On the computational complexity of dynamic slicing problems for program schemas

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The problem of deciding whether one point in a program is data dependent upon another is fundamental to program analysis and has been widely studied. In this paper we consider this problem at the abstraction level of program schemas, in which computations occur in the Herbrand domain of terms and predicate symbols, which represent arbitrary predicate functions, are allowed. Given a vertex \( l \) in the flowchart of a schema \( S \) having only equality assignments, and variables \( v, w \), we show that it is PSPACE-hard to decide whether there exists an execution of a program defined by \( S \) in which \( v \) holds the initial value of \( w \) at at least one occurrence of \( l \) on the path of execution, with membership in PSPACE holding provided there is a constant upper bound on the arity of any predicate in \( S \). We also consider the ‘dual’ problem in which \( v \) is required to hold the initial value of \( w \) at every occurrence of \( l \), for which the analogous results hold. Additionally, the former problem for programs with non-deterministic branching (in effect, free schemas) in which assignments with functions are allowed is proved to be polynomial-time decidable provided a constant upper bound is placed upon the number of occurrences of the concurrency operator in the schemas being considered. This result is promising since many concurrent systems have a relatively small number of threads (concurrent processes), especially when compared with the number of statements they have.

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1. INTRODUCTION

A schema represents the statement structure of a program by replacing real functions and predicates by symbols representing them. A schema, $S$, thus defines a whole class of programs which all have the same structure. Each program can be obtained from $S$ via a domain $D$ and an *interpretation* $i$ which defines a function $f^i : D^n \rightarrow D$ for each function symbol $f$ of arity $n$, and a predicate function $p^i : D^m \rightarrow \{T, F\}$ for each predicate symbol $p$ of arity $m$. As an example, Figure 1 gives a schema $S$, and the program $P$ of Figure 2 is defined from $S$ by interpreting the function symbols $f, g, h$ and the predicate symbol $p$ as given by $P$, with $D$ being the set of integers. The subject of schema theory is connected with that of program transformation and was originally motivated by the wish to compile programs effectively [Greibach 1975]. Many results on schema equivalence [Danicic et al. 2007; Laurence et al. 2004; 2003; Sabelfeld 1990; Luckham et al. 1970] and on applying schema formulation to program slicing [Laurence 2005; Danicic et al. 2005] have been published.

In this paper we are concerned with the relevance of schema theory to deciding data dependence. We only consider schema interpretations over the Herbrand domain of terms in the variables and function symbols. We consider the problem of deciding the following two properties, defined using a schema $S$, a variable $v$, a variable or function symbol $f$ and a vertex $l$ in the flowchart of $S$.

—(Existential data dependence.) If there is an executable path through $S$ that ends at $l$ at which point the term defined by $v$ contains the symbol $f$, then $SDD_S(f, v, l)$ is said to hold.

—(Universal data dependence.) If, for all executable paths through $S$, the term defined by $v$ contains the symbol $f$ whenever $l$ is reached, then $\forall DD_S(f, v, l)$ is said to hold.

We prove that if all assignments in $S$ are equality assignments, no concurrency constructs are allowed and only two while loops are permitted in $S$, one of which lies in the body of the other, then both problems defined by these properties are PSPACE-hard, with membership in PSPACE holding provided there is a constant upper bound on the arity of any predicate in $S$.

Additionally, we consider the existential data dependence problem in the case where assignments having function symbols are allowed, but where all schemas are

\[
\begin{align*}
  u &:= 1; \\
  \text{if } u > 1 & \text{ then } v := u + 1; \\
  \text{else } & v := 2;
\end{align*}
\]

Fig. 2. Program $P$
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free (that is, all paths are executable) and hence all branching is, in effect, non-deterministic. For arbitrary schemas under these conditions, deciding existential data dependence is easily shown to be PSPACE-complete, owing to a reduction from the finite intersection problem for deterministic finite state automata, but we prove that if a constant upper bound is placed upon the number of occurrences of the concurrency operator in the schemas being considered, then the problem becomes decidable in polynomial time.

To the authors’ knowledge, neither problem has been previously considered for arbitrary schemas. Both problems have been studied for programs of various types. In [Müller-Olm and Seidl 2001], it is proved that deciding existential data dependence (expressed in the paper as a slicing problem) is PSPACE-complete for programs having concurrency constructs, but only non-deterministic branching. Müller-Olm et al. have also considered a generalisation of our universal data dependence problem [Müller-Olm et al. 2005; Müller-Olm et al. 2005], defined by testing for equality between two terms at particular program points, but their programs use term inequality guards on edges in flowcharts, and apart from this restriction, their programs are non-deterministic. In [Müller-Olm and Rüthing 2001], an extensive classification of the complexity of deciding both our problems is given, but branching is non-deterministic and the domain is that of the integers in every case.

The complexity results are more promising than they might initially appear. This is because many concurrent systems have only relatively few threads even if they are quite large (in terms of lines of code). The results also suggest that it should be easier to ‘scale’ data dependence algorithms to large programs/schemas with only a few threads than to smaller programs/schemas with many threads. For schemas and programs that might not be free, data dependence calculated on the assumption that freeness holds provides an upper bound on the actual data dependence. As a result, if data dependence does not hold under the freeness assumption then we know it does not hold even if the program or schema under consideration is not free. This is important in areas such as security where we wish to show that the value of one variable \(x\), whose value is accessible, cannot depend on the value of another variable \(y\) whose value should be kept secret.

2. BASIC DEFINITIONS FOR SCHEMAS

Throughout this paper, \(F, P, V, L\) denote fixed infinite sets of function symbols, predicate symbols, variables and labels respectively. We assume a function

\[
\text{arity} : F \cup P \rightarrow N.
\]

The arity of a symbol \(x\) is the number of arguments referenced by \(x\). Note that in the case when the arity of a function symbol \(g\) is zero, \(g\) may be thought of as a constant.

**Definition 1 schemas.** We define the set of all schemas recursively as follows. Let

\[
l : \text{skip} \text{ is a schema.}
\]

An assignment \(l : y := f(x)\); where \(y \in V, f \in F, l \in L\) and \(x\) is a vector of \(\text{arity}(f)\) variables, is a schema. Similarly an equality assignment \(l : y := x\) for \(y, x \in V\) is a schema. From these all schemas may be ‘built up’ from the following constructs on schemas.
sequences: \( S' = U_1U_2 \ldots U_r \) is a schema provided that each \( U_i \) for \( i \in \{1, \ldots, r\} \) is a schema.

if schemas: \( S'' = l : p(x) \text{ then } T_1 \text{ else } T_2 \) is a schema whenever \( p \in \mathcal{P} \), \( l \in \mathcal{L} \), \( x \) is a vector of arity \( p \) variables, and \( T_1, T_2 \) are schemas.

don-deterministic branches. \( S''' = l : T_1 \sqcap T_2 \ldots \sqcap T_m \) is a schema whenever \( l \in \mathcal{L} \) and \( T_1, \ldots, T_m \) are schemas.

while schemas: \( S'''' = l : \text{ while } q(y) T \) is a schema whenever \( q \in \mathcal{P} \), \( l \in \mathcal{L} \), \( y \) is a vector of \( q \) variables, and \( T \) is a schema.

non-deterministic loops. \( S'''' = l : \text{ loop } T \) is a schema if \( l \in \mathcal{L} \) and \( T \) is a schema.

concurrent schemas. \( S''''' = l : T_1 \parallel T_2 \ldots \parallel T_m \) is a schema, where \( T_1, \ldots, T_m \) are schemas.

Thus a schema is a word in a language over an infinite alphabet.

The semantics of schemas are defined by their flowcharts, which are finite directed graphs. A directed graph \( G \) is a pair \((V, E)\) with \( E \subseteq V \times V \). We define \( V = \text{Vertices}(G) \), the set of vertices of \( G \).

**Definition 2.** Given a schema \( S \), we define a finite directed graph \( \text{Flowchart}(S) \) with an edge labelling function \( \text{edgeType}_S \) that associates to each edge of \( \text{Flowchart}(S) \) either \( \varepsilon \), a triple \((p, x, X)\) for a predicate \( p \), a vector \( x \) of variables and \( X \in \{T, F\} \), or an assignment, as follows. Unless otherwise stated below, \( \text{edgeType}_S \) maps to \( \varepsilon \).

1. If \( S \) is \( l : \text{skip} \) or \( l : y := f(x) \); or \( l : y := x \); then \( \text{Flowchart}(S) \) has vertex set \{\( \text{start}, l, \text{end} \)\} and edges \((\text{start}, l)\) and \((l, \text{end})\). Here \( \text{edgeType}_S(l, \text{end}) = y := f(x) \); or \( y := x \);, respectively.

2. If \( S = S_1S_2 \), then \( \text{Flowchart}(S) \) has vertex set \( \text{Vertices}(\text{Flowchart}(S_1)) \cup \text{Vertices}(\text{Flowchart}(S_2)) \) and contains every edge occurring in either \( S_1 \) or \( S_2 \), with the function \( \text{edgeType}_S \) returning the same value as in \( S_1 \) or \( S_2 \) respectively, except that \( \text{Flowchart}(S) \) does not have any edge \((l, \text{end})\) for a vertex \( l \) in \( S_1 \) or \( \text{(start), l} \) for a vertex \( l \) in \( S_2 \). Instead, it has an edge \((l_1, l_2)\) for each pair of edges \((l_1, \text{end})\) and \((\text{start}, l_2)\) in \( \text{Flowchart}(S_1) \) and \( \text{Flowchart}(S_2) \) respectively, with the function \( \text{edgeType}_S(l_1, l_2) = \text{edgeType}_{S_i}(l_1, \text{end}) \).

3. If \( S = S_1 \sqcap S_2 \ldots \sqcap S_m \), then \( \text{Flowchart}(S) \) has vertex set \( \text{Vertices}(\text{Flowchart}(S_1)) \cup \ldots \cup \text{Vertices}(\text{Flowchart}(S_m)) \cup \{l\} \) and contains all edges \((l', l'')\) lying in any \( \text{Flowchart}(S_k) \) such that \( l' \neq \text{start} \), with the function \( \text{edgeType}_S \) returning the same value as \( \text{edgeType}_{S_k} \) in the appropriate \( \text{Flowchart}(S_k) \), and also contains an edge \((l, l'')\) for each edge \((\text{start}, l'')\) in any \( \text{Flowchart}(S_k) \). Additionally, \( \text{Flowchart}(S) \) contains the edge \((\text{start}, l)\).

3'. If \( S = S_1 \text{ if } p(x) \text{ then } S_1 \text{ else } S_2 \), then \( \text{Flowchart}(S) \) is identical to \( \text{Flowchart}(l : S_1 \sqcap S_2) \) except that the edges \((l, l'')\) for each edge \((\text{start}, l'')\) in either \( \text{Flowchart}(S_1) \) or \( \text{Flowchart}(S_2) \) are mapped by \( \text{edgeType}_S \) to \((p, x, T)\) or \((p, x, F)\) respectively.

4. If \( S = l : \text{ while } q(y) T \), then \( \text{Flowchart}(S) \) has vertex set \( \text{Vertices}(T) \cup \{l\} \) and contains all edges \((l', l'')\) lying in \( \text{Flowchart}(T) \) such that \( l' \neq \text{start} \) and
If \( M \) — If \( \nu \) is started for all \( n \) — if each variable is a term, it is formally defined as follows:

\[
\text{Term values in the set of terms built from the sets of variables and function symbols.}
\]

The symbols upon which schemas are built are given meaning by defining the notions of a state and of an interpretation. It will be assumed that variables take terms; that is, each function symbol \( f \) is a term.

\( \text{Definition 3 state associated with a path through Flowchart}(S) \) for schema \( S \). Given a state \( d \), a schema \( S \) and a path \( \nu \) through Flowchart \( (S) \) whose first element is \( \text{start} \), we define the state \( M[\nu]_d \) recursively as follows.

- Each variable is a term,
- If \( f \in \mathcal{F} \) is of arity \( n \) and \( t_1, \ldots, t_n \) are terms then \( f(t_1, \ldots, t_n) \) is a term.

The function symbols represent the ‘natural’ functions with respect to the set of terms; that is, each function symbol \( f \) defines the function \( (t_1, \ldots, t_n) \mapsto f(t_1, \ldots, t_n) \) for all \( n \)-tuples of terms \( (t_1, \ldots, t_n) \). A state is a function \( \mathcal{V} \) into the set of terms. An interpretation \( i \) defines, for each predicate symbol \( p \in \mathcal{P} \) of arity \( m \), a function \( p^i : D^m \to \{T, F\} \). We define the natural state \( e : \mathcal{V} \to \text{Term}(\mathcal{F}, \mathcal{V}) \) by \( e(v) = v \) for all \( v \in \mathcal{V} \).

\( \text{Definition 3 state associated with a path through Flowchart}(S) \) for schema \( S \). Given a state \( d \), a schema \( S \) and a path \( \nu \) through Flowchart \( (S) \) whose first element is \( \text{start} \), we define the state \( M[\nu]_d \) recursively as follows.

- \( M[\text{start}]_d(v) = d(v) \) for all variables \( v \).
- If \( \nu = \mu \nu' \) for vertices \( l, l' \) in Flowchart \( (S) \) and \( \text{edgeType}(l, l') \) is not an assignment, then \( M[\nu]_d = M[\mu]_d \).
- If \( \nu = \mu \nu' \) for \( l, l' \in \text{Labels}(S) \) and \( S \) and \( \text{edgeType}(l, l') = y := f(x_1, \ldots, x_n) \), then \( M[\nu]_d(z) = M[\mu]_d(z) \) for all variables \( z \neq y \), and

\[
M[\nu]_d(y) = f(M[\mu]_d(x_1), \ldots, M[\mu]_d(x_n)),
\]

and cases having equality assignments are treated analogously.
DEFINITION 4 executable paths and free schemas. Given a schema \( S \) and an interpretation \( i \) and a path \( \nu \) through \( \text{Flowchart}(S) \) whose first element is \( \text{start} \), we say that \( \nu \) is compatible with \( i \) if given any prefix \( \mu l' \) of \( \nu \) such that edgeType\(_S\)(l, l') = (p, x\(_1\), ..., x\(_n\), X), \( p'(\mathcal{M}[\mu l_1]_c(x_1), \ldots, \mathcal{M}[\mu l_d]_d(x_n)) = X \) holds. A path whose first element is \( \text{start} \) is said to be executable if there exists an interpretation with which it is compatible. A schema is said to be free if every path whose first element is \( \text{start} \) is executable.

Since a schema \( S \) may contain the non-deterministic \( \text{loop} \), \( \cap \) and \( \| \) constructions, an initial state \( d \) and an interpretation \( i \) need not define a unique executable path in \( \text{Flowchart}(S) \) from \( \text{start} \) to \( \text{end} \). In the event that a unique path does exist, we denote it by \( \pi_S(i, d) \), and write \( \mathcal{M}[S]_d^i \) to denote the state \( \mathcal{M}[\pi_S(i, d)]_d^i \). If \( S \) is merely a sequence of assignments, so that the interpretation \( i \) is irrelevant, then we simply write \( \mathcal{M}[S]_d^i \).

2.2 The data dependence problems

We now formalise the two data dependence conditions with which we are concerned in this paper.

DEFINITION 5. Let \( S \) be a schema and let \( v \in \mathcal{V} \), let \( l \in \text{Vertices}(\text{Flowchart}(S)) \) and let \( f \in \mathcal{F} \cup \mathcal{V} \). The predicate \( \text{SDD}_S(f, v, l) \) is defined to hold if there is an executable path \( \mu \) through \( \text{Flowchart}(S) \) which starts at \( \text{start} \) and ends at \( l \) such that the term \( \mathcal{M}[\mu]_c(v) \) contains \( f \); and the predicate \( \forall \text{DD}_S(f, v, l) \) is defined to hold if for every executable path \( \mu \) through \( \text{Flowchart}(S) \) that starts at \( \text{start} \) and ends at \( l \), the term \( \mathcal{M}[\mu]_c(v) \) contains \( f \).

3. COMPLEXITY RESULTS FOR SCHEMAS HAVING ONLY EQUALITY ASSIGNMENTS

In this section, we prove the main \( \text{PSPACE} \)-hardness theorem of this paper, Theorem 7.

Notational conventions.

In the proof of Theorem 7, we will define schemas without indicating labels, and indicate paths simply by using sequences of predicates and \( \text{end} \). These schemas do not have the concurrency \( \| \) symbol and hence all vertices in the appropriate graph \( \text{Flowchart}(S) \) lie in \( \text{Labels}(S) \cup \{\text{start}, \text{end}\} \). In the cases where this convention is used, paths in the sense of Definition 2 are defined unambiguously.

Also, in order to save space, we will sometimes abbreviate sequences of equality assignments in schemas by using the quantifier \( \forall \). For example, in Fig. 3, the line \( \forall k \ t_k := s_{jk} \) is intended as a shorthand for the sequence

\[
t_0 := s_{j0}; t_1 := s_{j1}; \ldots; t_{m_j} := s_{jm_j};
\]

where \( m_j \) is the largest value of \( k \) for which \( s_{jk} \) is defined, in Lemma 6. The lines \( \forall j < m \ b_j := u_{\text{bad}}; \forall j \forall k \ s_{jk} := u_{\text{bad}}; \text{ and } \forall s \in F_1 \cup \ldots \cup F_m \ s := u_{\text{good}}; \) in Figure 4 have analogous meanings. We only use this notation in cases where the order of the assignments is immaterial, since no variable occurs on both the left side of one assignment in the sequence and the right side of another, and so the assignments commute.

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∀k t_k := s_{jk};
∀k s_{jk} := t_{x_j(t,k)};

Fig. 3. The schema $U_{jl}$ of Lemma 6. Here the $t_k$ are new variables used solely for copying and the function $\chi_j$ is defined by the state transition function $\eta_j$ of the automaton $A_j$ as follows; for any letter $\alpha_l$ and state $s_{jk}$, $\eta_j(\alpha_l, s_{jk}) = s_{x_j(t,k)}$. Observe that the value defined by a variable $s_{jk}$ after execution of $U_{jl}$ is the same as that defined by the variable $\eta_j(\alpha_l, s_{jk})$ before execution.

In Lemma 6 and Theorem 7, we will define finite state automata for which the word ‘state’ has its usual meaning; however we will also define schemes having variables which are the states of the automata, and thus the word state has the distinct meaning of a function from variables (automata-theoretic states) to elements of the domain (variables, in the case of schemas having only equality assignments). This should not cause confusion.

**Lemma 6.** Consider a set of $m$ deterministic finite state automata $A_1, \ldots, A_m$ for some $m \geq 0$, all using an alphabet $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$, with each automaton $A_j$ having state set $S_j = \{s_{j0}, \ldots, s_{jm}\}$, total transition function $\eta_j : \Sigma \times S_j \rightarrow S_j$ and final state set $F_j \subseteq S_j$.

For each automaton $A_j$ and each letter $\alpha_l \in \Sigma$, let $U_{jl}$ be the predicate-free schema in Fig. 3 and define $V_l = U_{1l} \ldots U_{ml}$. Let $l_1, l_2, \ldots, l_r \in \{1, 2, \ldots, n\}$ and define $\gamma = \alpha_{l_r} \alpha_{l_{r-1}} \ldots \alpha_{l_1} \in \Sigma^*$. Let $e_{final}$ be the state (in the program sense)

$$
\begin{cases}
  s_{jk} \mapsto u_{bad} & s_{jk} \in S_j - F_j \\
  s_{jk} \mapsto u_{good} & s_{jk} \in F_j
\end{cases}
$$

for new variables $u_{bad}$, $u_{good}$.

(1) If the schema $V_{l_1} \ldots V_{l_r}$ is executed from the state $e_{final}$, then after execution, the state

$$
\begin{cases}
  s_{jk} \mapsto u_{bad} & \eta_j(\gamma, s_{jk}) \in S_j - F_j \\
  s_{jk} \mapsto u_{good} & \eta_j(\gamma, s_{jk}) \in F_j
\end{cases}
$$

is reached.

(2) In particular, if each automaton $A_j$ has initial state $s_{j0}$, then

$$M[V_{l_1} \ldots V_{l_r}]_{e_{final}}(s_{j0}) = u_{good}$$

for all $j$ if and only if the word $\gamma$ is accepted by every automaton $A_j$.

**Proof.** (1) can be straightforwardly proved by induction on $r$. (2) follows immediately from (1) using the fact that for any $j$, $A_j$ accepts $\gamma$ if and only if $\eta_j(\gamma, s_{j0}) \in F_j$ holds.

**Theorem 7.** Let $S$ be a schema and let $v \in V$ and $f \in V$. Assume that $S$ has no concurrency or non-deterministic branching constructions and all its assignments are equality assignments.

(1) Suppose that $S$ is further restricted to membership of the class of schemas satisfying the following two conditions.
—No predicate occurs more than once in $S$.
—$S$ contains two while predicates, one of which lies in the body of the other.

Then the problems of deciding whether $SDD_S(f, v, \text{end})$ and $\forall DD_S(f, v, \text{end})$ hold are PSPACE-hard.

(2) Furthermore, if the conditions in Part (1) are replaced by the requirement that $S$ has no while predicates, then deciding $SDD_S(f, v, \text{end})$ becomes NP-hard and deciding $\forall DD_S(f, v, \text{end})$ becomes co-NP-hard.

Proof. For both parts of the proof, we consider $SDD_S$ first, and then indicate the proofs for $\forall DD_S$. We first prove (1), using a reduction from the intersection problem for finite state automata, which is known to be PSPACE-complete [Kozen 1977]. An instance of this problem comprises a set of $m$ deterministic finite state automata $A_1, \ldots, A_m$ for some $m \geq 0$, all using an alphabet $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$, with each $A_j$ having state set $S_j = \{s_{j0}, \ldots, s_{jm_j}\}$, total transition function $\eta_j : \Sigma \times S_j \rightarrow S_j$, initial state $s_{j0}$ and final state set $F_j \subseteq S_j$. The problem is satisfied if there is a word in $\Sigma^*$ which is accepted by every automaton $A_j$.

Given these automata, we will show that there is a schema $S$ and variables $u_{\text{good}}, a_m$ such that $SDD_S(u_{\text{good}}, a_m, \text{end})$ holds if and only if the intersection of the acceptance sets of all the automata $A_j$ is non-empty and that $S$ can be constructed in polynomial time, thus proving the Theorem. The schema $S$ is given in Fig. 4.

\begin{verbatim}
\forall j \ a_j := u_{\text{bad}};
\forall j \ b_j := u_{\text{bad}};
\text{while }Q_1(a_m) \ \{ \\
\forall j \ a_j := u_{\text{bad}};
\forall j < m \ b_j := u_{\text{bad}};
c := u_{\text{bad}};
\forall j \forall k \ s_{jk} := u_{\text{bad}};
\text{if }Q_2(b_m) \ \text{then } c := u_{\text{good}};
\text{else } \{ \\
\forall s \in F_1 \cup \ldots \cup F_m \ s := u_{\text{good}};
\text{while }Q_3(s_{10}, \ldots, s_{m0}) \ T_n \\
\} \\
\text{if }p_1(s_{10}) \ \text{then } a_1 := b_m;
\text{else } b_1 := c;
\text{if }p_2(s_{20}) \ \text{then } a_2 := a_1;
\text{else } b_2 := b_1;
\vdots \\
\vdots \\
\text{if }p_m(s_{m0}) \ \text{then } a_m := a_{m-1};
\text{else } b_m := b_{m-1};
\}
\end{verbatim}

Fig. 4. The schema $S$ used in the proof of Part (1) of Theorem 7. The schema $T_n$ is defined in Fig. 5.
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Fig. 5. The recursive definition of the schema $T_1$. Here $s$ is a vector whose entries are all the variables $s_{jk}$, in any fixed order, and $U_{j}$ is the schema in Fig. 3.

$\begin{align*}
  T_1 & \equiv \quad \text{if } q_i(s) \quad \text{then } V_i \\
  T_1 & \equiv V_i \\
  \text{where } \quad V_i & \equiv U_{1i} \ldots U_{mi}
\end{align*}$

Fig. 5. The recursive definition of the schema $T_1$. Here $s$ is a vector whose entries are all the variables $s_{jk}$, in any fixed order, and $U_{j}$ is the schema in Fig. 3.

$\begin{align*}
  T_1 & \equiv \quad \text{if } q_i(s) \quad \text{then } V_i \\
  T_1 & \equiv V_i \\
  \text{where } \quad V_i & \equiv U_{1i} \ldots U_{mi}
\end{align*}$

We now prove that $M$ is the next symbol through which $\pi$ can write the two occurrences of $q$. Suppose that this is impossible; that is, that there is a repeated $q_i$-predicate term along $\mu$ for some $q_i$, which $i$ would have to map to both $T$ and $F$. Thus we can write $\mu = \mu' q \mu'' q \mu'''$ such that every variable $s_{jk}$ defines the same value at the two occurrences of $q$. Assume that $Q_3$ occurs $z'$ times in $\mu'$ and $z''$ times in $\mu''$; clearly $z'' \geq 1$. Since no variable apart from the variables $s_{jk}$ occurs in the while schema guarded by $Q_3$, every variable $s_{j0}$ defines the same value after the path $\mu' q \mu''$ as after $\mu$, namely $u_{good}$. Thus by Part (2) of Lemma 6, the word $\alpha_{d_1} \alpha_{d_2} \ldots \alpha_{d_{z''}} \alpha_{d_{z''+1}} \alpha_{d_{z''+1}} \ldots \alpha_{d_1}$ is accepted by every automaton $A_j$, contradicting the minimality of $z$. Thus the definition of $i$ on $Q_3$ given above ensures that $\pi_S(i, e)$ follows the path $\mu p_1$ where required, provided that $i$ can defined appropriately on each predicate $q_i$.

Suppose that this is impossible; that is, that there is a repeated $q_i$-predicate term along $\mu$ for some $q_i$, which $i$ would have to map to both $T$ and $F$. Thus we can write $\mu = \mu' q \mu'' q \mu'''$ such that every variable $s_{jk}$ defines the same value at the two occurrences of $q$. Assume that $Q_3$ occurs $z'$ times in $\mu'$ and $z''$ times in $\mu''$; clearly $z'' \geq 1$. Since no variable apart from the variables $s_{jk}$ occurs in the while schema guarded by $Q_3$, every variable $s_{j0}$ defines the same value after the path $\mu' q \mu''$ as after $\mu$, namely $u_{good}$. Thus by Part (2) of Lemma 6, the word $\alpha_{d_1} \alpha_{d_2} \ldots \alpha_{d_{z''}} \alpha_{d_{z''+1}} \alpha_{d_{z''+1}} \ldots \alpha_{d_1}$ is accepted by every automaton $A_j$, contradicting the minimality of $z$. Thus we have shown that the interpretation $i$ can be defined so that $\pi_S(i, e)$ always follows the path $\mu$ whenever $\text{while} (Q_3(s_{10}, \ldots, s_{m0})) T_n$ is reached, and furthermore, every variable $s_{j0}$ defines the value $u_{good}$ at the end of $\mu$, and so $p_1$ is the next symbol though which $\pi_S(i, e)$ passes.

We now prove that $M[S]_i^1(a_m) = u_{good}$ holds. The definition of $i$ on $Q_1$ ensures that $\pi_S(i, e)$ passes at least once through the body of $Q_1$, and since $i$ maps
Q₂(bₘ) to T and each pⱼ(u_bad) to F, on the first passing of πₜ(i, e) through the body of Q₁, the assignment c := u_good; and all assignments to every bⱼ occur, and hence bₘ defines the value u_good when Q₁ is reached for the second time along πₜ(i, e). Since i maps Q₁(u_bad) to T, the path πₜ(i, e) then enters the body of Q₁ a second time, and since i maps Q₂(u_good) to F, this time πₜ(i, e) passes through Q₃. As proved above, πₜ(i, e) terminates within while (Q₃(s₁₀, . . . , sₙ₀)) Tₙ and every s_j₀ defines u_good when πₜ(i, e) then reaches p₁, and so πₜ(i, e) then passes through all the assignments a₁ := bₘ; and a_j := a_j₋₁; after which aₘ defines the value u_good. Since i maps Q₂(aₘ) to F, SDD_S(u_good, aₘ, end) holds, as required.

---(⇒). Conversely, suppose that SDD_S(u_good, aₘ, end) holds. Thus M[S]ₜₐₘ(uₘ) = u_good for some interpretation i holds. The only sequence of assignments which could copy u_good at the start of S to aₘ at the end consists, in order, of the assignment c := u_good; and those referencing every bⱼ for j < m followed by those referencing bₘ and every aⱼ for j < m, and so πₜ(i, e) must pass through all of these in turn. Furthermore, owing to the assignments setting c and b₁, . . . , bₘ₋₁ to u_bad, the assignments referencing c and every bⱼ for j < m must occur in a single passing through the body of Q₁, during which every s_j₀ defines u_bad when pⱼ is reached. Thus i must map every pⱼ(u_bad) to F. Similarly, owing to the assignments aⱼ := u_bad; the assignments referencing every aⱼ for j < m must also occur in a single passing through the body of Q₁, and so the predicate term defined by each pⱼ(s_j₀) must map to T, and so every s_j₀ must define a value distinct from u_bad simultaneously. The only possibility is u_good, and so at some point the path πₜ(i, e), must reach p₁ with each s_j₀ defining u_good, and thus must have passed through Q₁ since the last occurrence of Q₂. Let V_d₁ . . . V_dₜ be the sequence of schemas V_k occurring on πₜ(i, e) since this occurrence; then by Part (2) of Lemma 6, the word α_d₁α_d₋₁α_d₁ is accepted by every automaton A_j, as required.

To prove PSPACE-hardness of deciding the ∀DD_S relation, observe that the final value of the variable aₘ always lies in {u_good, u_bad} and so SDD_S(u_good, aₘ, end) ⇐ ⇒ ¬∀DD_S(u_bad, aₘ, end) holds. Thus deciding ∀DD_S(f, v, end) is co-PSPACE-hard and hence PSPACE-hard.

Thus we have proved Part (1) of the Theorem, and we now prove Part (2). To show NP-hardness of deciding whether SDD_S(f, v, l) holds, we use a polynomial-time reduction from 3SAT, which is known to be an NP-hard problem [Cook 1971]. An instance of 3SAT comprises a set X = {x₁, . . . , xₙ} and a propositional formula ρ = ∪ₖ=₁ⁿ yₖ₁ ∨ yₖ₂ ∨ yₖ₃, where each yⱼ is either xⱼ or ¬xⱼ for some k ≤ n. The problem is satisfied if there exists a valuation δ : X ∪ ¬X → {T, F} such that for each x ∈ X, {δ(x), δ(¬x)} = {T, F}, under which ρ evaluates to T. Given this instance of 3SAT we will construct a schema S which has no loops and contains variables u_bad, u₀, . . . , uₘ such that SDD_S(u₀, uₙ, end) holds if and only if ρ is satisfiable. The schema S is

∀j > 0 u_j := u_bad; D₁ . . . D_m,

where D₁ is as defined in Figure 6. Clearly S can be constructed in polynomial time from the given instance of 3SAT, as required.

Assume first that there exists a valuation δ : X → {T, F} under which ρ evaluates
Fig. 6. The definition of the schema $D_l$ used in the proof of Part (2) of Theorem 7.

to $T$. Define the interpretation $i$ to map $q_j()$ to $\delta(x_j)$ for each $q_j$. Then the path $\pi_S(i,e)$ clearly passes through at least one assignment $u_l := u_{l-1}$; within each $D_l$ in $S$, proving $SDD_S(u_0, u_n, \text{end})$ holds. Conversely, if $SDD_S(u_0, u_n, \text{end})$ holds, then there is an interpretation $i$ such that the path $\pi_S(i,e)$ passes through the sequence of assignments $u_1 := u_0$, $\ldots$, $u_n := u_{n-1}$; in turn, and hence passes through $u_l := u_{l-1}$; at least once within each $D_l$. Define the valuation $\delta$ as follows; $\delta(x_j) = T$ if and only if $i$ maps $q_j()$ to $T$. Clearly $\rho$ evaluates to $T$. Thus we have proved Part (2) of the Theorem for $SDD_S$.

To prove co-NP-hardness of deciding the $\forall DD_S$ relation under the restricted conditions in Part (2), observe that the final value of the variable $u_n$ always lies in $\{u_0, u_{bad}\}$ and so $SDD_S(u_0, u_n, \text{end}) \iff \forall DD_S(u_{bad}, u_n, \text{end})$ holds. Thus deciding $\forall DD_S(f, v, \text{end})$ is co-NP-hard.

In order to prove that our problems lie in PSPACE, we need to show that the successors of a vertex in Flowchart$(S)$ can be enumerated in polynomial time. This motivates Theorem 8.

THEOREM 8. Let $S$ be a schema.

(1) The vertices of Flowchart$(S)$ can be encoded as words in the alphabet Labels$(S) \cup \{\text{start}, \text{end}\}$ in which no element of Labels$(S)$ occurs more than once and start and end each occur not more than $|\text{Labels}(S)|$ times.

(2) Given any $l' \in \text{Vertices}(\text{Flowchart}(S))$, the set of all $l'' \in \text{Vertices}(\text{Flowchart}(S))$ for which $(l',l'')$ is an edge in Flowchart$(S)$, and the corresponding values of edgeType$(l',l'')$, can be computed in polynomial time.

PROOF. (1) We indicate the encoding by assuming that $S$ has the form $S = l : S_1||S_2||\ldots||S_m$; the encoding in the case of the other constructions given in Definition 2 is straightforward to infer. In the concurrent case, Flowchart$(S)$ has vertex set $\times_{i=1}^m (\text{Vertices}(\text{Flowchart}(S_i)) \cup \{\text{start}, l, \text{end}\}$ and a vertex of Flowchart$(S)$ can be encoded either by an element of $\{\text{start}, l, \text{end}\}$ (representing themselves) or by a word $w = w_1 \ldots w_m$, where each $w_i$ represents an element $l_i \in \text{Vertices}(\text{Flowchart}(S_i))$ and $w$ represents $(l_1, \ldots, l_m)$. The conditions given on the frequency of letters in $w$ follow easily from those for each $w_i$ and the fact we assume that no label occurs more than once in $S$.

(2) This follows easily by induction on the structure of $S$, using the encoding given in Part (1) of this Theorem.
Our other main theorem of this Section follows.

**Theorem 9.** Let $S$ be a schema and let $v \in \mathcal{V}$, let $l$ be a vertex of Flowchart$(S)$ and let $f \in \mathcal{V}$. Assume that all assignments in $S$ are equality assignments. Assume that there is a constant upper bound on the arity of any predicate symbol occurring in $S$. Then the problems of deciding whether $SDD_S(f, v, l)$ or $\forall DD_S(f, v, l)$ hold both lie in PSPACE.

**Proof.** We first prove decidability of $SDD_S(f, v, l)$ in PSPACE. We do this by constructing the following algorithm, which lies in NPSPACE. We non-deterministically guess a path beginning at start through the schema $S$ that realises the copying of the initial value of the variable $f$ onto $v$ at the vertex $l$. At each point in the algorithm we store not just the vertex and the state (with the domain restricted to the set of variables referenced in $S$) reached, but also a finite, initially empty set of equations of the form $p(y) = X$ for predicate $p$ occurring in $S$, variable vector $y$ whose components are referenced in $S$ and $X \in \{T, F\}$. If $n$ is an upper bound on the total number of predicates and variables occurring in $S$ and $b$ is the assumed constant upper bound on the arity of any predicate in the class of schemas under consideration, then the number of equations of this form is bounded by $2^n b + 1$ and thus the data stored at any point in the execution of the algorithm is polynomially bounded.

Whenever the algorithm crosses an edge $(l', l'')$ in Flowchart$(S)$ satisfying $\text{edgeType}_S(l', l'') = (q, x, X)$, the equation $q(y) = X$ is added to the set, where the vector $y = M[\mu] x$, with $\mu$ being the path traced by the algorithm up to the vertex $l'$. No equation is added to the set when an edge for which $\text{edgeType}_S$ returns $\epsilon$ or an assignment is crossed. Thus this equation set encodes the set of interpretations which are compatible with the path followed, in the sense that an interpretation $i$ is compatible with this path if and only if $p(y) = X$ is a consequence of $i$ for all equations $p(y) = X$ in the set.

The algorithm terminates and returns $false$ if the equation set acquires a pair of contradictory equations (that is, a pair $p(w) = T$, $p(w) = F$) at any point. It terminates and returns $true$ if $l$ is reached with the state mapping $v$ to $f$ without two contradictory equations having occurred in the set. By Theorem 8, this algorithm lies in NPSPACE. Since $PSPACE = NPSPACE$ holds, the problem of deciding $SDD_S(f, v, l)$ is thus in PSPACE.

To prove decidability of $\forall DD_S(f, v, l)$ in co-NPSPACE = PSPACE, we modify the algorithm as follows; termination with output $true$ occurs if $l$ is reached with the state not mapping $v$ to $f$.

4. **COMPLEXITY RESULTS FOR FREE SCHEMAS**

If we allow assignments with function symbols, and not just equalities, to occur in schemas, then deciding data dependence becomes harder, and the proof of membership in PSPACE for both problems in Theorem 9 does not appear to generalise. However, M"uller-Olm's result [M"uller-Olm and Seidl 2001] for non-deterministic programs can be used to prove, in Theorem 15 that deciding existential data dependence is PSPACE-complete for arbitrary free schemas. Additionally, we prove in...
Theorem 21 that if a constant bound is placed on the number of subschemas occurring in parallel in the class of schemas considered, then this problem becomes polynomial-time decidable.

**Definition 10.** A schema $S$ is free if every path through $Flowchart(S)$ starting at $start$ is executable.

**Lemma 11.** Given any schema $S$ without predicates, a variable $v$ and $f \in V \cup F$, the problem of deciding whether $SDD_S(f, v, \texttt{end})$ holds is PSPACE-hard.

**Proof.** This is [Miller-Olm and Seidl 2001, Theorem 2].

**Lemma 12.** Given any free schema $S$, a vertex $l$ in $Flowchart(S)$, a variable $v$ and $f \in V \cup F$, with $l$, $v$ and $f$ all occurring in $S$, there exists a free schema $S'$ which does not contain any $\texttt{loop}$ or $\sqcap$ constructions, and such that $SDD_S(f, v, l)$ holds if and only if $SDD_{S'}(f, v, l)$ does. Furthermore, $S'$ can be constructed in polynomial time from $S$.

**Proof.** Given $S$, we replace $\texttt{loop}$ or $\sqcap$ constructions with while and nested if statements respectively, in the following way. Let $z$ be a variable not occurring in $S$ and not equal to $v$ or $f$, let $h$ be any function symbol and let $q$ be any predicate symbol. Suppose that $m : \texttt{loop} T$ occurs in $S$; then we replace it by $m' : z := h(z); m : \texttt{while} q(z) \texttt{do} \{m'' : z := h(z); T\}$, for new labels $m', m''$. Similarly, an occurrence of $m : T_n \sqcap \ldots \sqcap T_1$ in $S$ can be replaced by the schema $m : P_n$, where we recursively define $P_1 \equiv z := h(z); T_1$ and $P_r \equiv \text{if } q(z) \text{ then } z := h(z); T_r \text{ else } z := h(z); P_{r-1}$ for $r > 1$, where we have omitted labels in the definitions of each $P_r$. Let $S'$ be the schema obtained from $S$ after all the $\texttt{loop}$ or $\sqcap$ constructions have been replaced. Since $z$ is never referenced in the original schema $S$, the new assignments to $z$ cannot interfere with the existing data dependence relations in $S$, and the length of any term defined by $z$ along a path through $S'$ must successively increase at each assignment to $z$, hence the introduction of the new while and if statements cannot cause repeated predicate terms to occur. Thus $S'$ is free if $S$ is. There is a natural correspondence between paths in $S$ and in $S'$, and thus $SDD_S(f, v, l)$ holds if and only if $SDD_{S'}(f, v, l)$ follows. Also, $S'$ can be constructed in polynomial time from $S$, proving the Lemma.

**Definition 13.** Given a schema $S$, $l, l' \in Vertices(Flowchart(S))$ and variables $v, v'$, we define the relation $(l, v) \rightarrow^S (l', v')$ to hold if either $edgeType(l, l')$ is an assignment to $v'$ that references $v$, or $v = v'$ and $edgeType(l, l')$ is not an assignment to $v'$.

**Lemma 14.** For any free schema $S$, a vertex $l$ in $Flowchart(S)$, a variable $v$ and $f \in F$, $SDD_S(f, v, l)$ holds if and only if there exist $m, n \in Vertices(Flowchart(S))$ and a variable $w$ such that $edgeType(m, n)$ is an assignment to $w$ with function symbol $f$ and $(n, w) \rightarrow^* (l, v)$ holds.

**Proof.** This follows immediately from the definition of $SDD_S(f, v, l)$.

**Theorem 15.** Given any free schema $S$, a vertex $l$ in $Flowchart(S)$, a variable $v$ and $f \in V \cup F$, the problem of deciding whether $SDD_S(f, v, l)$ holds is PSPACE-complete, and is PSPACE-hard even if $l = \texttt{end}$ and $S$ does not contain any $\texttt{loop}$ or $\sqcap$ symbols.
PROOF. The PSPACE-hardness result follows immediately from Lemmas 11 and 12.

To show membership in PSPACE, we first assume that \( f \in \mathcal{F} \), since if \( f \in \mathcal{V} \), then we can replace \( S \) by the schema \( S' \equiv f := g() \); \( S \) for a function symbol \( g \) not occurring in \( S \), for then \( SDD_S(f, v, l) \iff SDD_{S'}(g, v, l) \) holds, and \( S' \) can be constructed in polynomial time from the input. The result then follows from Lemma 14 as follows. We non-deterministically guess an edge \((m, n)\) in \( Flowchart(S) \) and a variable \( w \) such that \( \text{edgeType}(m, n) \) is an assignment to \( w \) with function symbol \( f \) and then decide whether \((n, w) \sim^*(l, v)\) holds. This can be done by guessing a path from \((n, w)\) to \((l, v)\) in the digraph whose vertices are pairs \((l', v')\) for \( l' \in \text{Vertices}(Flowchart(S)) \) and variables \( v' \) occurring in \( S \) and whose edges are given by the \( \sim \) relation. At any point in the algorithm, only the current pair \((l', v')\) is stored, rather than the entire graph. By Theorem 8, only polynomial space in the input is required for this, thus proving that the problem lies in \( \text{NPSPACE} = \text{PSPACE} \). □

We now consider the existential data dependence problem in which a constant upper bound is placed on the number of occurrences of \( \| \) in the schemas. Lemma 16 provides the crucial result in showing that in this case, the problem is polynomial-time bounded.

**Lemma 16.** Given any integer \( B \geq 0 \), let \( \chi_B \) be the set of all schemas in which \( \| \) occurs not more than \( B \) times. Then there exists a polynomial-time algorithm that when given a schema \( S \) in \( \chi_B \), constructs the graph \( Flowchart(S) \).

**Proof.**
For each \( B \geq 0 \), it suffices to prove that the set containing \( |\text{Vertices}(Flowchart(S))|\) for every schema \( S \) in \( \chi_B \) is polynomially bounded in terms of the number of letters needed to encode \( S \). Suppose that there are non-decreasing functions \( P_B : \mathbb{N} \rightarrow \mathbb{N} \) satisfying the following conditions, corresponding to the schema constructions listed in Definition 2. Then by induction on the structure of \( S \), for every schema in \( \chi_B \) encoded by a word of length \( n \), \( Flowchart(S) \) has not more than \( P_B(n) \) vertices.

1. \( P_B(n) \geq 3 \) if \( n \geq 1 \).
2. \( P_B(n_1 + n_2) \geq P_{C_1}(n_1) + P_{C_2}(n_2) \) if \( C_1 + C_2 = B \) and \( n_1, n_2 \geq 1 \).
3. \( P_B(n_1 + \ldots + n_m + 1) \geq P_{C_1}(n_1) + \ldots + P_{C_m}(n_m) + 1 \) if \( C_1 + \ldots + C_m = B \) and \( n_i \geq 1 \forall i \).
4. \( P_B(n + 1) \geq P_B(n) + 1 \) if \( n \geq 1 \).
5. \( P_B(n_1 + \ldots + n_m + m) \geq P_{C_1}(n_1) \ldots P_{C_m}(n_m) + 3 \) if \( C_1 + \ldots + C_m = B - m + 1 \) and \( B \geq m - 1 \geq 1 \) and \( n_i \geq 1 \forall i \).

Consider the functions \( Q_B : n \mapsto \max(3, n^{3(B+1)}) \) as candidates for \( P_B \). We will show that they satisfy (1–5), and hence \( Q_B(n) \) is an upper bound for the number of vertices in \( Flowchart(S) \) for any schema in \( \chi_B \) encoded by a word of length \( n \). Since \( Q_B(n) \leq n^{3(B+1)} + 3 \), the existence of the polynomial time bound required will follow.

Clearly the functions \( Q_B \) satisfy Cases (1, 4, 4'). To prove that they satisfy Case (3, 3'), observe that \( P_B(n_1 + \ldots + n_m + 1) = (n_1 + \ldots + n_m + 1)^{3(B+1)} \geq \)
\[\sum_{i \leq m} n_i^{3(B+1)} + 1 + 3(B+1) \sum_{i \leq m} n_i n_i^{3(B+1)-2},\]

where the summand in the last term is obtained from taking the 1 from one of the 3(B + 1) bracketed expressions, \(n_i\) from the first remaining one and \(n_1\) from each of the others. Since each \(P_i(n_i) \leq 3 + n_i^{3(B+1)}\), (3, \(3'\)) follows under the conditions stated. Case (2) is proved similarly, and hence it remains to prove (5). We have

\[P_B(n_1 + \ldots + n_m + m) = (n_1 + \ldots + n_m + m)^{3(B+1)} \geq (n_1 + 3)^{3(C_1+1)} \ldots (n_m + 3)^{3(C_m+1)}\]

(since \(\sum_{i \leq m} (C_i + 1) = B + 1\), each \(n_i \geq 1\) and \(m \geq 2\)

\[\geq (n_1 + 2)^{3(C_1+1)} \ldots (n_m + 2)^{3(C_m+1)} + 3 \geq Q,\]

thus proving the Lemma.

**Definition 17.** Given a set \(W\) and functions \(H_1, H_2 : W \rightarrow \{T, F\}\), we say that \(H_1\) precedes \(H_2\) if \(H_1(w) = T \Rightarrow H_2(w) = T\) holds for all \(w \in W\).

This definition of precedence is clearly a partial ordering. Observe that no chain of functions \(W \rightarrow \{T, F\}\) has more than \(|W|\) functions in it.

**Definition 18.** Let \(S\) be a schema. We define the set \(W_S\) to be the subset of \((\mathcal{V} \cup \mathcal{F}) \times \mathcal{V}\) for which both components occur in \(S\).

**Definition 19.** Let \(S\) be a schema. Then \(\text{DatDep}_S\) is the minimal function \(H\) from \(W_S \times \text{Vertices}(\text{Flowchart}(S))\) to \(\{T, F\}\) satisfying the following

1. \(H(v, v, \text{start}) = T\) for all \((v, v) \in W_S\).

2. If \(w\) is a variable, \((l, l')\) is an edge in \(\text{Flowchart}(S)\) and \(\text{edgeType}(l, l')\) is not an assignment to the variable \(w\), then \(H(f, w, l) = T \Rightarrow H(f, w, l') = T\) holds.

3. If \(x, y \in \mathcal{V}\) and \((l, l')\) is an edge in \(\text{Flowchart}(S)\) and \(\text{edgeType}(l, l')\) is an assignment to the variable \(y\) that references \(x\), then \(H(f, x, l) = T \Rightarrow H(f, y, l) = T\) holds. If in addition, the assignment \(\text{assign}_S(l, l')\) has function symbol \(h\), then \(H(h, y, l) = T\) holds.

**Theorem 20.**

Let \(S\) be a free schema and let \((f, v) \in W_S\). Let \(l \in \text{Vertices}(\text{Flowchart}(S))\). Then \(\text{SDD}_S(f, v, l) \iff \text{DatDep}_S(f, v, l)\) holds.

**Proof.** This follows immediately from Definition 19.

**Theorem 21.** Let \(B \geq 0\) and let \(S\) be a free schema in which every \(\parallel\) construction occurs not more than \(B\) times, and let \(f \in \mathcal{V} \cup \mathcal{F}\) and \(v \in \mathcal{V}\). Let \(l \in \text{Vertices}(\text{Flowchart}(S))\). Then it can be decided in polynomial time whether \(\text{SDD}_S(f, v, l)\) holds.

**Proof.** From Theorem 20 it suffices to prove that it can be decided in polynomial time whether \(\text{DatDep}_S(f, v, l)\) holds, under the restriction given on \(\parallel\) constructions. We compute \(\text{DatDep}_S(f, v, l)\) as follows, using the graph \(\text{Flowchart}(S)\). We may assume that \((f, v) \in W_S\), since otherwise \(\text{SDD}_S(f, v, l)\) can clearly be decided in polynomial time.

We approximate \(\text{DatDep}_S\) on the domain \(W_S \times \text{Vertices}(\text{Flowchart}(S))\) by a sequence of functions \(H_1, H_2, \ldots : W \rightarrow \{T, F\}\). Firstly, let \(H_1\) satisfy Condition (1) of Definition 19 for every \((v, v) \in W_S\) and let \(H_1(f, v, l) = F\) whenever \((f, v, l) \neq W_S\).
Given a function $H_i$ that does not satisfy every instance of Condition (2) or (3) of Definition 19, obtain the function $H_{i+1}$ by altering $H_i$ on one such instance, so that $H_i$ precedes $H_{i+1}$. From Lemma 16, the size of the domain $W_S \times Vertices(\text{Flowchart}(S))$ is polynomially bounded in terms of $S$, and therefore a maximal function $H_n$ is eventually reached with $n$ polynomially bounded in terms of $S$. In addition, each function $H_i$ can be encoded by listing the elements of its domain mapped to $T$, thus $H_n$ is computable in polynomial time. By induction on $i$, every $H_i$ precedes $DatDep_S$, and $H_n$ satisfies all three conditions in Definition 19, the minimality condition on $DatDep_S$ implies $H_n = DatDep_S$. □

5. CONCLUSIONS

We have extended conventional data dependency problems to arbitrary schemas and have shown that both the existential and universal data dependence problems lie in PSPACE for schemas without concurrency constructs and having only equality assignments, provided that there is a constant upper bound on the arity of any predicate symbol occurring in the schemas. We have also shown that without this upper bound, both problems are PSPACE-hard. This PSPACE-hardness result, Theorem 7, entails constructing a schema without this arity restriction; see the predicates $Q_3$ and $q_l$ in Figs. 4 and 5. This suggests that assuming this restriction may result in a lower complexity bound than PSPACE. Since schemas with predicates approximate the behaviour of real programs much more accurately than wholly non-deterministic programs which are normally used in program analysis, a reasonable class of schemas for which our two problems could be decided tractably would be of considerable interest.

In addition, we have proved that for free schemas, existential data dependence is decidable in polynomial time provided that a constant upper bound is placed on the number of occurrences of $\|$ in the schemas being considered. We have not attempted to prove an analogous result for the universal data dependence relation. This would be an interesting subject for future investigation.

As mentioned in the Introduction, many concurrent systems have only relatively few threads even if they have many lines of code, and therefore the bound on the number of occurrences of $\|$ is not particularly restrictive. The freeness hypothesis (equivalent to assuming non-deterministic branching) is common in program analysis, and its use ensures that no false positives for data dependence are computed. This is important in areas such as security where we wish to show that the value of one variable $x$, whose value is accessible, cannot depend on the value of another variable $y$ whose value should be kept secret.

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