Out-of-Band Ambiguity Analysis of Nonuniformly Sampled SAR Signals

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Abstract—High-resolution and wide-swath synthetic aperture radar (SAR) images can be obtained by multichannel sampling. However, a nonoptimum pulse-repetition frequency is associated with a nonuniform spatial sampling in azimuth. A spectrum reconstruction algorithm and a filter group algorithm can be used to reconstruct the spectrum in azimuth with a nonuniform sampling. In this letter, a comparison of spectrum reconstruction algorithm and filter group algorithm is given. Through a transformation matrix, the relationship between out-of-band signals and the out-of-band signals after reconstruction is built. Furthermore, an equivalent spectrum after reconstruction is obtained, and simulation results proved that it is precise. From the equivalent spectrum, it shows that multichannel SAR can be seen as uniform sampling system but with a different original signal. Finally, channel imbalance is discussed. Simulation results have proved that the imbalance targets have the same location with corresponding ambiguous targets.

Index Terms—Equivalent spectrum, nonuniformly sampling, out-of-band signals, signal sampling.

I. INTRODUCTION

HIGH-resolution and wide-swath synthetic aperture radar (SAR) image can be obtained by multichannel sampling [1]–[7]. Some algorithms are used to reconstruct the spectrum in azimuth. For example, Jenq presented an algorithm that perfectly reconstructs the spectrum from nonuniform spatial time samples when the original signal is band limited [2]. The spectrum reconstruction algorithm has been used in multichannel SAR systems, and a multichannel SAR image is given in [3]. Moreover, Gebert et al. put forward a reconstruction algorithm based on filter group algorithm [1]. Both Gebert et al. and Jenq’s methods could reconstruct band-limited signals from nonuniform sampling. Moreover, compressive sampling can be used for the reconstruction from nonuniformly sampled data [8] when the original signal is sparse. In this letter, a comparison of spectrum reconstruction algorithm (Jenq’s) and filter groups algorithm (Gebert et al.) is given. The reconstruction algorithm in this letter is based on Jenq’s method.

II. SPECTRUM RECONSTRUCTION

Considering a discrete signal \( x(t_q) \) sampled from an analog signal \( x(t) \) at nonuniformly spatial time intervals, then

\[
 t_q = qT + \Delta_q. \tag{1}
\]

The time offset \( \Delta_q \) has a periodic structure with period \( M \). In multichannel SAR, periodic structure \( M \) stands for the \( M \)-channels. A single channel is uniformly sampled with sampling rate of \( 1/MT \).

Let \( q = nM = m \), and then

\[
t_q = (nM + m)T + \Delta_{nM+m} = nMT + mT + \gamma_mT \tag{2}
\]

where \( n = \pm 1, 0, 1, \ldots \pm \infty \) and \( m = 0, 1, \ldots M-1 \). \( \gamma_m = \Delta_m/T \) is the timing offset measured in percentage of uniform sampling period \( T \). \( x(t_q) \) is sampled from analog signal \( x(t) \); the sampling function is

\[
p_s(t) = \sum_{q=-\infty}^{\infty} \delta(t - t_q) \]

\[
 = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \delta(t - nMT - mT + \gamma_mT) \tag{3}
\]

\[
x(t_q) = x(t) \times p_s(t). \tag{4}
\]

The Fourier transform of \( x(t) \) is \( F(w) \), and the Fourier transform of \( x(t_q) \) is \( F_{\text{non}}(w) \). Then

\[
F_{\text{non}}(w) = F(w) \ast P_s(w). \tag{5}
\]

Operator \( \ast \) stands for convolution. \( P_s(w) \) is the Fourier transform of \( p_s(t) \). It is derived from the Appendix that

\[
P_s(w) = \frac{1}{MT} \sum_{k=-\infty}^{\infty} e^{-jkw_s(mT+\gamma_mT)} \delta(w - kw_s) \tag{6}
\]

where \( w_s = 2\pi/MT \), then combine (5) and (6)

\[
F_{\text{non}}(w) = \frac{1}{MT} \sum_{k=-\infty}^{\infty} e^{-jkw_s(mT+\gamma_mT)} F(w - kw_s). \tag{7}
\]

Let

\[
A(k) = \frac{1}{M} \sum_{m=0}^{M-1} e^{-jkw_s(mT+\gamma_mT)} \tag{8}
\]

\[
F_{\text{non}}(w) = \frac{1}{T} \sum_{k=-\infty}^{\infty} A(k) F(w')|_{w'=w-kw_s}. \tag{9}
\]

In (9), the conclusion is the same as the corresponding conclusion in [2]. From Jenq’s method [2], we will get

\[
[F_{\text{non}}(w)] = [A][XC(w)]. \tag{10}
\]
where

\[
[F_{\text{non}}(w)] = \\
\begin{bmatrix}
F_{\text{non}}(w) \\
F_{\text{non}}(w + (M - 1)w_s) \\
\vdots \\
F_{\text{non}}(w + (M - 1)w_s) \\
X_c(w - \pi/T) \\
\vdots \\
X_c(w - \pi/T + (M - 1)w_s) \\
A(M/2) & \cdots & A(-M/2 + 1) \\
\vdots & \ddots & \vdots \\
A(M/2 + M - 1) & \cdots & A(M/2)
\end{bmatrix}
\]  \quad (11)

\[ [A] = \begin{bmatrix}
A(1) & \cdots & A(1) \\
\vdots & \ddots & \vdots \\
A(M - 1) & \cdots & A(M - 1) \\
\end{bmatrix}
\]  \quad (12)

\( A(k) \) is from (8); \( X_c(w) \) is the reconstructed spectrum.

Following, a fast algorithm of the spectrum reconstruction algorithm is given. \( F_{\text{non}}(w) \) can be written as follows:

\[
F_{\text{non}}(w) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} x[t_{nM+m}] e^{-jwnM+m}. \quad (13)
\]

Substituting (2) into (13) yields

\[
F_{\text{non}}(w) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} x[t_{nM+m}] e^{-jw(nM+m+\gamma)mT}. \quad (14)
\]

Divide the spectrum into \( M \) groups, it can be reformulated as

\[
F_{\text{non}}(w) = \sum_{m=0}^{M-1} e^{-jw(\gamma m + M \gamma)mT} \sum_{n=-\infty}^{\infty} x[t_{nM+m}] e^{-jwnMT}. \quad (15)
\]

From (15), it is shown that the inner parts are in the form of discrete Fourier transform (DFT), let

\[
X_m(w) = \sum_{n=-\infty}^{\infty} x[t_{nM+m}] e^{-jwnMT}. \quad (16)
\]

In fact, \( X_m(w) \) is the spectrum of the \( m \)th uniformly sampled group with uniformly sampled frequency \( 1/MT \). \( X_m(w) \) is an aliasing spectrum, for the sampling rate is not enough. In addition, \( X_m(w) \) is periodic, and then

\[
X_m(w + mw_s) = X_m(w) \quad (17)
\]

where \( w_s = 2\pi/MT \), \( m = 0, 1, 2 \ldots M - 1 \).

Combining (15)–(17) yields

\[
\begin{cases}
F_{\text{non}}(w) = \sum_{m=0}^{M-1} e^{-jw(\gamma m + M \gamma)mT} X_m(w) \\
\vdots \\
F_{\text{non}}(w + (M - 1)w_s) = e^{-jw((M - 1)(M - 1)w_s) + (M - 1)\gamma)mT} X_m(w).
\end{cases}
\]  \quad (18)

Vector\([X(w)]\) is built

\[
[X(w)] = [X_0(w), X_1(w) \ldots X_{M-1}(w)]^T. \quad (19)
\]

From (18), the relationship between vector \([F_{\text{non}}(w)]\) and vector \([X(w)]\) is obtained as

\[
[F_{\text{non}}(w)] = [R_w] [X(w)] \quad (20)
\]

\[ [X(w)] = [R_w] [X(w)] \quad (20) \]

Combining (10) and (20) yields

\[
[F_{\text{non}}(w)] = [R_w] [X(w)] = [A] [XC(w)] \quad (22)
\]

\[
[XC(k)] = [A]^{-1} [R_w] [X(w)]. \quad (23)
\]

Let

\[
[AR_w] = [A]^{-1} [R_w] \quad (24)
\]

\[
[XC(w)] = [AR_w] [X(w)]. \quad (25)
\]

where \( k = 0, 1, 2 \ldots N - 1 \). \([R_w]\) is defined as follows:

\[
[R_w] = \begin{bmatrix}
e^{jw\gamma 0T} & \cdots & e^{jw(M-1)\gamma(T-1)T} \\
\vdots & \ddots & \vdots \\
e^{jw((M-1)w_s)\gamma 0T} & \cdots & e^{jw((M-1)w_s)(M-1)\gamma(T-1)T}
\end{bmatrix} \quad (21)
\]

Combining (10) and (20) yields

\[
[F_{\text{non}}(w)] = [R_w] [X(w)] = [A] [XC(w)] \quad (22)
\]

\[
[XC(k)] = [A]^{-1} [R_w] [X(w)]. \quad (23)
\]

Let

\[
[AR_w] = [A]^{-1} [R_w] \quad (24)
\]

\[
[XC(w)] = [AR_w] [X(w)]. \quad (25)
\]

where \( k = 0, 1, 2 \ldots N - 1 \). In (25), the reconstructed spectrum is synthesized from the spectrum of every channel.

Suppose the sampling number of a single channel is \( N \). Totally, all processing procedure requires \( MN\log_2^N / 2 + M^2N \) complex multiplications, whereas in [2], it requires \( (MN)^2 + M^2N \) complex multiplications. Equation (25) can be seen as a fast algorithm for original spectrum reconstruction algorithm.

III. COMPARISON OF SPECTRUM RECONSTRUCTION ALGORITHM AND FILTERS GROUP ALGORITHM

Based on [5], the filter group algorithm is written in the form of a nonuniform model in this letter.

In Fig. 1, the impulse response of the \( m \)th filter is

\[
h_m(t) = \delta(t + \gamma mT + mT) \quad (26)
\]

\[
H_m(w) = e^{jw\gamma mT + mT} \quad (27)
\]

\[
[XC(w)] = [P(w)] [X(w)] \quad (28)
\]

\[
[P(w)] = [H(w)]^{-1} \quad (29)
\]

and

\[
[H(w)] = \begin{bmatrix}
e^{jw\gamma 0T} & \cdots & e^{jw(M-1)\gamma(T-1)T} \\
\vdots & \ddots & \vdots \\
e^{jw((M-1)w_s)\gamma 0T} & \cdots & e^{jw((M-1)w_s)(M-1)\gamma(T-1)T}
\end{bmatrix} \quad (30)
\]

Suppose that the two methods is equivalent, then

\[
[P(w)]^T = [AR_w]. \quad (31)
\]

Take (24) and (29) into (31), then

\[
([H(w)]^{-1})^T = [A]^{-1} [R_w] \quad (32)
\]

\[
[A] = [R_w] [H(w)]^T. \quad (33)
\]
Let $B_{uv}(k)$ to be the $u$th row and $v$th line of $[R_w][H(w)]^T$.

$$B_{uv}(k) = \left( \frac{1}{M} \right) \sum_{m=0}^{M-1} e^{-j(u-v)\gamma_{rm}(2\pi/M)} e^{-jkm(2\pi/M)}, \quad (34)$$

In [2], $w \in (0, \pi/T)$, if $w \in (-\pi/T, \pi/T)$; from (21), (30), and (33), we can get $B(k) = A(k)$. It is proved that the two methods are equivalent. When $[A]$ is an identity matrix, all channels are uniformly sampled. Matrix $[A]$ is a Hermitian matrix, and it reflects the degree of nonuniform sampling.

In a multichannel SAR system, the sampling time of two channels may be approximate or even coincident and it will be more computationally intensive in calculating the inverse of matrix. In (25), when calculating the reconstruction matrix $[A\bar{R}_w]$, it only needs to calculate matrix inverse $[A]$ once. Whereas, by the filter group algorithm, it needs to calculate matrix inverse $[H(w)]$ with every point of $w$. With the increases in $M$ or $N$, the spectrum reconstruction algorithm seems to have higher computational efficiency. The two algorithms are identical in mathematics and with the same application area.

IV. OUT-OF-BAND AMBIGUITY ANALYSIS FROM SPECTRUM RECONSTRUCTION ALGORITHM

An equivalent spectrum is used to express the out-of-band signal after nonuniformly sampled and reconstructed. Nonuniform sampling the original signal and reconstructed is equal to uniform sampling from the signal of equivalent spectrum.

Based on (9), a vector of out-of-band signals is built

$$[FB(w'_0)] = [F(w'_0 - \pi/T), F(w'_0 - \pi/T + w_s) \ldots F(w'_0 - \pi/T + (M - 1)w_s)]^T \quad (35)$$

where $w'_0 = w_0 + 2\pi a/T = w_0 + aMw_s$, $a \neq a$; $a$ is a frequency step factor of $2\pi/T$ and $w_0 \in [0,w_s)$. From (9) and (10), factor $k$ in vector $[FB(w'_0)]$ is with a step frequency shift of $aM$ compared with vector $[XC(w)]$. Thus, the relationship between vector $[F_{\text{non}}(w_0)]$ and vector $[FB(w'_0)]$ can be written as follows:

$$[F_{\text{non}}(w_0)] = [D][FB(w'_0)] \quad (36)$$

$$[D] = \begin{bmatrix} A(-aM + M/2) & \ldots & A(-aM - M/2 + 1) \\ \vdots & \ddots & \vdots \\ A(-aM + M/2 + M - 1) & \ldots & A(-aM + M/2) \end{bmatrix}. \quad (37)$$

From (10) and (36), then

$$[F_{\text{non}}(w_0)] = [D][FB(w'_0)] = [A][XC(w_0)]. \quad (38)$$

Let

$$[B] = [A]^{-1}[D]. \quad (39)$$

Then

$$[XC(w_0)] = [B][FB(w'_0)]. \quad (40)$$

An equivalent spectrum $F_{eq}(w') w' \in (-\infty, +\infty)$ is defined

$$[F_{eq}(w'_0)] = [B][FB(w'_0)] \quad (41)$$
SAR also can be seen as uniform sampling but with a different original signal (i.e., the signal from the equivalent spectrum).

In (41), a step factor of $k$ in matrix $[B]$ is with frequency shift of $kw_s$ ($k$PRF). The location of the extra target is decided by the frequency shift; thus

$$t_{\text{shift}} = kw_s/(2\pi Ka).$$

(42)

$K\alpha$ is the azimuth frequency module rate, and $w_s = 2\pi \text{PRF}$.

Next, a simulation demonstration of (42) is given. The original signal only in the first ambiguous zone, with frequency between $2\text{PRF} < f < 6 \times \text{PRF}$ (i.e., equivalent sampling rate of all channels is $4\text{PRF}$). It is nonuniformly sampled, reconstructed, and gone through a matched filter, as shown in Fig. 4.

In the first ambiguous zone (in (37), $a = 1$), $k \in \{1, 2, 3, 4, 5, 6, 7\}$. With the system parameter $Ka = 4513$ Hz/s and (42), the location of the ambiguous targets calculated by (42) is (0.4875, 0.9750, 1.4624, 1.9500, 2.4375, 2.9250, 3.4124), which corresponds to the results in Fig. 4.

V. CHANNEL IMBALANCE TARGETS

In a multichannel SAR system, there are gain and phase offsets between different channels. Supposing only one channel with offset, to other channels, the offset part of the signal is zero. For whole channels, the offset part of the signal can be regarded as the upsampling of the offset channel, and the upsampling factor is $M$.

Next, a pointlike target is taken into consideration, and a general result can be obtained by integrating all the pointlike targets. In simulation, a pointlike target consists of a linear frequency modulation (LFM) signal without SINC window. The sampling frequency of a single channel is 2200 Hz. The equivalent sampling frequency of all channels is 8800 Hz, $M = 4$, and the bandwidth of the original signal is from −3250 to 3250 Hz. The first channel is with offset of $\Delta G = 0.1 G_0$, $\Delta \Psi = 10^\circ$, and then goes through a matched filter, as shown in Fig. 5(a). The offset part of the signal goes through the matched filter, as shown in Fig. 5(b).

In Fig. 5(a) and (b), all corresponding imbalance targets are with same location and amplitude. Following, only the offset part of the signal is taken into account.

The data of the offset channel are downsampled from the original LFM signal, and the downsampling factor is $M = 4$.

To the whole offset signal, other channels are inserted with zero. The DFT of the original LFM signal is

$$X_{\text{original}}(k) = \sum_{q=0}^{MN-1} x_{\text{original}}(q)e^{-j 2\pi kq/MN}$$

(43)

where $k = 0, 1, 2 \ldots MN - 1$, and $f_s = 8800 \text{Hz}$. $N$ is the number of a single channel. The offset channel is down sampling from the original LFM signal

$$X_{\text{offset-single}}(k) = \sum_{n=0}^{N-1} x_{\text{offset-single}}(n)e^{-j 2\pi kn/N}$$

(44)

where $(k = 0, 1, 2 \ldots N - 1)$. The offset part of others is zero.

$$X_{\text{offset-whole}}(k) = \sum_{n=0}^{N-1} x_{\text{offset-whole}}(n)e^{-j 2\pi kn/N}$$

(45)

where $k = 0, 1, 2 \ldots MN - 1$. The relationship of the offset part of the whole channel with the original LFM can be obtained.

$$X_{\text{eq-offset}}(k) = X_{\text{original}}(k)$$

$$X_{\text{eq-offset}}(k + 3MN) = X_{\text{original}}(k + 3N).$$

(47)

The spectrum of the offset part of the original signal is shown as downsampling from the equivalent spectrum as shown in Fig. 6(a) with a sampling factor of $M = 4$.

In Fig. 6(a), equivalent spectrum is truncated into $2M - 1$ (here is 7) segments, named as (−3, −2, −1, 0, 1, 2, and 3), as shown in Table II.

The frequency shift of segment (0) is referring to 0 Hz. $\mu$ stands for the number of aliasing periods (8800 Hz). The frequency of segment (3) is between 13200 and 17600 Hz.
TABLE II

<table>
<thead>
<tr>
<th>Segment</th>
<th>Frequency-shift(Hz)</th>
<th>Band-width(Hz)</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-13200 Hz</td>
<td>3250 Hz</td>
<td>2</td>
</tr>
<tr>
<td>-2</td>
<td>-11000 Hz</td>
<td>5450 Hz</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>-2200 Hz</td>
<td>1050 Hz</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0 Hz</td>
<td>6500 Hz</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2200 Hz</td>
<td>1050 Hz</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>11000 Hz</td>
<td>5450 Hz</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>13200 Hz</td>
<td>3250 Hz</td>
<td>-2</td>
</tr>
</tbody>
</table>

After downsampling, the frequency of segment (3) is aliased between $-4400$ and $0$ Hz, added with $-17600$ Hz; hence, $\mu = -17600 \text{ Hz}/8800 \text{ Hz} = -2$.

The location of every segment is calculated as follows:

$$t_{\text{segment}} = (f_{\text{shift}} + \mu \times M \times PRF)/Ka$$

(48)

where $Ka$ is the azimuth frequency module rate.

$$t_{\text{seg}(-3)} = (-13200+2 \times 4 \times 2200)/4513 = 0.9750 \text{ s}.$$  (49)

Frequency shift of imbalance targets is integer times of PRF ($w_{\text{gl}} = 2\pi PRF$). The location of imbalance target is with the same location as corresponding ambiguous targets. When downsampled and compressed by matched filter, every segment corresponds to an imbalance target, as shown in Fig. 6(b). Using the amplitude of each target is determined by the bandwidth of the segment

$$A_{\text{segment}} = 20\log_{10} (BW_{\text{segment}}/BW_{\text{original}}).$$  (50)

In Fig. 6(b), $A_{\text{seg}(1)} = -49.20 - (-33.36) = -15.84 \text{ dB}$, and in (49)

$$A_{\text{seg}(1)} = 20\log_{10} (1050 \text{ Hz}/6500 \text{ Hz}) = -15.83 \text{ dB}.$$  (51)

If the first channel is set as the referred channel, the spectrum of channel $m$ should multiply the delay filter in (27); thus, the sampling time axis is based on the first channel. A general result can be obtained as follows:

$$A_{\text{seg}} = \int_{(f+\mu \times M \times PRF)\in f_{\text{seg}}} \left| \sum_{m=0}^{M-1} \sum_{j=0}^{M-1} G_{m}^{\text{offset}} H_{m}(f) G_{T}(f) \times (f) G_{R}(f) P_{mj}(f) \right| df$$

(52)

where $f_{\text{seg}}$ is the frequency range from the equivalent spectrum. $G_{m}^{\text{offset}}$ is the offset of the $m$th channel. $G_{T}(f)G_{R}(f)$ is the antenna pattern of the transmitter/receiver. When $G_{T}(f)G_{R}(f) = 1$ and $H(f)P(f) = 1$, then the bandwidth is proportional to the amplitude as in (50).

In (52), the form is similar with AASR in [4], whereas the frequency range should be based on the equivalent spectrum in this letter. Moreover, the equivalent spectrum is not unique, but all of them will lead to the same result after sampling.

### VI. Conclusion

In this letter, a comparison of spectrum reconstruction algorithm and filter group algorithm has been given. The two methods are equivalent, but spectrum reconstruction algorithm only needs to calculate the inverse of matrix once. Through a transformation matrix, the relationship between out-of-band signals and out-of-band signals after reconstruction is built. Then, an equivalent spectrum is built. Nonuniformly sampling from the original signal is equal to uniform sampling from the equivalent signal. At last, channel imbalance can be seen as the upsampling of the offset channel. The location and amplitude of the imbalance target can be estimated. The location of ambiguous target is same with corresponding imbalance targets.

## Appendix

In (3), the period of $p_{s}(t)$ is $MT$; the Fourier transform of it is

$$P_{s}(kw_{s}) = \frac{1}{T_{s}} \int_{-T_{s}/2}^{T_{s}/2} \sum_{m=0}^{M-1} \delta(t - mT - \gamma_{m}T) e^{-jkw_{s}t} dt$$

$$= \frac{1}{T_{s}} \sum_{m=0}^{M-1} e^{-jkw_{s}(mT + \gamma_{m}T)}, \ T_{s} = MT$$

$$p_{s}(t) = \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} e^{-jkw_{s}(mT + \gamma_{m}T)} e^{jkw_{s}t}$$

$$= \frac{1}{T_{s}} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} e^{jkw_{s}(t-mT-\gamma_{m}T)}$$

$$P_{s}(w) = \int_{-\infty}^{\infty} p_{s}(t) e^{-j\omega t} dt$$

$$= \frac{1}{T_{s}} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} \int_{-\infty}^{\infty} e^{-j\omega t} e^{jkw_{s}(t-mT-\gamma_{m}T)} dt$$

$$= \frac{1}{MT} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{M-1} e^{-j\omega w_{s}(M+mT+\gamma_{m})}$$

### REFERENCES


