(Q, r) Inventory policies in a fuzzy uncertain supply chain environment

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ABSTRACT

Managers have begun to recognize that effectively managing risks in their business operations plays an important role in successfully managing their inventories. Accordingly, we develop a (Q, r) model based on fuzzy-set representations of various sources of uncertainty in the supply chain. Sources of risk and uncertainty in our model include demand, lead time, supplier yield, and penalty cost. The naturally imprecise nature of these risk factors in managing inventories is represented using triangular fuzzy numbers. In addition, we introduce a human risk attitude factor to quantify the decision maker's attitude toward the risk of stocking out during the replenishment period. The total cost of the inventory system is computed using defuzzification methods built from techniques identified in the literature on fuzzy sets. Finally, we provide numerical examples to compare our fuzzy-set computations with those generated by more traditional models that assume full knowledge of the distributions of the stochastic parameters in the system.

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1. Introduction

Managers have begun to recognize that effectively managing risks in their business operations plays an important role in successfully managing their inventories. One of the most common risks is demand uncertainty, a phenomenon that is widely studied in the literature. Another risk that has attracted significant attention is supply uncertainty, especially the risk associated with direct material supplies and deliveries. This type of risk has escalated as Fortune 500 companies have sourced a great proportion of products from areas of the globe with low labor costs, such as China and India. Frequently, the hidden perils of such sourcing strategies are not fully considered (Elkins et al., 2005).

There is an abundant literature that models uncertainty in demand and/or lead time using probability distributions with known parameters. However, in many cases where there is little or no historical data available to the inventory decision maker, perhaps due to recent changes in the supply chain (SC) environment, probability distributions may simply not be available, or may not be easily or accurately estimated (Xie et al., 2006). Additionally, in some cases, it may not be possible to collect data on the random variables of interest because of certain system or time constraints. Furthermore, other critical supply chain parameters, in particular the various costs that impact the system, are often ill-defined and may vary from time to time. All of these situations raise challenges for using traditional inventory models in practice.

Fuzzy theory provides an alternate, flexible approach to handle such situations because it allows the model to easily incorporate various subject experts' advice in developing critical parameter estimates (Zimmermann, 2001). Recent research has employed fuzzy logic in modeling demand or lead time uncertainty in inventory systems. For example, Pai and Hsu (2003) apply fuzzy-set theory to continuous review reorder point problems, with the assumption that uncertainties may appear not only in demand over the lead time, but also in inventory holding costs. They obtain optimal (Q, r) policies that account for these uncertainties by computing derivatives of a Type-2 fuzzy-set of total cost. Petrovic et al. (2001) develop a fuzzy model for the newsvendor problem in an uncertain environment, where inventory cost and demand are represented by fuzzy sets. Vujosevic et al. (1996) develop an alternative EOQ formula that accounts for fuzzy inventory costs and demand parameters. This solution approach is extended in Mondal and Maiti (2002) to a multi-item fuzzy EOQ system by using a genetic algorithm. Chang (1999) derives a membership function of the total cost and EOQ in a production-inventory problem where the production quantity is a triangular fuzzy number.

In a broader setting, Wang and Shu (2005) develop a fuzzy decision methodology that provides an alternative framework to handle SC uncertainty and to determine inventory management strategies for a SC network with a parameter, $\rho (0 < \rho < 1)$, to describe the managerial (human) attitude toward risk. Also in a SC setting, Xie et al. (2006) develop a hierarchical, two-level coordination method to compute order-up-to policies in SCs under fuzzy demand. In their model, the entire SC system is decomposed into independent sub-systems optimized at the “follower level,” while...
coordination is implemented at the “leader level.” SC simulation models have also been developed by Petrovic et al. (1998, 1999) in a fuzzy uncertain environment for demand and material supply, using base-stock control policies. Finally, Yimer and Demirli (2004) describe the performance achieved by a fuzzy inventory control simulation model in a continuous review inventory system.

However, to the best knowledge of the authors, no research has simultaneously considered multiple fuzzy risk factors in (Q, r) inventory policies while also incorporating the decision maker’s risk attitude (i.e., the ρ factor introduced by Wang and Shu, 2005). In this paper, we present a (Q, r) system that accounts for the typical imprecision in modeling demand, lead time, supply yield, penalty cost, and inventory holding costs. Since fuzzy sets are based on human subjective opinion, however, the decision maker’s attitude towards risks driven by uncertainty affects the decision process and is, therefore, clearly worthy of study. We study the effects of three levels of risk attitude – pessimistic, neutral, and optimistic – extended from Wang and Shu (2005), and we provide corresponding managerial insights from this analysis.

The rest of the paper is organized as follows: In Section 2, we present several important fuzzy set-related definitions and develop important propositions of fuzzy arithmetic to be used in our (Q, r) computations. In Section 3, we present our approach to modeling SC uncertainty using fuzzy sets, and we provide our model assumptions and notations. We derive a fuzzy total cost function based on fuzzy arithmetic rules, and use a direct search algorithm and defuzzification functions. There are many defuzzification heuristics main that most appropriately describes a fuzzy-set, we need to use defuzzification functions. There are many defuzzification heuristics such as extreme value (EV), center of area (COA), and center of gravity (COG) defuzzification (Zimmermann, 2001). In this paper, we adopt COG defuzzification as it has a distinct geometrical meaning and strong probability analogy. The definition is given as follows:

**Definition 2.2.** If \( \tilde{P} = \{(x, \mu_{\tilde{P}}(x))\} \) is a Type-2 fuzzy-set if

\[
\mu_{\tilde{P}}(x) = \{(x, \mu_{\tilde{P}}(x)) | x \in X, \mu_{\tilde{P}}(x) \in [0, 1]\},
\]

where \( \mu_{\tilde{P}}(x) \) is the membership of a Type-1 fuzzy-set and \( \mu_{\tilde{P}}(x) \) is its membership function (Zimmermann, 2001).

Next, we give the definition when the element of the membership function is a fuzzy-set, but we can also define it when \( x \) is a fuzzy-set. In order to obtain a scalar from the fuzzy domain that most appropriately describes a fuzzy-set, we need to use defuzzification functions. For ease of exposition in this paper, we present some important definitions and propositions that will be used in the following sections.

**Definition 2.1.** If \( X \) is a collection of objects denoted generically by \( x \), then a fuzzy-set \( \tilde{P} \) on \( X \) is a set of order pairs

\[
\tilde{P} = \{(x, \mu_{\tilde{P}}(x)) | x \in X\},
\]

where \( \mu_{\tilde{P}}(x) \) is a membership function and \( \mu_{\tilde{P}}(x) \geq 0 \). If max \((\mu_{\tilde{P}}(x)) = 1\), \( \tilde{P} \) is called a fuzzy number (Zadeh, 1978).

If \( x \) is continuous on \( X \), then the fuzzy-set is continuous and the range of \( x \) is called the support of fuzzy-set \( \tilde{P} \). Continuous fuzzy sets can have various membership function shapes such as triangular and trapezoidal. In this paper, we use triangular fuzzy numbers, a special case of trapezoidal, due to its straightforward structure and computational simplicity. Throughout the paper, unless otherwise indicated, we use a bold capital letter with a wide tilde (e.g., \( \tilde{P} \)) to represent a generic fuzzy-set and a non-bold capital letter with a wide tilde (e.g., \( P \)) to denote a triangular fuzzy number. Fig. 1a shows a triangular fuzzy number \( \tilde{P} = (a, c, b) \). Its membership function, \( \mu_{\tilde{P}}(x) \), can be written as follows:

\[
\mu_{\tilde{P}}(x) = \begin{cases} 
0 & \text{if } x \leq a, \\
\frac{x-a}{c-a} & \text{if } a < x \leq c, \\
\frac{b-x}{b-c} & \text{if } c < x \leq b, \\
0 & \text{if } x > b. 
\end{cases}
\] (1)

The fuzzy-set in the above definition is called a Type-1 fuzzy-set, whose membership function is determined. When an exact membership function is difficult to determine, however, we need to use the concept of Type-2 fuzzy-set, which is used to handle and measure additional uncertainty. A Type-2 fuzzy-set is actually a fuzzy-set of a fuzzy-set, defined as follows:

**Definition 2.3.** COG defuzzification of \( \tilde{P} \) is defined by

\[
\text{defuzz}(\tilde{P})_{\text{COG}} = \frac{\int x\mu_{\tilde{P}}(x)\,dx}{\int \mu_{\tilde{P}}(x)\,dx}
\]

(Zimmermann, 2001).

Next, we give the definition for the \( \alpha \)-level cut of a fuzzy-set, and then use it in the definition of triangular fuzzy number arithmetic.

**Definition 2.4.** The \( \alpha \)-level cut of fuzzy-set \( \tilde{P} \) is defined as

\[
\tilde{P}_\alpha = \{x \in X | \mu_{\tilde{P}}(x) > \alpha \}, \quad \text{where } 0 \leq \alpha \leq 1
\]

(Zimmermann, 2001).

**Definition 2.5.** Let \( M \) and \( N \) be fuzzy numbers with membership functions \( \mu_{\tilde{M}}(x) \) and \( \mu_{\tilde{N}}(y) \), respectively, where \( x, y \in \mathbb{R} \). Let \( \circ \) be a binary operation, i.e., \( \circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \). The operation \( \circ \) can be extended to the fuzzy domain by the following formula:

\[
\mu_{\tilde{M} \circ \tilde{N}}(z) = \sup_{x+y=z} \min\left(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)\right)
\]

(Kaufmann and Gupta, 1988).

By applying the definition of an \( \alpha \)-level cut to (4) (see Fig. 2), we have the following exact arithmetic operations for continuous fuzzy sets:

\[
\begin{align*}
M + N &= [m_1 + m_2] + [n_1 + n_2] = [m_1 + n_1, m_2 + n_2], \\
M - N &= [m_1 - n_1, m_2 - n_2], \\
M \times N &= [m_1 \times n_1, m_2 \times n_2], \\
M \times N &= [m_1 \times n_1, m_2 \times n_2].
\end{align*}
\] (5)

For triangular fuzzy numbers, fuzzy arithmetic can be simplified. Let \( M = (a_1, c_1, b_1) \) and \( N = (a_2, c_2, b_2) \). It can be easily proved (see, e.g., Kaufmann and Gupta, 1988) that

\[
\begin{align*}
\tilde{M} + \tilde{N} &= (a_1 + a_2, c_1 + c_2, b_1 + b_2), \\
\tilde{M} - \tilde{N} &= (a_1 - a_2, c_1 - c_2, b_1 - b_2), \\
\tau\tilde{M} &= (\tau a_1, \tau c_1, \tau b_1), \quad \text{where } \tau \text{ is a crisp number.}
\end{align*}
\]

For multiplicative, inverse, and division operations, similar triplets cannot be used, since the resulting membership function is no
Proposition 2.6. Let formulas obtained by interval multiplication/division and inverse function numbers are given in the following propositions, which are ob-
tions of the product and division of two triangular fuzzy numbers can be carried out based on (5) because the arithmetic membership function of $e$ longer linear over its left and right segments (see Fig. 1b), and this could produce erroneous results, especially when multiplication and division operands are used repeatedly (Wang and Shu, 2005; Giachetti and Young, 1997). For the purpose of differentiation, however, we add a star superscript to denote a parabolic-triangular fuzzy number over support $[a, b]$ (See Fig. 1b).

By exact methods, operations among multiple triangular fuzzy numbers can be carried out based on (5) because the arithmetic has been converted to arithmetic operations on a certain membership interval from fuzzy triangular numbers. Membership functions of the product and division of two triangular fuzzy numbers are given in the following propositions, which are obtained by interval multiplication/division and inverse function formulas.

Proposition 2.6. Let $\tilde{M} = (a_1, c_1, b_1)$ and $\tilde{N} = (a_2, c_2, b_2)$. Then, the membership function of $\tilde{M} \times \tilde{N}$ is given by

$$
\mu_{\tilde{M} \times \tilde{N}}(x) = \begin{cases} 
\frac{e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8}{2} & 
\text{if } a_1 a_2 \leq x \leq c_1 c_2, \\
\frac{e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8}{4} & 
\text{if } c_1 c_2 \leq x \leq b_1 b_2.
\end{cases}
$$

(6)

where

- $e_1 = (c_1 - a_1)^{-1}$, $e_2 = -a_1 (c_1 - a_1)^{-1}$, $e_3 = -(b_1 - c_1)^{-1}$,
- $e_4 = b_1 (b_1 - c_1)^{-1}$, $e_5 = (c_2 - a_2)^{-1}$, $e_6 = -a_2 (c_2 - a_2)^{-1}$,
- $e_7 = -(b_2 - c_2)^{-1}$, $e_8 = b_2 (b_2 - c_2)^{-1}$.

Proof. The proof is straightforward from Definition 2.5. □

Fig. 1. (a) Triangular fuzzy number $\tilde{P}$. (b) Parabolic-triangular fuzzy number with parabolic membership function $\tilde{P}^*$.

Fig. 2. (a) $x$ cut of $\tilde{M}$. (b) $x$ cut of $\tilde{N}$.

Proposition 2.7. Let $\tilde{M} = (a_1, c_1, b_1)$ and $\tilde{N} = (a_2, c_2, b_2)$. Then, the membership function of $\tilde{M} \div \tilde{N}$ is given by

$$
\mu_{\tilde{M} \div \tilde{N}}(x) = \begin{cases} 
\frac{e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8}{2} & 
\text{if } a_1 / b_2 \leq x \leq c_1 / c_2, \\
\frac{e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8}{4} & 
\text{if } c_1 / c_2 \leq x \leq b_1 / a_2,
\end{cases}
$$

(7)

where

- $e_1 = (c_1 - a_1)^{-1}$, $e_2 = -a_1 (c_1 - a_1)^{-1}$, $e_3 = -(b_1 - c_1)^{-1}$,
- $e_4 = b_1 (b_1 - c_1)^{-1}$, $e_5 = (c_2 - a_2)^{-1}$, $e_6 = -a_2 (c_2 - a_2)^{-1}$,
- $e_7 = -(b_2 - c_2)^{-1}$, $e_8 = b_2 (b_2 - c_2)^{-1}$.

Proof. The proof is straightforward from Definition 2.5. □

To measure the possibility that a fuzzy-set belongs to another fuzzy set, we need to introduce the definitions of possibility and necessity measures. The definitions are given as follows:

Definition 2.8. Suppose $x$ is restricted by a fuzzy-set $\tilde{B}$ in the universe $U$. Further, suppose that the possible distribution of $x$, $\pi_x$, is taken to be equal to membership function $\mu_{\tilde{B}}$. Then, the possibility of event $x \in \tilde{P}$ is defined as

$$
\Pi(x \in \tilde{P}) = \sup_{u \in U} \min \left( \mu_{\tilde{B}}(u), \mu_{\tilde{P}}(u) \right).
$$

(8)

The dual measure of possibility, i.e., the necessity of event $x \in \tilde{P}$, is defined as

$$
N(x \in \tilde{P}) = \inf_{u \in U} \max \left( 1 - \mu_{\tilde{B}}(u), \mu_{\tilde{P}}(u) \right).
$$

(Yager, 1979; Wang and Shu, 2005).

Suppose $R$ is a crisp number, then $\Pi(\tilde{B} \leq R)$ represents the maximum likelihood of the event that $\tilde{B}$ is less than $R$, and $N(\tilde{B} \leq R)$
estimates the minimum likelihood of the event that $\mathbf{B} \leq R$ will occur. By Definition 2.8, we have
\[
\mathbb{I}(\mathbf{B} \leq R) = \mathbb{I}(x \in (-\infty, R]) = \sup_{u \in R} \left( \mu_\mathbf{B}(u) \right),
\]
\[
N(\mathbf{B} \leq R) = N(x \in (-\infty, R]) = \inf_{u \in R} \left( 1 - \mu_\mathbf{B}(u) \right),
\]
\[
\mathbb{I}(\mathbf{B} \geq R) = \mathbb{I}(x \in [R, +\infty)) = \sup_{u \in R} \left( \mu_\mathbf{B}(u) \right),
\]
\[
N(\mathbf{B} \geq R) = N(x \in [R, +\infty)) = \inf_{u \in R} \left( 1 - \mu_\mathbf{B}(u) \right).
\]

Since fuzzy estimates are based on human judgement, however, they should reflect some assessment of whether the decision maker tends toward a “looser” interpretation of the fuzzy estimate (possibility) or a “tighter” one (necessity). Accordingly, a human tend toward a “looser” interpretation of the fuzzy estimate they should reflect some assessment of whether the decision ma-

Subject to expert estimation, and it offers additional flexibility in describing these uncertainties by relaxing strict constraints. An interpretation of customer demand as denoted by the triangular fuzzy number $(a, b, c)$ could be, “Demand is about $c$, and it is possible range is from $a$ to $b$.” The parameters $(a, c, b)$ could be estimated by a firm’s marketing professionals through their knowledge and experience. Similarly, unreliable supplies (i.e., material delivery shortages) and/or delivery delays due to disruptions at the supplier can also be represented by fuzzy sets, perhaps parameterized by discussions with a firm’s procurement specialists. In this study, the domain of supplier risk pertains to the percentage of ordered quantities rejected by a buyer due to quality issues, as well as variation in delivery lead time due to operation disruptions. Using fuzzy sets, supplier risk evaluation represents a judgement by management of the supplier’s reliability. For example, a supplier might provide an average shortage in deliveries of about 2% of each order and/or it might deliver consistently between 0 and 2 days later than expected. In addition, other parameters such as the holding cost rate, stock-out penalty cost, ordering cost, and unit cost, if imprecise, can also be modeled by fuzzy sets.

In addition to the assumptions of a stochastic, one-supplier, one-retailer $(Q, r)$ system in Nahmias (1997), Table 1 lists the parameters and decision variables used in our model. Our goal is to find reasonable ways to approximate each of these parameters and compute a $(Q, r)$ policy that minimizes the average annual total cost, using whatever fuzzy-set theory and tools may assist in that process. Fig. 3 shows a sample path view of how some of these

### Table 1

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Unit of measure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Units</td>
<td>Reorder point</td>
</tr>
<tr>
<td>$Q$</td>
<td>Units</td>
<td>Reorder quantity</td>
</tr>
<tr>
<td>$D$</td>
<td>Units/time</td>
<td>Fuzzy demand of product, $(d_1, d_2, d_3)$</td>
</tr>
<tr>
<td>$k$</td>
<td>$$/order$</td>
<td>Fixed reorder cost</td>
</tr>
<tr>
<td>$h$</td>
<td>$$/unit/time$</td>
<td>Holding cost rate</td>
</tr>
<tr>
<td>$T$</td>
<td>Time</td>
<td>Inventory cycle length</td>
</tr>
<tr>
<td>$u_1$</td>
<td>$$/unit$</td>
<td>Unit purchasing cost</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$$/unit$</td>
<td>Extra handling cost for returned item per unit</td>
</tr>
<tr>
<td>$p$</td>
<td>$$/unit$</td>
<td>Back-order penalty cost, $(p_1, p_2, p_3)$</td>
</tr>
<tr>
<td>$s$</td>
<td>Percentage</td>
<td>Supplier yield (% of order accepted), $(s_1, s_2, s_3)$</td>
</tr>
<tr>
<td>$L$</td>
<td>Time</td>
<td>Supply lead time, $(l_1, l_2, l_3)$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Dimensionless</td>
<td>Extent of managerial optimism ($\rho &gt; 1$) or pessimism ($\rho &lt; 0$) toward maintaining positive stock</td>
</tr>
<tr>
<td>$F^*$</td>
<td>$$/Q$</td>
<td>Fuzzy set of total annual cost</td>
</tr>
</tbody>
</table>

Fig. 3. Cycle inventory by $(Q, r)$ policy at retailer site.
fuzzy parameters interact as inventory levels change over time. Note that \( D \) and \( L \) are fuzzy numbers, and therefore, the inventory cycle, \( T \), must also be a fuzzy number, \( T^* \). Moreover, \( T^* \) also depends on the values of \( Q \) and \( r \), whose optimal values depend on \( P \) and \( S \), as well as \( D \) and \( L \).

### 3.2. Fuzzy stock-out sets with human risk attitude

In this section, we develop a method to determine the fuzzy stock-out set based on possibility and necessity measures, and also taking into account the managerial attitude toward stock-out risk. From the definitions we present earlier, demand over the lead time, \( \tilde{D}_L \), is given by

\[
\tilde{D}_L = D \times \tilde{L} = (d_1 l_1, d_2 l_2, d_3 l_3)^*,
\]

whose membership function can be computed according to Proposition 2.6.

The possibility and necessity measures that a parabolic-triangular fuzzy number is greater than and less than a certain crisp number are depicted in Fig. 4. For a certain level of reorder point \( r \), by (10) and (11), we have the maximum and minimum estimations that the demand during the lead time is less than \( r \) (see Fig. 4a). Eq. (10) reflects an optimistic managerial attitude towards demand during lead time (i.e., expecting the lowest possible demand – or most optimistic that stock will cover demand over the lead time) while (11) reflects a pessimistic attitude towards demand during replenishment period (i.e., expecting the highest possible demand – or least optimistic that stock will cover demand over the lead time). By Eq. (14), we can weight both measures with a human risk attitude factor denoted by \( \rho \) (0 < \( \rho \) < 1). When the risk attitude tends towards optimism, \( \rho \) is larger, resulting in a larger reorder quantity. At \( \rho = 0.5 \), risk attitude towards the stock-out risk over the lead time is neutral. Given \( \mathcal{A} \) in Eq. (15) and \( r \), the number of units by which demand over the lead time exceeds available stock (i.e., the size of the stock-out) is given by

\[
\begin{align*}
  n &= 0 & \text{if } r \geq d_1 l_1, \\
  n &\in \{1, 2, \ldots, d_3 l_3 - r\} & \text{if } r < d_1 l_1.
\end{align*}
\]

Thus, the membership function of the discrete, fuzzy stock-out set \( \mathcal{A} \) is

\[
\mu_{\mathcal{A}}(n) = \begin{cases} 
  \rho \left( \tilde{D}_L \leq r \right) + (1 - \rho)N(\tilde{D}_L \leq r) & \text{if } n = 0, \\
  \mu_{\tilde{D}_L}(n + r) & \text{if } n \in \{1, 2, \ldots, \max\{0, d_3 l_3 - r\}\}.
\end{cases}
\]

### 3.3. Fuzzy annual total cost sets

In this section, we first discuss the fuzzy total cost structure for each \( n \in \mathcal{A} \), and then compute the total cost function for a given \((Q, r)\) policy. From Fig. 3, note that the length of a complete inventory cycle is \( T = Q/D \), in general. In our fuzzy system, with uncertain delivery yield, such that an expected usable shipment is of size \( qS \), the fuzzy inventory cycle length is \( T^* = qS/D \). Thus, the total fuzzy cost for a certain \( n \in \mathcal{A} \) is

\[
\tilde{F}(r, Q, n) = \frac{k + nP + u_2Q + (u_1 - u_2)qS}{T^*} + h\left(\frac{qS}{2} + \varphi\right),
\]

where

\[
\varphi = \begin{cases} 
  \frac{\sum_{j=1}^{n-1}(qS/2 + j)u_2(j)}{\sum_{j=1}^{n-1}u_2(j)} & \text{if } n = 0, \\
  -n & \text{if } n > 0.
\end{cases}
\]

\( \varphi \) is part of inventory holding cost calculated based on the lowest cycle inventory level. If it is positive, we add the safety stock. If there is a stock-out, we exclude the portion of time associated with the out-of-stock condition (Zipkin, 2000) and we assume full backlogging of demand, i.e., that no sales are lost. Other cost components in (18) include the purchasing cost for received goods, \( u_1QS \); the handling cost for rejected items to be returned, \( u_2Q(1 - S) \); and the penalty cost \( nP \).

By Definition 2.5, the parabolic-triangular membership function of \( \tilde{F}(r, Q, n) \) can be written as

\[
\mu_{\tilde{F}}(x) = \begin{cases} 
  q_1 + q_2 + \varphi & \text{if } D_1 \leq x \leq D_2, \\
  q_3 + q_4 + \varphi & \text{if } D_2 \leq x \leq D_3,
\end{cases}
\]

where

\[
\begin{align*}
  D_1 &= 0.5\rho qS + h\varphi + d_1kQ^{-1} + p_1nQ^{-1} + u_2 + (u_1 - u_2)S)S_1^{-1}, \\
  D_2 &= 0.5\rho qS + h\varphi + d_2kQ^{-1} + p_2nQ^{-1} + u_2 + (u_1 - u_2)S)S_2^{-1}, \\
  D_3 &= 0.5\rho qS + h\varphi + d_3kQ^{-1} + p_3nQ^{-1} + u_2 + (u_1 - u_2)S)S_3^{-1}, \\
  v_1 &= (S_5 - S_1)^{-1}, \\
  v_2 &= -S_5 v_1, \\
  v_3 &= -(S_5 - S_2)^{-1}, \\
  v_4 &= -S_5 v_3, \\
  v_5 &= Q((S_5 - S_1)(u_1 - u_2) + n(p_2 - p_1))^{-1}, \\
  v_6 &= -(k + p_1)nQ^{-1} + u_2 + (u_1 - u_2)S_1v_5, \\
  v_7 &= -Q((S_5 - S_2)(u_1 - u_2) + n(p_3 - p_2))^{-1}, \\
  v_8 &= -(k + p_2)nQ^{-1} + u_2 + (u_1 - u_2)S_2v_7.
\end{align*}
\]

![Figure 4](image-url)

**Fig. 4.** (a) Possibility and necessity measures of event \( \tilde{P} \leq r_1 \). (b) Possibility and necessity measures of event \( \tilde{P} \geq r_1 \).
In short, our fuzzy approach to compute the optimal \((Q, r)\) inventory policy is summarized as follows.

**QR policy search:**

1. Calculate demand over lead time \(\tilde{\tau}\) from (15) and determine stock-out set \(\bar{A}\) by (17).
2. Calculate fuzzy total cost for each possible stock-out by (18) for a given \(r\) and \(Q\).
3. Evaluate overall total cost by \(s\)-fuzzification and union procedures by (20).
4. Use DIRECT algorithm to search for the \((Q, r)\) policy associated with minimal overall total cost in a predetermined reorder point and reorder quantity region.

From the above algorithm, we see that the computational complexity is proportional to the size of the possible stock-out sets for a certain \((Q, r)\) policy. Therefore, the efficiency of a direct search depends on the range of the predetermined sets for reorder points and reorder quantities. Thus, in some cases, the computational effort required to solve the fuzzy model might exceed the effort required to solve a traditional model that assumes a known normal distribution for demand with known parameters, especially when the range of demand over lead time is wide and the penalty cost is small (i.e., when the target service level is low). Our model should still provide a valuable and reasonably computed reference for inventory practice, however, since most inventory decision makers pursue high service levels.

Below, we give two remarks regarding the risk attitude and total cost.

**Remark 3.1.** Under a pessimistic attitude toward stock-outs over the replenishment period, average stock-outs during the replenishment period are larger than under an optimistic attitude, ceteris paribus.

**Proof.** Let decision maker 1 hold a more optimistic attitude toward stock-out over the replenishment period, \(\rho_1\), than of decision maker 2, \(\rho_2\). Therefore, \(\rho_1 > \rho_2\). Since it is always true that \(\Pi(\tilde{\tau}_1 \leq r) \geq N(\tilde{\tau}_2 \leq r)\), it follows that \(\mu_\bar{A}(0)|_{\rho_1} \geq \mu_\bar{A}(0)|_{\rho_2}\). Therefore, according to **Definition 2.3**, \(\text{defuzz}(\bar{A})|_{\rho_1} \leq \text{defuzz}(\bar{A})|_{\rho_2}\). Especially when \(r > d_0\), it can be easily shown from defuzzification of discrete sets that

\[
0 \leq \frac{\text{defuzz}(\bar{A})|_{\rho_2} - \text{defuzz}(\bar{A})|_{\rho_1}}{\text{defuzz}(\bar{A})|_{\rho_1}} \leq \frac{1}{\sum_{i=1}^{r-j} \mu(i + r)}.
\]

**Remark 3.2.** Triangular fuzzy number \(\bar{P} = (a, c, b)\) with membership function \(\mu_\bar{P}(x), x \in [a, b]\), has the same defuzzification value as triangular fuzzy-set \(\tilde{P} = \{(x, \mu_\tilde{P})\} < \alpha \leq 1\), and this value is \(\frac{a + c + b}{3}\).

**Proof.** The proof follows directly from **Definitions 2.1 and 2.3**.

From **Remark 3.1** we see that because of subjective opinion involved in estimating input parameters, different risk attitude levels will lead to different stock-out levels in the computation, which may cause the final decisions of different decision makers to diverge from one another. **Remark 3.2** provides some insight to the \(s\)-fuzzification procedure, although in some cases, weighting membership functions will not change the defuzzification of the fuzzy set; rather, it will cause a different shape of the unionized membership function. Note that due to the complexity of the total cost structure, analytical comparison of optimal policies with different human risk factors could be very complicated for our model.
Next, we would like to extend the formulation given by expression (18) from triangular fuzzy sets to two types of generic fuzzy sets: continuous and discrete. For continuous, unimodal fuzzy sets, in reality the membership function could be too complex to show in a mathematical form. Commonly encountered continuous fuzzy sets: continuous and discrete. For continuous, unimodal fuzzy sets, it is straightforward to repeat (4) in sequence. Membership functions generally can be obtained by taking inverse functions of α-cut functions over applicable intervals. For generic discrete fuzzy sets, it is straightforward to repeat (4) in sequence. If the three discrete fuzzy sets of demand, supplier yield, and penalty cost rate are given by $S = (s_i, \mu_{-}^{s_i} \mu_{+}^{s_i})$, $P = (p_j, \mu_{-}^{p_j} \mu_{+}^{p_j})$, and $D = (d_k, \mu_{-}^{d_k} \mu_{+}^{d_k})$, where $i,j,k$ are positive integers, then the membership function of discrete set $F(r, Q, n)$ is

$$F_{r}(r, Q, n) = [f_{r}^{+}, f_{r}^{-}],$$

where

$$f_{r}^{+} = \frac{d_{i}^{r}}{Q_{s_{i}}} (k + np_{j}^{r} + c_{2}Q + (c_{1} - c_{2})Q_{s_{i}^{r}}) + \mu_{s_{i}}^{r} \left(\frac{Q_{s_{i}}^{2}}{2} + \varphi\right),$$

$$f_{r}^{-} = \frac{d_{i}^{r}}{Q_{s_{i}}} (k + np_{j}^{r} + c_{2}Q + (c_{1} - c_{2})Q_{s_{i}^{r}}) + \mu_{s_{i}}^{r} \left(\frac{Q_{s_{i}}^{2}}{2} + \varphi\right).$$

Table 2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Traditional model</th>
<th>Fuzzy-set model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead time</td>
<td>$L - 0.5$ years</td>
<td>Same</td>
</tr>
<tr>
<td>Ordering cost</td>
<td>$k = 50$/order</td>
<td>Same</td>
</tr>
<tr>
<td>Supplier holding cost</td>
<td>$h = 52$/unit/year</td>
<td>Same</td>
</tr>
<tr>
<td>Supplier yield</td>
<td>$S = 0.8$</td>
<td>$S = 0.7$</td>
</tr>
<tr>
<td>Penalty cost</td>
<td>$P = 25$/unit</td>
<td>$P = 20$, $25$, $30$</td>
</tr>
<tr>
<td>Unit cost</td>
<td>$c_{1} = 10$/unit</td>
<td>Same</td>
</tr>
<tr>
<td>Short material handling cost</td>
<td>$c_{2} = 52$/unit</td>
<td>Same</td>
</tr>
</tbody>
</table>

4. Numerical example

In this section, we illustrate the use of our fuzzy $(Q, r)$ model through a simple example adapted from a popular textbook (Nahmias, 1997). The example problem (Harvey’s specialty shop: p. 287), as stated in the text, assumes normally distributed demand with known parameters. We also examine a skewed demand case represented by a comparable Gamma distribution, since in reality probability density functions corresponding to human activities are often skewed to the right (Law and Kelton, 2000). These more traditional demand assumptions, known distributions with known parameters, serve as baseline cases to our fuzzy models. Then we perform some sensitivity analysis for key parameters under different risk attitude levels.

4.1. Example problem

The values of all parameters besides demand in the traditional stochastic formulation of our example problem (Nahmias, 1997) and the corresponding fuzzy-set formulation are given in Table 2. Regarding demand, in the example problem, the normally distributed demand has a mean of 200 units and a standard deviation of 25√2 annually (see Fig. 6a). Since fuzzy-set possibility distributions are weaker representations of uncertainty than traditional probability distributions, moving from possibility to probability means a loss of information. Thus, to build a corresponding fuzzy

$$
\mu_{F(r, Q, n)}(f) = \sup_{f_{r}} \left( \mu_{r}^{s_{i}} \mu_{r}^{p_{j}} \mu_{r}^{d_{k}} \right),
$$

where

$$f_{r} = (k + np_{j} + c_{2}Q + (c_{1} - c_{2})Q_{s_{i}^{r}}) \frac{d_{i}}{Q_{s_{i}}} + \mu_{s_{i}}^{r} \left(\frac{Q_{s_{i}}^{2}}{2} + \varphi\right).$$

Similarly, using the s-fuzzification procedure and Definition 2.3, we can defuzzify the total annual cost for a given $(Q, r)$ policy associated with a certain risk attitude and search for a minimum. It is worth noting that, for non-triangular fuzzy sets, the fuzzy membership function of the total cost function could be very complicated.
distribution, we utilize the triangular approach from Law and Kelton (2000), suggested for specifying a traditional stochastic distribution in the absence of data. Using this approach, the middle value and boundaries of our triangular fuzzy-set are determined by the mode and given quantiles of the corresponding probability distribution, respectively. For our triangular fuzzy number parallel to the normal demand distribution in the Nahmias example, with mean 200 and standard deviation 25√2, the most likely value is \(d_2 = 200\), with minimal and maximal values of \(d_1 = 95\) and \(d_3 = 305\), obtained from the 0.15% and 99.85% quantiles, respectively, of the corresponding normal distribution. Similarly, for comparable right skewed demand, we use fuzzy number \(D = (95, 200, 410)\), obtained from Gamma distribution \(\Gamma(16, 13.3)\) (see Fig. 2b).

To specify the distribution of demand over the lead time for symmetric fuzzy number \(\tilde{D} = (95, 200, 305)\), since \(L = 0.5\) years (a crisp number), \(\tilde{D} \times L = (48, 100, 153)\). Then, given \(Q, r,\) and \(\rho\) (risk attitude), we can specify the fuzzy stock-out set and the fuzzy total cost set. Consider \(r = 110, Q = 140,\) and \(\rho = 0.5\), for example. By (17), the possible stock-out units are \(n \in \{0, 1, \ldots, 43\}\), and the membership function of the stock-out set \(\tilde{A}\) is

\[
\mu_{\tilde{A}}(n) = \begin{cases} 
0.59, & \text{if } n = 0, \\
-0.0189(120 + n) + 2.8868 & \text{if } n \in \{1, 2, \ldots, 43\}.
\end{cases}
\]

By (18) and (19), we can compute the fuzzy total cost set for each stock-out possibility and unify them. Fig. 7a illustrates 44 parabolic-triangular fuzzy numbers of total cost for the 44 possible stock-out cases, and Fig. 7b and c illustrate the \(s\)-defuzzification procedure on the aggregated total cost fuzzy-set.

In Table 3, the row labeled “Fuzzy” lists optimal policies for the three risk attitude levels, found using the global DIRECT algorithm, for the symmetric-demand fuzzy scenario for our example problem. The row labeled “Traditional” displays the optimal results for the normal baseline case. The result shows that the total cost obtained from the symmetric-demand fuzzy scenario is about 20% higher than that from its baseline case. Similarly, optimal \((Q, r)\) policies for the skewed fuzzy demand and its corresponding baseline Gamma scenario are listed in Table 4. The result indicates that the total cost obtained from fuzzy sets is around 40% higher than its baseline case. For symmetric-demand, optimal \((Q, r)\) policies from fuzzy scenarios are closer to the stochastic cases than for skewed demand. We should be careful, however, to put these results in the proper perspective. The larger cost outcomes for the fuzzy-number models are consistent with intuition: More “fuzziness” means less information about system parameters and outcomes, which in turn means higher cost from less well-matched inventory and demand levels. The important conclusion we should draw from our examples, though, is that our fuzzy method provides a means of actually quantifying the cost of not knowing the

### Table 3

<table>
<thead>
<tr>
<th>Model</th>
<th>Demand</th>
<th>Risk attitude</th>
<th>(\tilde{r})</th>
<th>(\tilde{Q})</th>
<th>(\tilde{f})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuzzy</td>
<td>(D = (95, 200, 305))</td>
<td>(\rho = 0)</td>
<td>141</td>
<td>139</td>
<td>2.6378e+03</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\rho = 0.5)</td>
<td>140</td>
<td>142</td>
<td>2.6374e+03</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(\rho = 1)</td>
<td>140</td>
<td>139</td>
<td>2.6368e+03</td>
</tr>
<tr>
<td>Traditional</td>
<td>(D \sim N(\mu = 200, \sigma = 25\sqrt{2}))</td>
<td>-</td>
<td>143</td>
<td>138</td>
<td>2.4067e+03</td>
</tr>
</tbody>
</table>

**Fig. 7.** Total cost structure when \(r = 110, q = 140,\) and \(\rho = 0.5\).
underlying demand distribution and parameters. Indeed, this could serve as an effective means of quantifying the value of an information system that could accurately estimate demand distributions and their parameters, in addition to providing accurate estimates of various cost parameters defining the inventory system. Finally, we also observe that optimal policies and total cost do not appear to vary much for different risk attitude levels, a phenomenon we explore further in the next section.

4.2. Sensitivity analysis

In this section, we use the same example problem introduced in Section 4.1 to conduct some sensitivity analysis on two key inputs to the fuzzy-set model: the demand parameters and the penalty cost. In addition, we study how the stock-out risk factor affects the optimal \((Q, r)\) policies.

Fig. 8 shows plots of the optimal reorder points, reorder quantities, and total costs under optimistic, neutral, and pessimistic risk attitude levels with \(P = 10/\theta + (20, 25, 30)\). From the plots, as expected, we observe that as \(P\) increases, reorder points and total annual costs increase as well. Interestingly, when \(\theta < 0\), the reorder quantity, reorder point, and total annual cost follow similar but non-synchronized patterns for the three types of risk attitude, from optimistic to neutral to pessimistic. This pattern can be explained in two ways. On one hand, a low penalty cost leads to correspondingly less safety stock buffer against uncertainty in demand over the lead time, but higher average cycle stock instead. On the other hand, the optimistic attitude demonstrates the most pronounced shift from cycle stock to safety stock to the increasing trend (\(\theta = 1\) and \(\theta = 0.5\)) due to its risk aversion, attempting to buffer most directly against the uncertainty during the lead time. Under the pessimistic case the opposite situation occurs. Also, for penalty cost levels beyond a certain point (\(\theta = 0\)), we see that the difference in inventory policies among different risk attitudes is trivial.

Fig. 9 shows plots of the optimal reorder points, reorder quantities, and total costs under optimistic, neutral, and pessimistic risk attitude levels with the following triangular fuzzy numbers:

\[
D = \begin{cases} 
(95, 200, 410) & \text{if } \xi > 0, \\
(95, 16, 20) & \text{if } \xi < 0.
\end{cases}
\]

With every unit change of \(\xi\), the skewness of the fuzzy demand set is increased by 0.03\%, as measured by the fuzziness function introduced in Yager (1979). When \(\xi > 0\), the demand set is more skewed to the right with average demand increasing. While for \(\xi < 0\), the situation is opposite. Fig. 9 shows very strong positive correlation between demand and reorder point, total cost, reorder quantity respectively when \(\xi > 0\). When the demand set moves towards symmetry (i.e., decreasing \(\xi\)), we observe that reorder points are not much different, but total costs and reorder quantities decrease steadily. The phenomena can be explained by high stock-out

![Fig. 8](image-url)

**Fig. 8.** Optimal policies with variation of penalty rate for three typical risk attitude.
penalty cost, which we believe also accounts for the close results among the three different risk attitude levels from Fig. 8.

5. Conclusion and remarks

In managing the supply chain, it may not be possible to provide precise estimates of all of the critical parameters that influence inventory decision making. Such parameters include the distribution of demand, the distribution of lead time, uncertainty in supplies, and difficult to estimate costs like the penalty cost for supply shortfalls. In this paper, we develop an approach to computing \((Q, r)\) inventory policy parameters based on fuzzy sets that account for imprecise estimates in key parameters, and for managerial risk attitudes. We present a method to optimize this model, present analytical results related to it, and demonstrate the model’s use via numerical examples and sensitivity analysis.

An important advantage of our fuzzy \((Q, r)\) model is that it can be used for virtually any empirical demand or lead time distribution, whereas non-normal demand or lead time distributions could significantly complicate the computation of policy parameters in a traditional, stochastic model of a \((Q, r)\) system. In addition, since fuzzy modeling involves a manager’s – or some functional experts’ – subjective estimates of key parameters, accounting for attitudes toward stock-out risk is important, and fuzzy logic provides a flexible and straightforward means of doing this.

We are currently utilizing our model and the computational approaches developed in this paper in a number of ways. First, we are adding a service level constraint as an alternative to an explicit penalty cost. Also, we are evaluating \((Q, r)\) policies in multi-level supply chains under a fuzzy risk environment to improve supply chain performance through various coordination mechanisms. Finally, we are comparing the performance of the fuzzy model and the traditional stochastic model in situations where some underlying data is available regarding demand and lead time, but where this data does not clearly fit any of the stochastic probability distributions that are traditionally used in inventory analysis.

Acknowledgement

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References


Fig. 9. Optimal policies with variation of demand for three typical risk attitude.