideal of \( P \) generated by \( \text{lm}(G) \).

**Proposition 4.1.** Let \( G \subset P \). Then \( \text{lm}(\Sigma \cdot G) = \Sigma \cdot \text{lm}(G) \). In particular, if \( I \) is
a \( \Sigma \)-ideal of \( P \) then \( \text{LM}(I) \) is also \( \Sigma \)-ideal.

**Proof.** Since \( P \) is endowed with a \( \Sigma \)-ordering, one has that \( \text{lm}(\sigma \cdot f) = \sigma \cdot \text{lm}(f) \) for
any \( f \in P, f \neq 0 \) and \( \sigma \in \Sigma \). Then, \( \Sigma \cdot \text{lm}(I) = \text{lm}(\Sigma \cdot I) \subset \text{lm}(I) \) and therefore
\( \text{LM}(I) = \langle \text{lm}(I) \rangle \) is a \( \Sigma \)-ideal. \( \square \)

**Definition 4.2.** Let \( I \subset P \) be a \( \Sigma \)-ideal and \( G \subset I \). We call \( G \) a Gröbner \( \Sigma \)-basis
of \( I \) if \( \text{lm}(G) \) is a \( \Sigma \)-basis of \( \text{LM}(I) \). In other words, \( \Sigma \cdot G \) is a Gröbner basis of \( I \)
as \( P \)-ideal.

Since the monoid \( \Sigma \) is assumed isomorphic to \( \mathbb{N}^r \) that is \( \Sigma \)-ideals are partial
difference ideals, we may say that Gröbner \( \Sigma \)-bases are partial difference Gröbner bases [9]. Another possible name is \( \Sigma \)-equivariant Gröbner bases [4]. Simplicity and
generality lead us to the previous definition that already appeared in [18].

Let \( f, g \in P, f, g \neq 0 \) and put \( \text{lt}(f) = cm, \text{lt}(g) = dn \) with \( m, n \in M \) and \( c, d \in K \).
If \( l = \text{lcm}(m, n) \) we define as usual the \( S \)-polynomial \( \text{spoly}(f, g) = (l/cm)f-(l/dn)g \).
Clearly \( \text{spoly}(f, g) = -\text{spoly}(g, f) \) and \( \text{spoly}(f, f) = 0 \).

**Proposition 4.3.** For all \( f, g \in P, f, g \neq 0 \) and for any \( \sigma \in \Sigma \) one has \( \sigma \cdot \text{spoly}(f, g) = \text{spoly}(\sigma \cdot f, \sigma \cdot g) \).

**Proof.** Since \( \Sigma \) acts on the variable set \( \text{X}(\Sigma) \) by injective maps, it is sufficient to
note that \( \sigma \cdot \text{lcm}(m, n) = \text{lcm}(\sigma \cdot m, \sigma \cdot n) \) for all \( m, n \in M \) and \( \sigma \in \Sigma \). \( \square \)

The following definition is a standard tool in Gröbner bases theory.

**Definition 4.4.** Let \( f \in P, f \neq 0 \) and \( G \subset P \). If \( f = \sum_i f_i g_i \) with \( f_i \in P, g_i \in G \)
and \( \text{lm}(f) \geq \text{lm}(f_i) \text{lm}(g_i) \) for all \( i \), we say that \( f \) has a Gröbner representation
respect to \( G \).

Note that if \( f = \sum_i f_i g_i \) is a Gröbner representation then \( \sigma \cdot f = \sum_i (\sigma \cdot f_i)(\sigma \cdot g_i) \)
is also a Gröbner representation, for any \( \sigma \in \Sigma \). In fact, since \( \prec \) is a \( \Sigma \)-ordering
of \( P \) one has that \( \text{lm}(f) \geq \text{lm}(f_i) \text{lm}(g_i) \) implies that \( \text{lm}(\sigma \cdot f) = \sigma \cdot \text{lm}(f) \geq
(\sigma \cdot \text{lm}(f_i))(\sigma \cdot \text{lm}(g_i)) = \text{lm}(\sigma \cdot f_i) \text{lm}(\sigma \cdot g_i) \) for all \( i \). A celebrated result from Bruno
Buchberger [5] is the following.

**Proposition 4.5** (Buchberger’s criterion). Let \( G \) be a basis of the ideal \( I \subset P \).
Then, \( G \) is a Gröbner basis of \( I \) if and only if for all \( f, g \in G, f, g \neq 0 \) the \( S \)-
polynomial \( \text{spoly}(f, g) \) has a Gröbner representation with respect to \( G \).
Usually the above result, see for instance [3], is stated when \( P \) is a polynomial algebra with a finite number of variables and \( G \) is a finite set. In fact, such assumptions are not needed since Noetherianity is not used in the proof, but only the existence of a monomial ordering for \( P \). See also the comprehensive Bergman’s paper [2] where the “Diamond Lemma” is proved without any restriction on the finiteness of the variable set. We want now to prove a generalization of the Buchberger’s criterion for Gröbner \( \Sigma \)-bases of \( P \). For this purpose it is useful to introduce the following notations.

**Definition 4.6.** Let \( \sigma = \prod_i \sigma_i^{\alpha_i}, \tau = \prod_i \sigma_i^{\beta_i} \in \Sigma \). We denote \( \gcd(\sigma, \tau) = \prod_i \sigma_i^{\gamma_i} \) where \( \gamma_i = \min(\alpha_i, \beta_i) \), for any \( i \).

**Proposition 4.7** (\( \Sigma \)-criterion). Let \( G \) be a \( \Sigma \)-basis of a \( \Sigma \)-ideal \( I \subset P \). Then, \( G \) is a Gröbner \( \Sigma \)-basis of \( I \) if and only if for all \( f, g \in G, f, g \neq 0 \) and for any \( \sigma, \tau \in \Sigma \) such that \( \gcd(\sigma, \tau) = 1 \), the \( S \)-polynomial \( \text{spoly}(\sigma \cdot f, \tau \cdot g) \) has a Gröbner representation with respect to \( \Sigma \cdot G \).

**Proof.** We prove that \( \Sigma \cdot G \) is a Gröbner basis of \( I \) and we make use of the Proposition \ref{prop:reduce}. Then, consider any pair of elements \( \sigma \cdot f, \tau \cdot g \in \Sigma \cdot G \) where \( f, g \in G, f, g \neq 0 \) and \( \sigma, \tau \in \Sigma \). Put \( \delta = \gcd(\sigma, \tau) \) and hence \( \sigma = \delta \sigma', \tau = \delta \tau' \) with \( \sigma', \tau' \in \Sigma, \gcd(\sigma', \tau') = 1 \). By Proposition \ref{prop:reduce} we have \( \text{spoly}(\sigma \cdot f, \tau \cdot g) = \delta \cdot \text{spoly}(\sigma' \cdot f, \tau' \cdot g) \). By hypothesis, assume that \( \text{spoly}(\sigma' \cdot f, \tau' \cdot g) = h = \sum_{\nu} f_{\nu}(\nu \cdot g_{\nu}) \), with \( \nu \in \Sigma, f_{\nu}, g_{\nu} \in G \), is a Gröbner representation with respect to \( \Sigma \cdot G \). Since \( \prec \) is a \( \Sigma \)-ordering of \( P \), we conclude that we have also the Gröbner representation \( \text{spoly}(\sigma \cdot f, \tau \cdot g) = \delta \cdot h = \sum_{\nu}(\delta \cdot f_{\nu})(\delta \nu \cdot g_{\nu}) \).

A standard procedure in the Buchberger’s algorithm is the following.

**Algorithm 4.1** **Reduce**

- **Input:** \( G \subset P \) and \( f \in P \).
- **Output:** \( h \in P \) such that \( f - h \in \langle G \rangle \) and \( h = 0 \) or \( \text{lm}(h) \notin \text{LM}(G) \).
- \( h := f \);
- **while** \( h \neq 0 \) and \( \text{lm}(h) \in \text{LM}(G) \) **do**
  - choose \( g \in G, g \neq 0 \) such that \( \text{lm}(g) \) divides \( \text{lm}(h) \);
  - \( h := h - (\text{lt}(h)/\text{lt}(g))g \);
- **end while:**
- **return** \( h \).

Note that the termination of **REDUCE** is provided since \( \prec \) is a monomial ordering of \( P \). In particular, even if \( G \) is an infinite set, there are only a finite number of elements \( g \in G, g \neq 0 \) such that \( \text{lm}(g) \) divides \( \text{lm}(h) \) and hence \( \text{lm}(g) \leq \text{lm}(h) \).

It is well-known that if \( \text{REDUCE}(f, G) = 0 \) then \( f \) has a Gröbner representation with respect to \( G \). Moreover, if \( \text{REDUCE}(f, G) = h \neq 0 \) then clearly one has \( \text{REDUCE}(f, G \cup \{h\}) = 0 \). Therefore, from Proposition \ref{prop:reduce} it follows immediately the correctness of the following algorithm.
Algorithm 4.2 SigmaGBasis

Input: $H$, a $\Sigma$-basis of a $\Sigma$-ideal $I \subset P$.
Output: $G$, a Gröbner $\Sigma$-basis of $I$.

$G := H$;
$B := \{(f,g) \mid f, g \in G\}$;

while $B \neq \emptyset$ do
    choose $(f,g) \in B$;
    $B := B \setminus \{(f,g)\}$;
    for all $\sigma, \tau \in \Sigma$ such that $\gcd(\sigma, \tau) = 1$ do
        $h := \text{Reduce}(\text{spoly}(\sigma \cdot f, \tau \cdot g), \Sigma \cdot G)$;
        if $h \neq 0$ then
            $B := B \cup \{(g, h), (h, h) \mid g \in G\}$;
            $G := G \cup \{h\}$;
        end if;
    end for;
end while;
return $G$.

Clearly, all well-known criteria (product criterion, chain criterion, etc) can be applied to SigmaGBasis to shorten the number of S-polynomials to be considered. In fact, one can understand this algorithm as the usual Buchberger’s procedure applied to the basis $\Sigma \cdot H$, where an additional criterion to avoid “useless pairs” is given by Proposition 4.7. Owing to Non-Noetherianity of the ring $P$, note that the termination of SigmaGBasis is not provided in general and this is, in fact, one of the main problems in differential/difference algebra. Nevertheless, in the next section we introduce some suitable grading for the algebra $P$ which provides that a truncated version of the algorithm SigmaGBasis with homogeneous input stops in a finite number of steps. Some variant of the algorithm SigmaGBasis appeared in [9] and before in [17, 18] for the ordinary difference case.

5. Gradings of $P$ compatible with $\Sigma$-action

We want now to introduce some gradings of the algebra $P = K[X(\Sigma)]$ which are compatible with $\Sigma$-action and formation of least common multiples in $M = \text{Mon}(P)$. As before, we fix a monomial order $<$ of $\Sigma$. We start extending the structure $(\Sigma, \max, \cdot)$ in the following way.

**Definition 5.1.** Let $0$ be an element disjoint by $\Sigma$ and put $\hat{\Sigma} = \Sigma \cup \{0\}$. Then, we define a commutative idempotent monoid $(\hat{\Sigma}, +)$ with identity $0$ that extends the monoid $(\Sigma, \max)$ (with identity $1$) by imposing that $0 + \sigma = \sigma$, for any $\sigma \in \Sigma$. Moreover, we define a commutative monoid $(\hat{\Sigma}, \cdot)$ with identity $1$ extending the monoid $(\Sigma, \cdot)$ by putting $0 \cdot \sigma = 0$, for all $\sigma \in \hat{\Sigma}$. Since multiplication clearly distributes over addition, one has that $(\hat{\Sigma}, +, \cdot)$ is a commutative idempotent semiring, also known as commutative dioid [13].

Note that the faithful monoid representation $\rho : \Sigma \to \text{End}_K(P)$ can be extended to $\hat{\Sigma}$ where $\rho(0) : P \to P$ is the algebra endomorphism such that $x_i(\sigma) \mapsto 0$, for all $x_i(\sigma) \in X(\Sigma)$.

**Definition 5.2.** Let $w : M \to \hat{\Sigma}$ be the unique mapping such that
(i) \( w(1) = 0; \)
(ii) \( w(mn) = w(m) + w(n), \) for any \( m, n \in M; \)
(iii) \( w(x_i(\sigma)) = \sigma, \) for all \( i \geq 0 \) and \( \sigma \in \Sigma. \)

Note that (i),(ii) state that \( w \) is a monoid homomorphism from the free commutative monoid \((M, \cdot)\) to \((\hat{\Sigma}, +)\). We call \( w \) the weight function of \( P. \)

More explicitly, if \( m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k} \) is any monomial of \( P \) different from \( 1 \) then \( w(m) = \delta_1 + \cdots + \delta_k = \max_\Sigma(\delta_1, \ldots, \delta_k). \) We denote \( M_\sigma = \{ m \in M \mid w(m) = \sigma \} \) and define \( P_\sigma \subseteq P \) the subspace spanned by \( M_\sigma, \) for any \( \sigma \in \hat{\Sigma}. \)

Because \( w : (M, \cdot) \to (\hat{\Sigma}, +) \) is a monoid homomorphism, one has that \( P = \bigoplus_{\sigma \in \hat{\Sigma}} P_\sigma \) is a grading of the algebra \( P \) over the commutative monoid \((\hat{\Sigma}, +). \) If \( f \in P_\sigma \) we say that \( f \) is a \( w \)-homogeneous element and we put \( w(f) = \sigma. \) Recall that for any \( \sigma \in \Sigma \) we denoted \( P(\sigma) = K[X(\sigma)] \) which is a subalgebra of \( P = K[X(\Sigma)] \) isomorphic to \( K[X]. \) If we put \( P(0) = P_0 = K \) then one has that \( P(\sigma) = \bigoplus_{\tau \leq \sigma} P_\tau = \bigotimes_{\tau \leq \sigma} P(\tau) \) is a subalgebra of \( P. \) Since \( < \) is a well-ordering of \( \Sigma, \) if \( X \) is a finite set then the sequence \( \{ P(\sigma) \mid \sigma \in \hat{\Sigma} \} \) is a filtration of \( P \) consisting of Noetherian subalgebras. Finally, note that \( P(1) = P_0 \oplus P_1 \) is isomorphic to \( K[X]. \)

**Proposition 5.3.** The weight function satisfies the following properties:
(i) \( w(\sigma \cdot m) = \sigma w(m), \) for any \( \sigma \in \Sigma \) and \( m \in M; \)
(ii) \( w(\text{lcm}(m,n)) = w(mn) = w(m) + w(n), \) for all \( m, n \in M. \) Then, \( m \mid n \) implies that \( w(m) \leq w(n). \)

**Proof.** If \( m = 1 \) then \( w(\sigma \cdot m) = w(m) = 0 = \sigma w(m). \) If otherwise \( m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k} \) with \( \delta_1 > \cdots > \delta_k \) then \( \sigma \cdot m = x_{i_1}(\sigma \delta_1)^{\alpha_1} \cdots x_{i_k}(\sigma \delta_k)^{\alpha_k} \) where \( \sigma \delta_1 > \cdots > \sigma \delta_k \) since \( < \) is a monomial ordering of \( \Sigma. \) We conclude that \( w(\sigma \cdot m) = \sigma \delta_1 = \sigma w(m). \) To prove (ii) it is sufficient to note that the weight of a monomial does not depend on the exponents of the variables occurring in it. \( \square \)

Note that the property (i) implies that the map \( w \) is a homomorphism with respect to the action of \( \Sigma \) on \( M \) and \( \hat{\Sigma}. \) In other words, one has that \( \sigma P_\tau \subseteq P_{\sigma \tau} \) for any \( \sigma \in \Sigma, \tau \in \hat{\Sigma}. \) Moreover, the property (ii) means that \( w \) is also a monoid homomorphism from \((M, \text{lcm})\) to \((\hat{\Sigma}, +). \)

**Definition 5.4.** Let \( I \) be an ideal of \( P. \) We call \( I \) a \( w \)-graded ideal if \( I = \sum_\sigma I_\sigma \) with \( I_\sigma = I \cap P_\sigma. \) In this case, if \( I \) is also a \( \Sigma \)-ideal then \( \sigma \cdot I_\tau \subseteq I_{\sigma \tau} \) for all \( \sigma \in \Sigma, \tau \in \hat{\Sigma}. \)

Owing to the \( w \)-grading of \( P, \) one can show that a truncated version of the algorithm SigmaGBasis admits termination. If \( f, g \in P, f \neq g \) are \( w \)-homogeneous elements then the S-polynomial \( h = \text{spoly}(f, g) \) is clearly \( w \)-homogeneous too. Moreover, by property (ii) of Proposition 5.3, we have that \( w(h) = w(f) + w(g) \) and hence if \( w(f), w(g) \leq \delta \) then also \( w(h) \leq \delta, \) for some \( \delta \in \Sigma. \) By means of this remark, one obtains immediately the following result.

**Proposition 5.5** (Truncated termination over the weight). Let \( I \subseteq P \) be a \( w \)-graded \( \Sigma \)-ideal and fix \( \delta \in \Sigma. \) Assume \( I \) has a \( w \)-homogeneous basis \( H \) such that \( H_\delta = \{ f \in H \mid w(f) \leq \delta \} \) is a finite set. Then, there is a \( w \)-homogeneous Gröbner \( \Sigma \)-basis \( G \) of \( I \) such that \( G_\delta \) is also a finite set. In other words, if we consider for the algorithm SigmaGBasis a selection strategy of the S-polynomials based on their weights ordered by \( <, \) we obtain that the \( \delta \)-truncated version of SigmaGBasis stops in a finite number of steps.
Proof. First of all, note that the algorithm SIGMABASIS computes essentially a subset $G$ of a Gröbner basis $\Sigma \cdot G$ obtained by applying the Buchberger algorithm to the basis $\Sigma \cdot H$ of $I$. Moreover, by Proposition 5.5 the elements of $\Sigma \cdot H$ and hence of $\Sigma \cdot G$ are all w-homogeneous. Denote $H'_\delta = \{ \sigma \cdot f \mid \sigma \in \Sigma, f \in H, \sigma w(f) \leq \delta \}$. Since $<$ is a monomial order of $\Sigma$ and $H'_\delta$ is a finite set one has that $H'_\delta$ is also a finite set. We consider therefore $X'_\delta$ the finite set of variables of $P$ occurring in the elements of $H'_\delta$ and define $P(\delta) = K[X'_\delta] \subset P$. In fact, the $\delta$-truncated algorithm SIGMABASIS computes a subset of a Gröbner basis of the ideal $I^{(\delta)} \subset P^{(\delta)}$ generated by $H'_\delta$. By Noetherianity of the finitely generated polynomial ring $P^{(\delta)}$ we clearly obtain termination. \hfill \Box

Clearly the above result provides algorithmic solution to the ideal membership for finitely generated w-graded $\Sigma$-ideals. Note that if $r = 0$ that is $\Sigma = \{1\}$ then the algorithm SIGMABASIS coincides with classical Buchberger’s algorithm and Proposition 5.5 states that if $I$ is a finitely generated ideal of $P = P_0 \oplus P_1 = K[x_0, x_1, \ldots]$ then $I$ has also a finite Gröbner basis. According with the above proof, this is a consequence of the fact that the Buchberger’s algorithm runs over the finite number of variables occurring in the generators of $I$.

Another useful grading of $P$ can be introduced in the following way. Consider the set $\hat{\mathbb{N}} = \mathbb{N} \cup \{-\infty\}$ endowed with the binary operations max and +. Then $(\hat{\mathbb{N}}, \text{max}, +)$ is clearly a commutative idempotent semiring (or commutative dioid or max-plus algebra). Define $\deg : \hat{\Sigma} \to \hat{\mathbb{N}}$ the mapping such that $\deg(0) = -\infty$ and $\deg(\sigma) = \sum_i \alpha_i$, for any $\sigma = \prod_i \sigma_i^{\alpha_i}$. Clearly $\deg$ is a monoid homomorphism from $(\hat{\Sigma}, \cdot)$ to $(\hat{\mathbb{N}}, +)$.

**Definition 5.6.** Let $\text{ord} : M \to \hat{\mathbb{N}}$ be the unique mapping such that

(i) $\text{ord}(1) = -\infty$;

(ii) $\text{ord}(mn) = \max(\text{ord}(m), \text{ord}(n))$, for any $m, n \in M$;

(iii) $\text{ord}(x_i(\sigma)) = \deg(\sigma)$, for all $i \geq 0$ and $\sigma \in \Sigma$.

Clearly (i),(ii) state that $\text{ord}$ is a monoid homomorphism from $(M, \cdot)$ to $(\hat{\mathbb{N}}, \text{max})$. We call $\text{ord}$ the order function of $P$.

For any monomial $m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k}$ different from 1 we have that $\text{ord}(m) = \max(\text{deg}(\delta_1), \ldots, \text{deg}(\delta_k))$. Clearly, the order function defines a grading $P = \bigoplus_{d \in \mathbb{N}} P_d$ of the algebra $P$ over the commutative monoid $(\hat{\mathbb{N}}, \text{max})$. Define $P^{(d)} = \bigoplus_{i \leq d} P_i = \bigotimes_{\text{deg}(\sigma) \leq d} P(\sigma)$ which is a subalgebra of $P$. Then, if $X$ is a finite set we have that the sequence $\{ P^{(d)} \mid d \in \hat{\mathbb{N}} \}$ is a filtration of $P$ with Noetherian subalgebras where $P^{(0)} = P_{-\infty} \oplus P_0$ is isomorphic to $K[X]$. If the monomial order $<$ of $\Sigma$ is compatible with $\deg$ that is $\deg(\sigma) < \deg(\tau)$ implies that $\sigma < \tau$ for any $\sigma, \tau \in \Sigma$, note that $\text{ord}(m) = \deg(w(m))$ for all $m \in M$. Moreover, the weight and order functions clearly coincide when $r = 1$.

**Proposition 5.7.** The order function verifies the following:

(i) $\text{ord}(\sigma \cdot m) = \deg(\sigma) + \text{ord}(m)$, for any $\sigma \in \Sigma$ and $m \in M$;

(ii) $\text{ord}(\text{lcm}(m, n)) = \text{ord}(mn) = \max(\text{ord}(m), \text{ord}(n))$, for all $m, n \in M$. Therefore, if $m | n$ then $\text{ord}(m) \leq \text{ord}(n)$.

*Proof. If $m = 1$ one has $\text{ord}(\sigma \cdot m) = \text{ord}(m) = -\infty = \deg(\sigma) + \text{ord}(m)$. If otherwise $m = x_{i_1}(\delta_1)^{\alpha_1} \cdots x_{i_k}(\delta_k)^{\alpha_k}$ then $\sigma \cdot m = x_{i_1}(\sigma \delta_1)^{\alpha_1} \cdots x_{i_k}(\sigma \delta_k)^{\alpha_k}$ and hence...*
ord(\sigma \cdot m) = \max(\deg(\sigma \delta_1), \ldots, \deg(\sigma \delta_k)) = \deg(\sigma) + \max(\deg(\delta_1), \ldots, \deg(\delta_k)) = \deg(\sigma) + \text{ord}(m). \quad \text{Property (ii) follows immediately as in Proposition 5.3} \quad \Box

**Definition 5.8.** Let \( I \) be an ideal of \( P \). We call \( I \) a \( \Sigma \)-graded ideal if \( I = \sum_i I_i \) with \( I_i = I \cap P_i \). If \( I \) is also a \( \Sigma \)-ideal then \( \sigma \cdot I_i \subset I_{\text{deg}(\sigma) + i} \) for any \( \sigma \in \Sigma \) and \( i \in \mathbb{N} \).

Consider now \( f, g \in P, f \neq g \) \( \sigma \)-homogeneous elements. The S-polynomial \( h = \text{spoly}(f, g) \) is clearly \( \sigma \)-homogeneous and \( \text{ord}(h) = \max(\text{ord}(f), \text{ord}(g)) \). Then \( \text{ord}(f), \text{ord}(g) \leq d \) implies that \( \text{ord}(h) \leq d \), for some \( d \in \mathbb{N} \) and one proves the following result as for Proposition 5.5.

**Proposition 5.9** (Truncated termination over the order). Let \( I \subset P \) be an \( \sigma \)-graded \( \Sigma \)-ideal and fix \( d \in \mathbb{N} \). Assume \( I \) has an \( \sigma \)-homogeneous basis of \( H \) such that \( H_d = \{ f \in H \mid \text{ord}(f) \leq d \} \) is a finite set. Then, there is an \( \sigma \)-homogeneous \( \Sigma \)-basis \( G \subset I \) such that \( G_d \) is also a finite set. In other words, if we consider for SIGMA\text{GBASIS} a selection strategy of the S-polynomials based on their orders, we have that the \( d \)-truncated version of SIGMA\text{GBASIS} terminates in a finite number of steps.

By means of weight and order functions one has criterions, also in the non-graded case, that provide that a \( \sigma \)-homogeneous \( \Sigma \)-basis is eventually finite complete one even if it has been computed within some bounded weight or order for the algebra \( P \) that is over a finite number of variables. This is of course important because actual computations can be only performed in such a way. As before, we fix a monomial ordering \( \prec \) of \( \Sigma \).

**Definition 5.10.** Let \( \prec \) be a monomial \( \Sigma \)-ordering of \( P \). We call \( \prec \) compatible with the weight function if \( w(m) < w(n) \) implies that \( m \prec n \), for all \( m, n \in M \). In a similar way, one defines when \( \prec \) is compatible with the order function.

**Proposition 5.11.** Let \( \prec_w \) be a weight \( \Sigma \)-ordering as in Definition 3.6. Then \( \prec_w \) is compatible with the weight function. In particular, if the monomial order \( \prec \) of \( \Sigma \) is compatible with \( \deg \) then \( \prec_w \) is also compatible with the order function.

**Proof.** Let \( m = m(\delta_1) \cdots m(\delta_k), n = n(\delta_1) \cdots n(\delta_k) \) two monomials of \( P \) with \( m(\delta_i), n(\delta_i) \in M(\delta_i) \) and \( \delta_1 > \ldots > \delta_k \). Assume \( m \prec_w n \) that is \( m(\delta_j) = n(\delta_j) \) if \( j < i \) and \( m(\delta_i) < n(\delta_i) \) for some \( 1 \leq i \leq k \). If \( i > 1 \) or \( m(\delta_i) \neq n(\delta_i) \) then clearly \( w(m) = w(n) = \delta_1 \). Otherwise, we conclude \( w(m) < \delta_1 = w(n) \). Moreover, if \( \prec \) is compatible with \( \deg \) then \( \text{ord}(m) = \deg(w(m)) < \deg(w(n)) = \text{ord}(n) \) implies that \( w(m) < w(n) \) and hence \( m \prec_w n \).

For any \( \delta \in \Sigma, d \in \mathbb{N} \) define now \( \Sigma_\delta = \{ \sigma \in \Sigma \mid \sigma \leq \delta \} \) and \( \Sigma_d = \{ \sigma \in \Sigma \mid \deg(\sigma) \leq d \} \).

**Proposition 5.12** (Finite \( \Sigma \)-criterion). Assume the \( \Sigma \)-ordering of \( P \) is compatible with the weight function. Let \( G \subset P \) be a finite set and denote \( I \) the \( \Sigma \)-ideal generated by \( G \). Moreover, define \( \delta = \max_{\prec \{ w(\text{lm}(g)) \mid g \in G \}} \). Then, \( G \) is a \( \Sigma \)-Gröbner basis of \( I \) if and only if for all \( f, g \in G \) and for any \( \sigma, \tau \in \Sigma \) such that \( \gcd(\sigma, \tau) = 1 \) and \( \gcd(\sigma \cdot \text{lm}(f), \tau \cdot \text{lm}(g)) \neq 1 \) the S-polynomial \( \text{spoly}(\sigma \cdot f, \tau \cdot g) \) has a \( \Sigma \)-Gröbner representation with respect to the finite set \( \Sigma_{\delta} \cdot G \). In the same way, if the \( \Sigma \)-ordering of \( P \) is compatible with the order function and \( d = \max\{ \text{ord}(\text{lm}(g)) \mid g \in G \} \), then \( G \) is a \( \Sigma \)-Gröbner basis of \( I \) when the above S-polynomials have a \( \Sigma \)-Gröbner representation with respect to \( \Sigma_{2d} \cdot G \).
Proof. Let spoly(σ · f, τ · g) = h = \sum_\nu f_\nu (\nu \cdot g_\nu) be a Gröbner representation with respect to Σ \cdot G that is lm(h) \geq lm(f_\nu (\nu \cdot \text{lm}(g_\nu))) for all \nu. We want to bound the elements \nu \in Σ with respect to the ordering <. Put m = \text{lm}(f), n = \text{lm}(g) and hence \text{lm}(σ \cdot f) = σ \cdot m, \text{lm}(σ \cdot g) = σ \cdot n. By product criterion, we can assume that u = \gcd(σ \cdot m, τ \cdot n) \neq 1. Then, there is a variable \x_i(σ\alpha) = \x_i(\tau\beta) that divides u where \x_i(\alpha) divides m and hence α \leq w(m) \leq δ and \x_i(\beta) divides n and therefore β \leq w(n) \leq δ. Then σα = τβ and one has that σ | β, τ | α because gcd(σ, τ) = 1. We conclude that σ, τ ≤ δ and if v = \text{lcm}(σ \cdot m, τ \cdot m) then w(v) = \max(σw(m), τw(n)) ≤ δ^2. Clearly v > \text{lm}(h) ≥ ν \cdot \text{lm}(g_\nu) and hence δ^2 ≥ w(v) ≥ νw(\text{lm}(g_\nu)) ≥ ν. In a similar way, one argues for the order function.

The above criterion implies that with respect to Σ-orderings compatible with weight or order functions one has an algorithm able to compute a finite Gröbner Σ-basis, whenever this exists, in a finite number of steps. In practice, this results in an adaptative procedure that keeps the bound \Σ-basis, whenever this exists, in a finite number of steps. In practice, this results in weight or order functions one has an algorithm able to compute a finite Gröbner closure, these techniques usually imply computational advantages (see for instance [3]). Note that for Σ-ideals one can develop these methods for both w-grading and ord-grading, but we will concentrate on the second one. Univariate homogenizations are usually more efficient than multivariate ones because leading monomials are preserved by the homogenization process.

Let t be a new variable disjoint by X. Define \bar{X} = X ∪ \{t\}, \bar{X}(Σ) = \bar{X} × Σ, \bar{P} = K[\bar{X}(Σ)] and finally \bar{M} = \text{Mon}(\bar{P}). Consider the algebra endomorphism \varphi : \bar{P} → \bar{P} such that \x_i(σ) ↦ \x_i(σ) and t(σ) ↦ 1, for all i, σ. Clearly \varphi^2 = \varphi and \bar{P} = \varphi(\bar{P}). Moreover, one has that \varphi is a Σ-algebra endomorphism. Then \varphi defines a bijective correspondence between all Σ-ideals of \bar{P} and Σ-ideals of \bar{P} containing \ker \varphi = (t(1) - 1)_Σ.

Definition 6.1. Denote by \mathcal{N} = N_{\text{ord}} the largest ord-graded Σ-ideal contained in \ker \varphi that is the ideal generated by all ord-homogeneous elements \mathcal{F} ∈ \bar{P} such that \varphi(f) = 0.

Proposition 6.2. The ideal \mathcal{N} ⊂ \bar{P} is generated by the elements

(i) t(σ) − t(τ) for all σ, τ ∈ Σ, σ \neq τ, \deg(σ) = \deg(τ);
(ii) t(σ) t(τ) − t(σ), x(σ) t(τ) − x(σ) for any σ, τ ∈ Σ, \deg(σ) ≥ \deg(τ).

Proof. Let \mathcal{F} ∈ \bar{P} be a ord-homogeneous element such that \varphi(\mathcal{F}) = 0. Since the polynomials of type (i),(ii) clearly belongs to \mathcal{N}, we have to prove that \mathcal{F} is congruent to 0 modulo (i),(ii). Assume first that all variables of \mathcal{F} belong to t(Σ) = \{t(σ) | σ ∈ Σ\}. Recall that if m = t(\delta_1)^a_1 \cdots t(\delta_k)^a_k is any monomial of \mathcal{F} then d = \text{ord}(\mathcal{F}) = \max(\deg(\delta_1), \ldots, \deg(\delta_k)). Therefore, one has that \mathcal{F} is congruent modulo (ii) to
\( f' = \sum_j c_j t(\tau_j) \) where \( \tau_j \in \Sigma, \text{deg}(\tau_j) = d \) and \( c_j \in K, \sum_j c_j = 0 \). By applying identity (i) it follows that \( f' \) is congruent to \( (\sum_j c_j)t(\sigma) = 0 \) for some fixed \( \sigma \) such that \( \text{deg}(\sigma) = d \).

Consider now the general case when the variables of \( f' \) belong to \( \bar{X}(\Sigma) \). Fix \( \sigma \in \Sigma \) such that \( \text{deg}(\sigma) = d \). Modulo the identities (i),(ii), one has that \( f' \) is congruent to a polynomial \( f'' \) whose monomials are either of type \( m \in M \) such that \( \text{ord}(m) = d \) or of type \( t(\sigma)n \) where \( n \in M, \text{ord}(n) < d \). We show that in fact \( f'' = 0 \). Denote \( f'' = t(\sigma)g - h \) where \( g, h \) are polynomials in \( P \). \( h \) is ord-homogeneous and \( \text{ord}(h) = d \). Since \( 0 = \varphi(f'') = g - h \) one has that \( f'' = (t(\sigma) - 1)g \). If we assume \( g \neq 0 \) then the monomials \( n \) of \( g \) are such that \( \text{ord}(n) = d \) which is a contradiction.

We want now to define a bijective correspondence between all \( \Sigma \)-ideals of \( P \) and some class of ord-graded \( \Sigma \)-ideals of \( \bar{P} \) containing \( N \).

**Definition 6.3.** Let \( I \) be any \( \Sigma \)-ideal of \( P \). We define \( I^* \subset \bar{P} \) the largest ord-graded \( \Sigma \)-ideal contained in the preimage \( \varphi^{-1}(I) \) that is \( I^* \) is the ideal generated by all ord-homogeneous elements in \( \varphi^{-1}(I) \). Clearly \( N \subset I^* \). We call \( I^* \) the ord-homogenization of the \( \Sigma \)-ideal \( I \).

**Definition 6.4.** Let \( f \in P \) and denote \( f = \sum_d f_d \) the decomposition of \( f \) in its ord-homogeneous components. We define \( \text{topord}(f) = d' = \max\{d\} \). If \( f \in K \) that is \( d' = -\infty \) we put \( f^* = f \). Otherwise, we denote \( f^* = t(\sigma)f \) where \( \sigma \in \Sigma \) such that \( \text{deg}(\sigma) = d' \). We call \( \text{topord}(f) \) the top order of \( f \) and \( f^* \) its ord-homogenization.

**Proposition 6.5.** Let \( I \) be a \( \Sigma \)-ideal of \( P \). Then \( I^* = \langle f^* \mid f \in I, f \neq 0 \rangle + N \).

**Proof.** Denote \( J = \langle f^* \mid f \in I \rangle + N \). Clearly \( J \subset I^* \). Let \( g \in I^* \) be a ord-homogeneous element and define \( f = \varphi(g) \in I \). If \( f = 0 \) then \( g \in N \subset J \). Otherwise, denote \( d = \text{topord}(f) \) and \( d' = \text{ord}(g) \). Since clearly \( d' \geq d \) one has that \( g \) is congruent modulo \( N \) to \( h = t(\sigma)f \), where \( \sigma \in \Sigma \) such that \( \text{deg}(\sigma) = d' \). Hence, if \( d' = d \) then \( h \) is congruent exactly to \( f^* \). Otherwise, the polynomial \( h \) is congruent to \( t(\sigma)f^* \). In both cases, we conclude that \( g \) is congruent modulo \( N \subset J \) to an element of \( J \) and therefore \( g \in J \).

If \( I \subset P \) is a \( \Sigma \)-ideal one has clearly that \( \varphi(I^*) = I \). Moreover, if \( J \subset \bar{P} \) is a ord-graded \( \Sigma \)-ideal containing \( N \) then in general \( J \subset \varphi(J)^* \).

**Definition 6.6.** Let \( N \subset J \subset \bar{P} \) be a ord-graded \( \Sigma \)-ideal. Define \( J' = \varphi(J)^* = \langle \varphi(f^*) \mid f \in J, f \notin N, f \text{ ord-homogeneous} \rangle + N \). Then \( J \subset J' \subset \bar{P} \) is a ord-graded \( \Sigma \)-ideal that we call the saturation of \( J \).

**Definition 6.7.** Let \( J \subset \bar{P} \) be a ord-graded \( \Sigma \)-ideal containing \( N \). We say that \( J \) is saturated if \( J \) coincides with its saturation \( \varphi(J)^* \) that is for any ord-homogeneous element \( f \in J, f \notin N \) one has that \( \varphi(f)^* \in J \). If \( I \) is a \( \Sigma \)-ideal of \( P \) then its ord-homogenization \( I^* \) is clearly a saturated ideal.

Therefore, a bijective correspondence is given between all \( \Sigma \)-ideals of \( P \) and the saturated ord-graded \( \Sigma \)-ideals of \( \bar{P} \) containing \( N \).

We want now to analyze the behaviour of Gröbner \( \Sigma \)-bases under homogenization and dehomogenization. Note that the arguments of Proposition 6.2 implies clearly that the polynomials (i),(ii) are in fact a Gröbner basis of the ideal \( N \) with respect to any monomial ordering of \( \bar{P} \). For this reason we introduce the following notion.
Definition 6.8. A monomial $m \in M$ is said normal modulo $N$ if $m \in M$ or $m = t(\sigma)n$ with $n \in M$, $\sigma \in \Sigma$ such that $d = \deg(\sigma) > \ord(n)$. Moreover, we assume that $t(\sigma) = \min \{ t(\tau) \mid \deg(\tau) = d \}$. A polynomial $f \in \bar{P}$ is in normal form modulo $N$ if all its monomials are normal modulo $N$.

Definition 6.9. Let $\prec$ be a $\Sigma$-ordering of $\bar{P}$ compatible with the order function. We call $\prec$ a ord-homogenization $\Sigma$-ordering if $t(\sigma)m \prec n$ for all $m, n \in M, \sigma \in \Sigma$ such that $\deg(\sigma) = \ord(n) > \ord(m)$.

It is easy to define one of the above orderings. Fix for instance the lex or $\degrevlex$ monomial order on the polynomial ring $\bar{P}(1) = K[x_0(1), x_1(1), \ldots, t(1)]$ where $x_0(1) > x_1(1) > \ldots > t(1)$. Moreover, fix a monomial ordering on $\Sigma$ which is compatible with $\deg$ and define the weight $\Sigma$-ordering $\prec_w$ of $\bar{P}$ as in Definition 6.8. Clearly $\prec_w$ is a ord-homogenization $\Sigma$-ordering.

From now on, we assume $\bar{P}$ be endowed with a ord-homogenization $\Sigma$-ordering.

Proposition 6.10. Let $p, q \in M$ be two normal monomials modulo $N$ such that $\ord(p) = \ord(q)$. Then $p \prec q$ implies that $\varphi(p) \prec \varphi(q)$.

Proof. By definition, the monomials $p, q$ are of type $m \in M$ or $t(\sigma)m$ with $\deg(\sigma) > \ord(m)$. Since $\prec$ is a ord-homogenization order, when comparing two of such monomials of the same order one has only the following cases: $m \prec n$, $t(\sigma)m \prec t(\sigma)n$ or $t(\sigma)m \prec n$. Then, we have to prove $\varphi(p) = m \prec n = \varphi(q)$ only when $t(\sigma)m \prec n$. This follows immediately from $\prec$ is compatible with the order function and $\ord(m) \prec \ord(n)$. $\square$

From now on, for any $f \in P, f \neq 0$ we denote by $f^*$ the normal form of $t(\sigma)f$ modulo $N$ where $\sigma \in \Sigma, \deg(\sigma) = \topord(f)$.

Proposition 6.11. Let $f \in P, f \neq 0$ be a ord-homogeneous polynomial in normal form modulo $N$. Then $\lm(\varphi(f)) = \varphi(\lm(f))$. Moreover, we have that $\lm(f^*) = \lm(f)$ for all $f \in P, f \neq 0$.

Proof. The first part of the statement follows immediately from Proposition 6.10. Moreover, if $\sigma \in \Sigma, \deg(\sigma) = \topord(f)$ then $t(\sigma)$ cannot appear in the leading monomial of $f^*$ and hence $\lm(f^*) = \lm(f)$. $\square$

Definition 6.12. Let $N \subset J \subset \bar{P}$ be a $\Sigma$-ideal. Moreover, let $G \subset J$ be a subset of polynomials in normal form modulo $N$. We say that $G$ is a Gröbner $\Sigma$-basis of $J$ modulo $N$ if $G \cup N$ is a Gröbner $\Sigma$-basis of $J$.

Proposition 6.13. Let $N \subset J \subset \bar{P}$ be a ord-graded $\Sigma$-ideal. If $G$ is a ord-homogeneous Gröbner $\Sigma$-basis of $J$ modulo $N$ then $\varphi(G)$ is a Gröbner $\Sigma$-basis of $\varphi(J)$.

Proof. Since $G$ is a Gröbner $\Sigma$-basis of $J$ modulo $N$ we have that for any ord-homogeneous polynomial $f \in J, f \neq 0$ in normal form modulo $N$ there is an element $g \in G$ and $\sigma \in \Sigma$ such that $\sigma \cdot \lm(g) \mid \lm(f)$. Then, by applying the $\Sigma$-algebra endomorphism $\varphi$ one obtains that $\sigma \cdot \lm(\varphi(g)) \mid \lm(\varphi(f))$ that is $\varphi(G)$ is a Gröbner $\Sigma$-basis of $\varphi(J)$. $\square$

Proposition 6.14. Let $I \subset P$ be a $\Sigma$-ideal and let $G$ be a Gröbner $\Sigma$-basis of $I$. Then $G^* = \{ g^* \mid g \in G \}$ is a ord-homogeneous Gröbner basis of $I^*$ modulo $N$. Moreover, one has that $\lm(G^*) = \lm(G)$.
Proof. Let \( f' \in I^* \) be a ord-homogeneous element in normal form modulo \( N \) and put \( f = \varphi(f') \in I \). Then, either \( f' = f^* \) or \( f' = t(\sigma)f^* \) with \( \text{ord}(f') = \deg(\sigma) > \text{ord}(f^*) = \text{topord}(f) \). Since \( G \) is a Gröbner \( \Sigma \)-basis of \( I \) there is \( g \in G, \tau \in \Sigma \) such that \( \tau \cdot \text{lm}(g) \mid \text{lm}(f) \). By Proposition 6.11 one has that \( \text{lm}(f) = \text{lm}(f^*) \) and \( \text{lm}(g) = \text{lm}(g^*) \). Therefore, \( \tau \cdot \text{lm}(g^*) \) divides \( \text{lm}(f^*) \) and this monomial clearly divides \( \text{lm}(f') \). □

By the above propositions we obtain immediately what follows.

Corollary 6.15. Let \( N \subset J \subset \bar{P} \) be a ord-graded \( \Sigma \)-ideal and denote \( J' = \varphi(J)^* \) its saturation. Moreover, let \( G \) be a ord-homogeneous Gröbner \( \Sigma \)-basis of \( J \) modulo \( N \). Then \( G' = \varphi(G)^* = \{ \varphi(g)^* \mid g \in G \} \) is a ord-homogeneous Gröbner \( \Sigma \)-basis of \( J' \) modulo \( N \). Moreover, we have \( \text{lm}(G') = \text{lm}(\varphi(G)) \).

Let \( I \subset P \) be any \( \Sigma \)-ideal. The previous results suggest an alternative method to calculate a Gröbner \( \Sigma \)-basis of \( I \) which is based only on ord-homogeneous computations. Assume \( H \) is any \( \Sigma \)-basis of \( I \) and denote as before \( H^* = \{ f^* \mid f \in H \} \). Clearly \( J = \langle H^* \rangle_{\Sigma} + N \) is a ord-homogeneous \( \Sigma \)-ideal of \( \bar{P} \) containing \( N \) such that \( \varphi(J) = I \). Assume now we compute \( G \) a ord-homogeneous Gröbner \( \Sigma \)-basis of \( J \) modulo \( N \). Then, \( \varphi(G) \) is a Gröbner \( \Sigma \)-basis of \( I \). Note that by using a ord-based selection strategy for the S-polynomials, the Gröbner \( \Sigma \)-basis \( G \) can be obtained order by order automatically minimal that is \( \sigma \cdot \text{lm}(f) \) not divides \( \text{lm}(g) \) for all \( f, g \in G, f \neq g \) and \( \sigma \in \Sigma \). This is clearly a computational advantage, but since generally \( \text{lm}(G) \neq \text{lm}(\varphi(G)) \) one has that \( \varphi(G) \) may be not minimal. In the worst case, the ideal \( J \) may have an infinite and hence uncomputable minimal Gröbner \( \Sigma \)-bases but \( I \) just a finite one. This is clearly not the case when one considers a saturated ideal \( J' = I^* \) since we have \( \text{lm}(G') = \text{lm}(\varphi(G')) \) when \( G' \) is a minimal Gröbner \( \Sigma \)-basis of \( J' \). Note that this nice property depends on the fact that we deal with a univariate homogenization. A drawback is that if one computes the saturation \( J' \) by means of the ideal \( J \) according to Corollary 6.15 one has again to compute a Gröbner \( \Sigma \)-basis of \( J \). Then, a better approach consists in computing “on the fly” the Gröbner \( \Sigma \)-basis of \( J' \) starting from the generating set \( \{ f^* \mid f \in H \} \).

In other words, any time that a new generator \( g \) of the ord-homogeneous Gröbner \( \Sigma \)-basis arises from the reduction of an S-polynomial, we saturate \( g \) that is we substitute this polynomial with \( \varphi(g)^* \). In formal terms, the algorithm one obtains is the following one.
Algorithm 6.1 SigmaGBasis2

Input: $H$, a $\Sigma$-basis of a $\Sigma$-ideal $I \subset \mathcal{P}$.
Output: $\varphi(G)$, a Gröbner $\Sigma$-basis of $I$ such that $\text{lm}(G) = \text{lm}(\varphi(G))$.

$G := H^*$;
$B := \{(f, g) \mid f, g \in G\}$;

while $B \neq \emptyset$ do
  choose $(f, g) \in B$;
  $B := B \setminus \{(f, g)\}$;
  for all $\sigma, \tau \in \Sigma$ such that $\text{gcd}(\sigma, \tau) = 1$ do
    $h := \text{Reduce}(\text{spoly}(\sigma \cdot f, \tau \cdot g), \Sigma \cdot G \cup N)$;
    $h = \varphi(h)\ast$
    if $h \neq 0$ then
      $B := B \cup \{(g, h), (h, h) \mid g \in G\}$;
      $G := G \cup \{h\}$;
    end if;
  end for;
end while;
return $\varphi(G)$.

Proposition 6.16. The algorithm SigmaGBasis2 is correct.

Proof. Note that at each step we are inside an ideal $J$ such that $\varphi(J) = I$ that is 
whose saturation is $J' = I^*$. Moreover, for any ord-homogeneous element $h \in \bar{P}$ 
which is in normal form modulo $N$ one has that $h' = \varphi(h)\ast$ divides $h$. This implies 
that if an S-polynomial is reduced to zero by adding $h$ to the basis $G$, the same 
holds if we substitute $h$ with $h'$. In case of termination, owing to the set $G$ is a ord-
homogeneous Gröbner $\Sigma$-basis of $J$ modulo $N$ whose elements are all saturated, by 
Corollary 6.15 we may conclude that $J = J'$ and hence $\varphi(G)$ is a Gröbner $\Sigma$-basis 
of $I$ such that $\text{lm}(G) = \text{lm}(\varphi(G))$. 

About termination or just termination up to some order $d$, this is not provided 
in general for the above algorithm. The reason is that even if all computations are 
ord-homogeneous, because of the saturation $h = \varphi(h)\ast$ that may decrease the order 
we can’t be sure at some suitable step that we will not get additional elements of 
order $\leq d$ in the steps that follow.

7. Examples and testing

In this section we present a set of tests for the algorithms SigmaGBasis and 
SigmaGBasis2 which is based on an experimental implementation of them in the 
language of Maple. This is actually the first implementation of algorithms for the 
computation of Gröbner bases of linear and non-linear partial difference ideals. Note 
that for the linear case one has the packages LDA (Linear Difference Algebra) [12] 
and Ore_algebra[shift_algebra] in the Maple distribution. The main idea that lead us 
when coding the proposed algorithms is that they can be considered variants of the 
classical Buchberger algorithm where some amount of computations can be avoided 
by means of the symmetry defined by the monoid $\Sigma$. In fact, as explained in the 
previous sections, a “basic” approach to calculate a Gröbner $\Sigma$-basis of a $\Sigma$-ideal $I$ 
generated by a $\Sigma$-basis $H$ consists in applying the Buchberger algorithm to the basis
One obtains therefore a Gröbner basis $G'$ of $I$ from which a Gröbner $\Sigma$-basis $G \subset G'$ can be extracted such that $\Sigma \cdot \lm(G) = \lm(G')$. Clearly, chain and coprime criterions can be used in the usual way in the procedure. Then, the algorithm SigmaGBasis can be understood as the variant that prescribes the application also of the $\Sigma$-criterion (Proposition 4.7) to the S-polynomials $\spoly(\sigma \cdot f, \tau \cdot g)$ and to add the set of all shifts $\Sigma \cdot h$ to the current basis when a new element $h$ arises from the reduction of an S-polynomial. Then, the Gröbner $\Sigma$-basis of $I$ is simply the union of the initial basis $H$ with the new elements $h$. Recall that the procedure is correct only if one uses a monomial $\Sigma$-ordering. Clearly, from the set $\Sigma$ is infinite it follows that actual computations can be only performed with a finite subset of $\Sigma$ that is over a finite set of variables of $P = K[X(\Sigma)]$. Typically, one fixes a bound $d$ for the degree of the elements of $\Sigma$ that is for the order of the variables $x_i(\sigma)$. Owing to the finite $\Sigma$-criterion (Proposition 5.12), a basis obtained with a monomial ordering compatible with the order function is certified to be a complete Gröbner $\Sigma$-basis if the order bound is at least the double of the maximum top order of its elements.

In addition to the basic procedure for the computation of Gröbner $\Sigma$-bases and the algorithm SigmaGBasis, for the experiments we consider also a variant of the latter method where the $\Sigma$-criterion is suppressed but one continues to shift the reduced form of the S-polynomials. This procedure is tested to the aim of understanding the contribution of any of the implemented strategies. Finally, we propose an implementation of the algorithm SigmaGBasis2 based on the saturation of a $\Sigma$-ideal with respect to the grading defined by the order function. In practice, once one has homogenized the initial generators, the saturation $\varphi(h)^*$ is performed before the application of shifting, for each new element $h$ obtained by the reduction of an S-polynomial. In output one returns the dehomogenization of the computed basis. Note that this procedure is correct only if one uses a $\Sigma$-ordering which is compatible with the order function and if the polynomials are kept in normal form modulo $N$ during the computations.

The monomial $\Sigma$-orderings of $P$ that we consider for the tests are defined in the following way. One has initially to fix a monomial ordering for $\Sigma$ and we choose $\text{degrevlex}$ in order to provide compatibility with the degree. Then, one fixes a monomial ordering, for us $\text{lex}$, over the subring $P(1) = K[X(1)]$ or $P(x_0) = K[x_0(\Sigma)]$ that is extended as a block ordering to the polynomial ring $P = K[X(\Sigma)]$ according to the choice of a variable ranking based on weight or index respectively. We distinguish these two cases in the examples by the letters “w” and “i”. The integer that comes before these letters refers to the fixed order bound. Note that the algorithm SigmaGBasis2 is compatible only with rankings of type weight.

For the basic variant of the Gröbner $\Sigma$-bases algorithm, one can clearly use any implementation of the Buchberger algorithm as, for instance, the one contained in the package Groebner of Maple. We have preferred instead to develop ourselves all different variants in order to have the same implementation and hence the same efficiency, for the fundamental subroutines of the algorithms. In this way, for the basic version we have been also able to access to important parameters of the computation as the total number of S-polynomial reductions. This number is for us the sum of the actual S-polynomials with the initial generators that are interreduced. Note that our implementation of the Buchberger algorithm is in fact generally comparable with the built-in one of Maple. For instance, the test falkow-6w-basic takes
less than 11 hours, but using Groebner[Basis] it takes more than two days. Other parameters that are considered for the experiments are the number of input and output generators. Note that for the basic algorithm we count generators and not \( \Sigma \)-generators. Finally, the parameter “minout” refers to the number of elements of a minimal Gröbner \( \Sigma \)-basis. All examples have been computed with Maple 12 running on a server with a four core Intel Xeon at 3.16GHz and 64 GB RAM. The timings are given in hour-minute-second format.

<table>
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<th>Example</th>
<th>in</th>
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We give now some details about the examples we have used. All the examples are based on systems of ordinary and partial difference equations which are of interest in literature. For instance, the tests falkow are obtained by the discretization of the Falkovich-Karman differential equation which is a non-linear two-dimentional one describing transonic flow in gas dynamics. The discretization we used are equations (41) in [11]. Then, the navier examples are based on equations \( e_1, e_2, e_3, e_4 \) of the system (13) in the paper [10] that are a finite difference scheme corresponding to the discretization (9) of the Navier-Stokes equations for two-dimensional viscous incompressible fluid flows. The tests heat are the discretization of the one-dimensional heat equation as described in the equations (10) and (11) of [20]. Finally, eq26 and eq27 are the equations (2.6) and (2.7) at page 24 of [13] which are examples of ordinary difference equations that have periodic solutions.
By analyzing the experiments, it is sufficiently clear that the strategy implemented in SigmacGBasis is the safest one and hence on the average, the most efficient one. In fact, by decreasing the number of S-polynomials this strategy avoids the dramatical effects of involved reductions as for the tests falkow-6w and navier-8w. For simpler examples the four strategies appear essentially equivalent. The algorithms SigmacGBasis and SigmacGBasis2 lead to practically identical computations but the latter method suffers of some overhead which is probably due to our still experimental implementation. For instance, even if the normal form modulo the ideal $N$ is described in the Definition 6.8 in our implementation we obtain it by computations that is by adding a Gröbner basis of $N$ to the input basis for SigmacGBasis2.

The proposed algorithms usually provide only partial informations about the structure of Gröbner Σ-bases since they are in general infinite. Nevertheless, it is interesting to note that by means of the finite Σ-criterion we have been able to certify that the examples falkow, navier and heat have finite bases with respect to the weight ranking. In particular, the elements of the Gröbner basis of falkow have maximum top order equal to 4 and hence they are certified in order 8 in about 4 minutes. The example navier has max top order equal to 6 and its certification is obtained in order 12 in less than one hour. Finally, the example heat has max top order 2 and it gets certification in order 4 in 0 seconds.

8. Conclusions and future directions

This paper shows that one can not only generalize in a systematic way the Gröbner bases theory and related algorithms to the algebras of partial difference polynomials but also make these methods really work by introducing suitable gradings for such algebras. In fact, weight and order functions provided a Noetherian subalgebras filtration that implies termination and completeness certification for actual computations that are performed within some bounded degree that is over a finite number of variables. We have then developed the first experimental implementation of a variant of the Buchberger algorithm for non-linear partial difference ideals that is able to perform computations for ideals arising from the discretization of real world systems of non-linear differential equations.

Since the algebras of partial difference polynomials are free objects in the category of commutative algebras endowed with the action of a monoid Σ isomorphic to $\mathbb{N}^r$, a natural future direction in this research consists in extending the proposed methods to other types of monoid symmetry over commutative algebras as the ones used, for instance, in algebraic statistic [4]. Starting from Gröbner bases, classical directions are the computation of the Hilbert series and free resolutions that one may generalize to partial difference ideals or other types of invariant ideals. The computation of the kernels of homomorphisms between free Σ-algebras is also important to work with concrete Σ-algebras. Finally, we aim to have the proposed algorithms implemented in the kernel of computer algebra systems in order to tackle involved problems related with the discretization of systems of partial differential equations [9] [11] [12].

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