Design of robust controllers for position servos using $H$-infinity theory

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Linear feedback controllers for position servos are designed using a frequency domain optimization method based on $H_\infty$ theory. This method aims to produce a low-order controller which is 'robust', in the sense that the closed-loop system is guaranteed to meet performance objectives in the presence of unmodelled dynamics. A detailed design of a controller for a hydraulic position servo, including experimental results, is presented.

1 INTRODUCTION

$H_\infty$ control theory was developed during the 1980s as an approach to the problem of designing robust multi-variable feedback control systems, that is systems that perform well in spite of inevitable modelling imperfections. In this approach, the control design problem is formulated as a constrained optimization problem involving the frequency response magnitude of the closed-loop system. The theory has already reached a remarkable level of maturity, as evidenced by the availability of software which implements state-space algorithms to solve the general $H_\infty$ problem (1), and the appearance of detailed descriptions of the method in recent engineering-level textbooks (2). However, although $H_\infty$ is one of the most active areas of control-theoretical research, actual applications are only just beginning to be described in the literature.

In the field of electromechanical servos, a method for the design of robust controllers has recently been proposed by Yamamoto et al. (3), using a combination of pole placement and trial-and-error tuning of the gain. Pohjolainen and Virvalo (4) have designed a pneumatic position servo using a control system optimization method developed by Pohjolainen and Mäkelä (5). This method optimizes the performance of the feedback system. The more general 'mixed sensitivity' problem, which involves the optimization of both performance and robustness, may be solved using the state-space algorithm of Glover and Doyle (6). This method has been applied to the design of a digital hydraulic position servo by Chiang and Safonov (1), yielding an inner loop with proportional feedback and an outer loop with a sixth-order compensator.

In this paper, a practical design procedure for position controllers is proposed which yields robust controllers of low order. The emphasis is on simplicity of use; the designer selects a small number of physically meaningful parameters to define the specifications, and the computer uses the modern $H_\infty$ algorithms to find the optimal design, or to report that the specifications are impossible and should be redefined. The details of the procedure are shown in an example involving a hydraulic position servo test bench.

2 $H_\infty$ OPTIMAL CONTROL

In this section, some standard definitions and results of $H_\infty$ control theory are outlined, and the general $H_\infty$ mixed sensitivity problem is explained. The material is mainly from references (1) and (2). For the sake of generality, all results in this Section are given for multi-variable systems, even though the servo problem considered in this paper is a single-input/single-output system.

2.1 Basic definitions

The name $H_\infty$ (H-infinity) refers, for control engineering purposes, to the set of exponentially stable transfer functions. A transfer matrix (a matrix of transfer functions) is said to be stable when every one of its elements is a stable transfer function.

$H_\infty$ methods of control design, broadly speaking, work by minimizing the norms of certain closed-loop transfer matrices. The $H_\infty$ norm of a stable transfer matrix $G$ is the largest value of the spectral norm of the frequency response matrix, that is

$$||G||_\infty = \sup_\omega \sigma(G(i\omega))$$ (1)

where $\sigma(\cdot)$ denotes the largest singular value of a matrix.

2.2 Performance specification

The standard feedback control system is shown in Fig. 1 and is described by the transfer matrix equations

$$e = v - y, \quad y = Gu, \quad u = Fe,$$ (2)

where $G$ represents the plant (the system to be controlled), $F$ represents the controller, the command reference signal is $v$, actuator noise is $e$, the control signal is $u$, the plant output is $y$, and the tracking error is $e$. The roles of the filters $W_1$ and $W_3$ are explained later.
The closed-loop transfer matrix
\[ S = (I + GF)^{-1} \]
is the transfer matrix from the reference signal \( v_1 \) to the tracking error \( \epsilon \), and is known as the sensitivity. The closed-loop system is said to be stable when all four transfer matrices \( S, SG, FS \) and \( FSG \) are stable. (This definition of stability rules out unstable pole-zero cancellations.)

One objective of the control design is to make the sensitivity as small as possible in certain specified frequency ranges; this objective can be represented by the inequality
\[ \sigma(S(i\omega)) \leq \sigma(W_1^{-1}(i\omega)) \quad \forall \omega \] (3)
where \( W_1^{-1} \) is a frequency-dependent function which the designer may use to specify the desired 'shape' of the sensitivity function. Standard frequency-domain criteria for tracking error may be represented by \( W_1^{-1} \), as explained in the papers by Kwakernaak (7) and Helton (8), among others.

The performance objective, equation (3), is implied by the \( \mathcal{H}_\infty \) inequality
\[ \|W_1S\|_\infty \leq 1 \] (4)
This is the performance constraint in the control design problem.

### 2.3 Stability robustness

Robustness is defined by the condition that the closed-loop system is stable for any plant \( G \) from the set
\[ \{ G : G = (I + \Delta_m)G_0, \Delta_m \in \mathcal{X}_\infty, \|\Delta_mW_3^{-1}\|_\infty < 1 \} \] (5)
where \( G_0 \) is a fixed nominal plant, \( W_3 \) is a fixed weighting transfer matrix, and \( \Delta_m \) is any stable transfer matrix whose 'weighted' frequency response is bounded according to \( \|\Delta_mW_3^{-1}\|_\infty < 1 \). This is known as a multiplicative perturbation model of plant uncertainty.

If, as is usual, the controller is chosen so that the closed-loop system with \( G = G_0 \) (no model error) is stable, then, according to the small gain theorem, a sufficient condition to guarantee stability of the closed-loop system with uncertain plant as described above is that
\[ \|W_3T_0\|_\infty \leq 1 \] (6)
where \( T_0 := G_0F(I + G_0F)^{-1} \) is the nominal complementary sensitivity of the system. This is the stability robustness constraint in the control design problem.

### 2.4 Mixed sensitivity problem

The two design objectives of performance and robustness can be combined into a single mixed sensitivity problem as follows. The problem is to find a controller \( F \) which stabilizes the nominal plant \( G_0 \) and which satisfies the \( \mathcal{H}_\infty \) inequality
\[ \|Q\|_\infty := \left\| \begin{bmatrix} W_1S_0 & rW_3T_0 \end{bmatrix} \right\|_\infty \leq 1. \] (7)
where \( S_0 := (I + G_0F)^{-1} \) and the real scalar frequency function \( r \) satisfies
\[ r(\omega) \geq 1 \]
at all frequencies. If such a controller is found, then both the condition
\[ \|rW_3T_0\|_\infty \leq 1 \] (8)
(which is a strengthened version of expression (6), and the nominal performance condition
\[ \|W_1S_0\|_\infty \leq 1 \] (9)
will be automatically satisfied.

In the mixed sensitivity problem, there is a conflict between the goals of optimizing performance and robustness. This conflict arises because the identity \( T_0 = 1 - S_0 \) implies that making the sensitivity small at some frequency leads to a complementary sensitivity which is not small, that is \( S_0 \approx 0 \Rightarrow T_0 \approx 1 \), and vice versa. The trade-off between performance and robustness objectives is managed using the weight functions \( W_1 \) and \( W_3 \).

It will be noted that the inequality (9) is not the same as the original performance condition expression (4), which involved the true plant. Thus, the two-block \( \mathcal{H}_\infty \) problem described here does not, strictly speaking, ensure that the performance condition set out in Section 2.2 is met, and it is, in principle, possible that the true sensitivity \( S \) may turn out to be quite different from the nominal \( S_0 \) which is predicted by the design. In other words, the two-block problem does not synthesize a controller to meet explicit \textit{a priori} performance robustness conditions. It is possible to compute a controller which ensures performance robustness using a diagonally scaled four-block \( \mathcal{H}_\infty \) problem. This is known as the Doyle 'structured singular value' (or '\( \mu \)-synthesis') method and is explained in Chapter 3.12 of reference (2). For this work, however, only the simpler two-block formulation will be presented.

With the two-block formulation, the performance robustness can be controlled to some extent by specifying excess robustness at critical frequencies using the function \( r(\omega) \). The effect of changing \( r \) can be estimated using the following theorem, which gives formu-
Theorem 1 Let S, T₀, and S₀ be defined as in the text. If
\[
\frac{1}{1 + \sigma(\Delta_m T_0)} \leq \frac{\sigma(S)}{\sigma(S_0)} \leq \frac{1}{1 - \sigma(\Delta_m T_0)}
\]
(10)
and
\[
\frac{\sigma(T - T_0)}{\sigma(S_0)} = \frac{\sigma(S - S_0)}{\sigma(S_0)} \leq \frac{1}{1 - \sigma(\Delta_m T_0)}
\]
(11)
then
\[
|\sigma(S) - 1| \leq \frac{\sigma(\Delta_m T_0) + \sigma(T_0)}{1 - \sigma(\Delta_m T_0)}
\]
(12)

When expressions (5) and (7) hold,
\[
\sigma(\Delta_m T_0) \leq \sigma(W_3 T_0) \leq \frac{1}{r}
\]

The bounds given in theorem 1 then continue to hold when \(\Delta_m T_0\) is replaced by \(W_3 T_0\) or \(r^{-1}\). It follows that, at any given frequency, making \(r \gg 1\) will ensure good performance robustness at that frequency. In practice, however, the bounds of theorem 1 can be rather slack, so that this approach is of limited usefulness.

2.5 Solution of \(\mathcal{H}_\infty\) problem

The mixed sensitivity problem of expression (7) is a special case of the general four-block \(\mathcal{H}_\infty\) problem shown in Fig. 2. The generalized plant is a transfer matrix \(P\) partitioned so that \(z = P_{11}v + P_{12}u\) and \(y = P_{21}v + P_{22}u\).

The \(\mathcal{H}_\infty\) problem is to find \(F\) that satisfies
\[
\|\hat{Q}\|_\infty = \|P_{11} + P_{12}F(I - P_{22}F)^{-1}P_{21}\|_\infty \leq 1
\]
(13)
subject to the constraint that the closed-loop system remains stable.

Comparing Figs. 1 and 2, it can be seen that a generalized plant for the mixed sensitivity problem of expression (7) is given by
\[
P = \begin{bmatrix}
W_1 & -\varepsilon W_1 G_0 & -W_1 G_0 \\
0 & \varepsilon W_1 G_0 & \varepsilon W_1 G_0 \\
1 & -\varepsilon G_0 & -G_0
\end{bmatrix}
\]
(14)

Here the blocks with \(\varepsilon\) correspond to the actuator noise input \(\varepsilon v_2\) as shown in Fig. 1. As is proposed by O’Young et al. (10), including this additional input is a good way to avoid difficulties arising from poles on the imaginary axis; such poles in the nominal plant prevent basic mathematical operations (‘model matching transformations’) in the \(\mathcal{H}_\infty\) solution algorithm from being carried out. It can be shown that, on choosing \(\varepsilon\) small enough, the norm \(\|\hat{Q}\|_\infty\) can be made into an upper bound estimate of \(\|Q\|_\infty\) in expression (7) which is as tight as desired, that is \(\|\hat{Q}\|_\infty \geq \|Q\|_\infty\) and \(\lim_{\varepsilon \to 0} \|\hat{Q}\|_\infty = \|Q\|_\infty\).

The two-Riccati equation formula developed by Glover and Doyle (6) can be used to solve the \(\mathcal{H}_\infty\) problem of expression (13). The MatLab robust control toolbox routine \(\text{hinfkgjd}\) is an implementation of the Glover–Doyle algorithm. It returns as output a class of state-space matrices of \(F\), or a message indicating that no solution exists. The class of solutions is parametrized by a transfer matrix \(U\) which is stable and satisfies \(\|U\|_\infty \leq 1\), but is otherwise arbitrary. The controller corresponding to \(U = 0\) is known as the minimum-entropy controller.

In general, the order of the minimum-entropy controller computed by the Glover–Doyle algorithm is at least equal to the sum of the number of states of \(G_0\), \(W_1\), and \(W_2\); pole-zero cancellations may make the controller order smaller. Thus, when high-order models are used for the nominal plant and for the weights, the resulting \(\mathcal{H}_\infty\)-optimal controller may be of excessively high order for practical realization. The solution adopted in this work is to use low-order models for the nominal plant and for the performance and robustness specification weights from the very outset. This is simpler than the more common approach [see reference (1) and Chapter 8.3.9 of reference (2)], which is to apply model reduction algorithms after the design is complete, at the cost of some degradation in the performance and/or robustness.

3 CONTROL DESIGN METHOD FOR POSITION SERVOS

3.1 Definition of basic transfer functions

The standard open-loop model for a position servo has the form
\[
G_0 = \frac{K_0 \omega_0^2}{s(s^2 + 2\zeta_0 \omega_0 s + \omega_0^2)}
\]
(15)
where \(K_0\), \(\omega_0\), and \(\zeta_0\) are positive constants.

The control design objectives are as follows:

1. The velocity error constant
\[
K_v = \lim_{s \to 0} sF(s)G_0(s)
\]
(16)
should be as large as possible. (This implies that the tracking error asymptotically goes to zero and that the static gain of the controller should be maximized.)

2. The servo should give a smooth step response without overshoot and with a settling time \(t_s\) as small as possible. (Maximizing \(K_v\) is approximately equivalent to minimizing \(t_s\).)

3. The response characteristics should not be sensitive to modelling errors due to changing operating points, non-linearities, delays, higher-order modes etc. In particular, the closed-loop system should be stable at all operating points.
The stability requirement implies that the sensitivity function will have a zero at the origin, because if $S(0) \neq 0$, then $GS$ would have a pole at the origin. The performance weighting function $W_p$ in expression (4) is therefore chosen to match this pole approximately with a corresponding pole near the origin. A simple transfer function with this property is

$$W_1 = \frac{\beta s + \gamma}{s + \delta y}$$

(17)

The parameter $0 < \beta < 1$ is the high-frequency limiting value of $W_1$, and serves to constrain the maximum resonance peak in the frequency response of $S_0$. The parameter $\gamma > 0$ gives the $0 \text{dB}$ crossing of $W_1$ (and thus of $S_0$), when $\delta$ is small. If $\delta = 0$, the parameter $\gamma$ also defines the minimum velocity error constant of the nominal system, because then expressions (9), (15), (16) and (17) together imply $\gamma < K_\gamma$.

The multiplicative model error $\Delta_m$ may be estimated by plotting the frequency response of $(G - G_0)/G_0$ for various operating points. This may be done by assuming that $G$ is a linear model of the same form as the nominal plant (equation (15)) but with different parameters. The corresponding $\Delta_m$ is stable, in accordance with the assumptions for stability robustness in Section 2.3. Alternatively, $\Delta_m$ may be estimated from the frequency responses of nonlinear numerical or laboratory test benches at various operating points. It is not necessary to determine $\Delta_m$ with great accuracy, but only to estimate its essential features, namely its peaks and limiting behaviour at high and low frequencies.

The typical multiplicative error $\Delta_m$ for a position servo modelled as in equation (15) has a frequency response magnitude which is small at low frequencies, crosses $0 \text{dB}$ at a frequency slightly higher than the first resonance frequency $\omega_0$, and thereafter grows at approximately $60 \text{dB} \text{per decade}$. This typical shape has been observed by the authors in the results of frequency response tests of pneumatic and hydraulic servos, and a similar shape is reported by Yamamoto et al. (3) for a linear positioning actuator driven by a voice coil motor. A good estimate for such a shape of frequency response is provided by the third-order Butterworth filter

$$W_3^{-1} = \frac{\omega_0^3}{s^3 + 2s^2\lambda + 2s\lambda^2 + \lambda^3}$$

(18)

where $\omega > 1$ is the reciprocal of the magnitude of the modelling error at low frequencies, and $\lambda$ is the cut-off frequency. This $W_3$ is improper, but $W_3G_0$ is proper, and it is $W_3G_0$ that is used in the generalized plant in equation (14).

Figure 3 shows the performance and robustness specification weights on a frequency response diagram. The control design is entirely specified by choosing the five parameters $\alpha, \beta, \gamma, \delta, \lambda$ that determine the shapes of the weight functions. These in turn constrain the sensitivity and complementary sensitivity functions that determine the performance and robustness properties of the closed-loop system.

### 3.2 Control design method

Once the transfer functions $G_0$, $W_1$, and $W_0$ are specified, it is routine to compute a control transfer function $F$ satisfying the $\mathcal{H}_\infty$ constraint of expression (7). The complete procedure is as follows:

1. Estimate the model parameters $K_0$, $\omega_0$, $\zeta$. It is appropriate to select the largest $K_0$ and the smallest $\omega_0$ among the expected operating conditions of the servo, as these choices tend to minimize the model error.

2. To determine $\Delta_m$, as described in Section 3.1.

3. Select the weight function parameters. For $\delta$ one chooses a small positive real number. The robustness parameters $\alpha$ and $\lambda$ may be chosen using interactive computer graphics by simply 'clicking' at a point on the screen to select the break point for the Butterworth filter. By plotting the model error estimates on the same graph, one can ensure that $|W_3(\omega)| \leq |\Delta_m(\omega)|$ at all frequencies, and visually estimate the excess robustness $\rho(\omega)$.

4. The remaining weight parameter $\gamma$ is computed at the same time as the controller. A bisection method is used to determine the largest $\gamma$ value for which a solution to the $\mathcal{H}_\infty$ problem of expression (13) exists.

5. A prefilter should be inserted to remove high-frequency components from the tracking signal. The prefilter bandwidth should match that of the closed-loop system. A simple choice is the first-order low-pass filter

$$F = \frac{\gamma}{s + \gamma}$$

(19)

When the minimum-entropy solution is used, the resulting feedback controller will be of order 3. This is less than the sum of the orders of $G_0$, $W_1$, and $W_0$ because of pole-zero cancellations in the controller. The two zeros of the controller will be the resonance poles of $G_0$. [This agrees with the general theoretical results (11).]

Once the feedback controller is specified, the closed-loop frequency responses may be plotted. The possible deviation between the nominal system and the true one can be estimated using the formulas of theorem 1, or using open-loop frequency response data if available, or, of course, by testing the actual closed-loop system.

### 3.3 Proportional feedback is also $\mathcal{H}_\infty$ optimal

The following brief discussion is a side path which leads to some interesting insights on $\mathcal{H}_\infty$ theory. Consider the following question: given a closed-loop control system designed by some other method, can the sensi-
tivity and robustness functions be improved on, uniformly across frequency, by $\mathcal{H}_\infty$ optimization? This problem was considered by Pohjolainen and Mäkelä (5) for the case of $S_0$ alone. A complete solution has been presented by Lenz et al. (12), who show how a large class of practical controllers, including typical stable controllers, are $\mathcal{H}_\infty$ optimal in this strong sense. Their criteria involve simply counting the zeros of certain characteristic polynomials.

Proportional feedback is the simplest and most common form of control for a position servo. Applying lemma 4.3 of reference (12) to equation (15), it is easily shown that proportional feedback for a position servo is indeed $\mathcal{H}_\infty$ optimal in this strong sense. In other words, there does not exist any linear controller which will, at all frequencies, give better sensitivity $S_0$ and robustness $1 - S_0$ than a proportional feedback. Any improvement in some frequency range will therefore be at the expense of degradation in some other frequency range. This is the usual tradeoff in control design problems. The $\mathcal{H}_\infty$ method is a way of performing this tradeoff by shaping the frequency response according to explicit criteria.

4 DESIGN OF HYDRAULIC SERVO CONTROLLER

4.1 Plant data

The laboratory test bench comprises a hydraulic cylinder and piston driving a sliding mass. The cylinder is of unsymmetric type. The controller is a digital microcomputer connected to a sensor of the position $y$ of the mass, and to the servo valve. The main parameters of the test bench are as follows:

<table>
<thead>
<tr>
<th>Cylinder and load</th>
<th>32 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piston diameter</td>
<td>20 mm</td>
</tr>
<tr>
<td>Stroke</td>
<td>1000 mm</td>
</tr>
<tr>
<td>Load</td>
<td>220 kg</td>
</tr>
<tr>
<td>Limiting static friction</td>
<td>400 N</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Servo valve and amplifier</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal flow rate</td>
</tr>
<tr>
<td>Natural frequency</td>
</tr>
<tr>
<td>Damping factor</td>
</tr>
<tr>
<td>Input signal limits</td>
</tr>
<tr>
<td>Combined hysteresis and threshold</td>
</tr>
<tr>
<td>Charge pump pressure</td>
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<table>
<thead>
<tr>
<th>Controller</th>
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<tbody>
<tr>
<td>Transducer resolution</td>
</tr>
<tr>
<td>Microcontroller</td>
</tr>
<tr>
<td>D/A transducer</td>
</tr>
</tbody>
</table>

In this paper, the valve command signal $u$ is given as a percentage of the maximum saturation voltage (7.5 V).

The open-loop velocity frequency response is shown in Fig. 4. It was computed from a sine sweep test of amplitude 5 per cent applied at $y_0 = 500$ mm (centre of the stroke) and at $y_0 = 100$ mm. From these curves one may see the presence of a resonance peak at 20–30 Hz. The damping ratio, computed using the resonance peak amplitude method, is found to be 0.2–0.35. The servo has a lower natural frequency and lower damping at operating points near the centre of the stroke. Further sine sweep experiments using higher amplitude inputs yield similar results, but with higher apparent damping ratios (up to 0.80 when the excitation amplitude is 50 per cent). From Fig. 4, and from open-loop step response tests (not shown), the value of $K_0$ is estimated to be in the range 17–21 mm/s%.

From the aforementioned test results, the parameters for the nominal model are selected as

$K_0 = 21$ mm/s%
$\omega_0 = 20$ Hz = 126 rad/s
$\zeta_0 = 0.3$

The corresponding model error $\Delta_m$, estimated using the experimental frequency response in expression (5), is shown in Fig. 6.

The parameters for the robustness constraint function $W_3$ in equation (18) are selected as $\alpha = 2$ and $\lambda = 70$ rad/s, ensuring a loop bandwidth (the lowest frequency beyond which $|T_0| < 1/\sqrt{2}$) of at most 17 Hz. The curve of $W_3^{-1}$ is shown in Fig. 6. As is required for robustness, it lies below the estimated values of $\Delta_m^{-1}$, the reciprocal of the model error.

4.2 $\mathcal{H}_\infty$ controller

The performance constraint function $W_1$ is of the form of equation (17) with $\beta = 0.10$. The largest $\gamma$ for which the $\mathcal{H}_\infty$ solution exists is computed to be $\gamma = 35$ rad/s, and the corresponding controller transfer function is

$$F = \frac{1029.4s^2 + 206972s + 1.6255 \cdot 10^7}{s^3 + 680.54s^2 + 105984s + 8.5345 \cdot 10^6}$$

which has the units %/mm when $s$ is in rad/s. This controller gives a velocity error constant $K_v = 40\text{s}^{-1}$. The nominal closed-loop frequency response magnitudes are shown as solid lines in Figs. 5 and 6. The broken lines show the design constraints. Note how the performance constraint $W_1$ governs at frequencies below 10 Hz, but the robustness constraint $W_3$ governs at frequencies above 15 Hz. A discrete-time version of $F(s)$ is computed using a Tustin transform (with 5 ms sampling period), yielding the discrete-time control transfer function

$$\frac{1.1784z^2 - 0.1481z^2 - 0.8879z + 0.4386}{z^3 - 1.0405z^2 + 0.2963z + 0.0493}$$
The step response of the closed-loop system with the prefilter (expression (19)) is shown in Figs. 7 and 8. The response is seen to be adequately smooth and rapid. The responses at other operating points are similar; the response at one end of the stroke is shown in Figs. 9 and 10. For comparison, the same figures show the step response of the hydraulic servo with only a proportional feedback whose gain is equal to the static gain of the \( \mathcal{H}_\infty \) controller, that is 1.9 per cent per millimetre. This comparison shows how, in this case, the dynamic \( \mathcal{H}_\infty \) controller can have the same level of performance (as measured by \( K_v \)) as a simple static feedback and at the same time be more robust.

Figure 11 shows the experimentally determined magnitude of the closed-loop frequency response \( T \). It was computed using standard sine-sweep methods; sampling effects were neglected because the sampling frequency (200 Hz) was well above the closed-loop bandwidth (17 Hz). \( |T| \) is seen to be close to the nominal (design) curve \( |T_0| \), and to lie within the bounds predicted by theorem 1. These bounds are rather slack, however.

5 DISCUSSION

This paper has shown how \( \mathcal{H}_\infty \) control theory can be used to design a position servo controller. The controller is designed to meet frequency-domain specifications optimally on both performance and robustness. The order of the controller is only three, which is a reason-
able size for such applications. Laboratory results indicate that the method yields an acceptable controller whose behaviour is about the same at all operating points.

The controller produced by the proposed method is of a rather standard form, and the same controller could have been produced by other methods of control design, such as an $\mathcal{H}_\infty$ method (Kalman observer), a pole-placement method (3), or the Horowitz method (13). In the proposed method, however, only a small number of parameters need be chosen, the design specifications are essentially graphical, and the optimal controller is computed using reliable state-space algorithms.

Further work is required on the following:

1. The continuous-time $\mathcal{H}_\infty$ design was implemented using a discrete-time controller, based on an ad hoc discretization using the bilinear transform. For the laboratory test bench used in this investigation, the sampling rate was set above the closed-loop bandwidth, so that sampling effects were negligible. Thus, it made little difference which discretization procedure was employed. For problems where the sampling rate is not so high, the discretization effect should be analysed. If, as here, the controller is designed using continuous-time $\mathcal{H}_\infty$ theory, then the discretization error and its effect on stability can be estimated in a manner analogous to the plant modeling robustness analysis (14). If discrete-time $\mathcal{H}_\infty$ theory is used, as suggested in reference 1, the intersample behaviour may also be studied using $\mathcal{H}_\infty$ methods (15).

2. The prefilter (expression (19)) was chosen using an ad hoc criterion. It would be more consistent with the design philosophy if the prefilter were computed at the same time as the feedback controller using explicit frequency-domain criteria. This can be done by formulating the problem for the two-degree-of-freedom control system configuration, where the controller is a two-input/one-output transfer matrix and the system equations are

$$u = [F_1 \ F_2] \begin{bmatrix} p \\ y \end{bmatrix}, \ y = Gu$$

in place of equations (2). However, the mixed-sensitivity problem for such a configuration does not lead to the same sort of performance-robustness trade-off seen in this work. In fact, it has been shown on p. 219 of reference (16) that, for any given robustness constraint of the type considered here, it is possible to achieve as high a level of nominal performance (as represented by $\gamma$, for example) as desired. Obviously, this result indicates that the standard mixed-sensitivity formulation is not useful for two-degree-of-freedom configurations, since there is no assurance of performance robustness as in theorem 1. The authors are currently investigating the synthesis of performance-robust two-degree-of-freedom controllers for position servos.

3. The free parameter $U(s)$ which appears in the $\mathcal{H}_\infty$ solution may be used as an additional tuning parameter.

ACKNOWLEDGEMENT

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REFERENCES

APPENDIX

Proof of theorem 1

To begin, a basic inequality for $A_m T_o$ when $(\|A_m T_o\| =)$ $\bar{\sigma}(A_m T_o) < 1$ is derived. Let $w$ be a non-zero vector. Then

$$\| (I + A_m T_o) w \| \geq \|w\| - \|A_m T_o\| \cdot \|w\|
= (1 - \|A_m T_o\|) \|w\| > 0$$

so that the equation $(I + A_m T_o) w = 0$ has no non-zero solutions. Then $(I + A_m T_o)^{-1}$ exists. From the inequality

$$1 = \| (I + A_m T_o)(I + A_m T_o)^{-1} \|
\geq \| (I + A_m T_o)^{-1} \| - \|A_m T_o\| \cdot \| (I + A_m T_o)^{-1} \|$$

it follows that

$$\| (I + A_m T_o)^{-1} \| \leq \frac{1}{1 - \|A_m T_o\|}$$

that is

$$\bar{\sigma} [(I + A_m T_o)^{-1}] \leq \frac{1}{1 - \bar{\sigma}(A_m T_o)} \quad (20)$$

The above inequality is now applied to prove the robustness formulas of theorem 1. The inequalities (10) follow directly from the identity

$$S(I + A_m T_o) = S_o \quad (21)$$

and inequality (20). The identity (21) can also be rearranged to give

$$S_o - S = S_o - S_o (I + A_m T_o)^{-1}
= S_o A_m T_o (I + A_m T_o)^{-1}$$

which together with expression (20) gives inequality (11). Finally, the identity

$$I - S = (T_o + A_m T_o) (I + A_m T_o)^{-1}$$

together with expression (20), gives expression (12).