On obtaining minimal variability
OWA operator weights

Robert Fullér
Department of Operations Research
Eötvös Loránd University
Pázmány Péter sétány 1C,
P. O. Box 120, H-1518 Budapest, Hungary
e-mail: rfuller@cs.elte.hu
and
Institute for Advanced Management Systems Research
Åbo Akademi University,
Lemminkäinengatan 14B, Åbo, Finland
e-mail: rfuller@mail.abo.fi

Péter Majlender
Turku Centre for Computer Science
Institute for Advanced Management Systems Research
Åbo Akademi University,
Lemminkäinengatan 14B, Åbo, Finland
e-mail: peter.majlender@mail.abo.fi
Abstract

One important issue in the theory of Ordered Weighted Averaging (OWA) operators is the determination of the associated weights. One of the first approaches, suggested by O’Hagan, determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of orness; algorithmically it is based on the solution of a constrained optimization problem. Another consideration that may be of interest to a decision maker involves the variability associated with a weighting vector. In particular, a decision maker may desire low variability associated with a chosen weighting vector. In this paper, using the Kuhn-Tucker second-order sufficiency conditions for optimality, we shall analytically derive the minimal variability weighting vector for any level of orness.

Keywords: Multiple criteria analysis, Fuzzy sets, OWA operator, Lagrange multiplier
1 Introduction


Definition 1.1. An OWA operator of dimension $n$ is a mapping $F: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector $W = (w_1, \ldots, w_n)^T$ of having the properties $w_1 + \cdots + w_n = 1$, $0 \leq w_i \leq 1$, $i = 1, \ldots, n$, and such that

$$F(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i b_i,$$

where $b_j$ is the $j$th largest element of the collection of the aggregated objects $\{a_1, \ldots, a_n\}$.

A fundamental aspect of this operator is the re-ordering step, in particular an aggregate $a_i$ is not associated with a particular weight $w_i$ but rather a weight is associated with a particular ordered position of aggregate. When we view the OWA weights as a column vector we shall find it convenient to refer to the weights with the low indices as weights at the top and those with the higher indices with weights at the bottom. It is noted that different OWA operators are distinguished by their weighting function.

In [6], Yager introduced a measure of orness associated with the weighting vector $W$ of an OWA operator, defined as

$$\text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i,$$

and it characterizes the degree to which the aggregation is like an or operation. It is clear that $\text{orness}(W) \in [0, 1]$ holds for any weighting vector.

It is clear that the actual type of aggregation performed by an OWA operator depends upon the form of the weighting vector [9]. A number of approaches have been suggested for obtaining the associated weights, i.e., quantifier guided aggregation [6, 7], exponential smoothing [3] and learning [10].

Another approach, suggested by O'Hagan [5], determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of orness. This approach is based on the solution of the following mathematical programming problem:

$$\begin{align*}
\text{maximize} \quad & \text{disp}(W) = -\sum_{i=1}^{n} w_i \ln w_i \\
\text{subject to} \quad & \text{orness}(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leq \alpha \leq 1 \\
& w_1 + \cdots + w_n = 1, \quad 0 \leq w_i, \ i = 1, \ldots, n.
\end{align*}$$

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In [4] we transferred problem (1) to a polinomial equation which is then was solved to determine the maximal entropy OWA weights.

Another interesting question is to determine the minimal variability weighting vector under given level of orness [8]. We shall measure the variance of a given weighting vector as

$$D^2(W) = \sum_{i=1}^{n} \frac{1}{n} \cdot (w_i - E(W))^2 = \frac{1}{n} \sum_{i=1}^{n} w_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} w_i \right)^2 = \frac{1}{n} \sum_{i=1}^{n} w_i^2 - \frac{1}{n^2},$$

where $E(W) = (w_1 + \cdots + w_n)/n$ stands for the arithmetic mean of weights.

To obtain minimal variability OWA weights under given level of orness we need to solve the following constrained mathematical programming problem

$$\text{minimize} \quad D^2(W) = \frac{1}{n} \sum_{i=1}^{n} w_i^2 - \frac{1}{n^2}$$

subject to $\text{orness}(w) = \sum_{i=1}^{n} \frac{n - i}{n - 1} \cdot w_i = \alpha, \; 0 \leq \alpha \leq 1,$

$$w_1 + \cdots + w_n = 1, \; 0 \leq w_i, \; i = 1, \ldots, n.$$

Using the Kuhn-Tucker second-order sufficiency conditions for optimality, we shall solve problem (2) analytically and derive the exact minimal variability OWA weights for any level of orness.

2 Obtaining minimal variability OWA weights

Let us consider the constrained optimization problem (2). First we note that if $n = 2$ then from $\text{orness}(w_1, w_2) = \alpha$ the optimal weights are uniquely defined as $w_1^* = \alpha$ and $w_2^* = 1 - \alpha$. Furthermore, if $\alpha = 0$ or $\alpha = 1$ then the associated weighting vectors are uniquely defined as $(0, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)^T$, respectively, with variability

$$D^2(1, 0, \ldots, 0, 0) = D^2(0, 0, \ldots, 0, 1) = \frac{1}{n} - \frac{1}{n^2}.$$

Suppose now that $n \geq 3$ and $0 < \alpha < 1$. Let us

$$L(W, \lambda_1, \lambda_2) = \frac{1}{n} \sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} + \lambda_1 \left( \sum_{i=1}^{n} \frac{n - i}{n - 1} w_i - \alpha \right) + \lambda_2 \left( \sum_{i=1}^{n} w_i - 1 \right).$$
denote the Lagrange function of constrained optimization problem (2), where \( \lambda_1 \) and \( \lambda_2 \) are real numbers. Then the partial derivatives of \( L \) are computed as
\[
\frac{\partial L}{\partial w_j} = \frac{2w_j}{n} + \lambda_1 + \frac{n - j}{n - 1} \cdot \lambda_2 = 0, \quad 1 \leq j \leq n, \quad (3)
\]
\[
\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{n} w_i - 1 = 0,
\]
\[
\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{n - i}{n - 1} \cdot w_i - \alpha = 0.
\]

We shall suppose that the optimal weighting vector has the following form
\[
W = (0, \ldots, 0, w_p, \ldots, w_q, 0 \ldots, 0)^T \quad (4)
\]
where \( 1 \leq p < q \leq n \) and use the notation
\[
I_{\{p,q\}} = \{p, p + 1, \ldots, q - 1, q\},
\]
for the indexes from \( p \) to \( q \). So, \( w_j = 0 \) if \( j \notin I_{\{p,q\}} \) and \( w_j \geq 0 \) if \( j \in I_{\{p,q\}} \).

For \( j = p \) we find that
\[
\frac{\partial L}{\partial w_p} = \frac{2w_p}{n} + \lambda_1 + \frac{n - p}{n - 1} \cdot \lambda_2 = 0
\]
and for \( j = q \) we get
\[
\frac{\partial L}{\partial w_q} = \frac{2w_q}{n} + \lambda_1 + \frac{n - q}{n - 1} \cdot \lambda_2 = 0
\]
That is,
\[
\frac{2(w_p - w_q)}{n} + \frac{q - p}{n - 1} \cdot \lambda_2 = 0
\]
and therefore, the optimal values of \( \lambda_1 \) and \( \lambda_2 \) (denoted by \( \lambda_1^* \) and \( \lambda_2^* \)) should satisfy the following equations
\[
\lambda_1^* = \frac{2}{n} \left[ \frac{n - q}{q - p} \cdot w_p - \frac{n - p}{q - p} \cdot w_q \right] \quad \text{and} \quad \lambda_2^* = \frac{n - 1}{q - p} \cdot \frac{2}{n} \cdot (w_q - w_p). \quad (5)
\]
Substituting \( \lambda_1^* \) for \( \lambda_1 \) and \( \lambda_2^* \) for \( \lambda_2 \) in (3) we get
\[
\frac{2}{n} \cdot w_j + \frac{2}{n} \left[ \frac{n - q}{q - p} \cdot w_p - \frac{n - p}{q - p} \cdot w_q \right] + \frac{n - j}{n - 1} \cdot \frac{n - 1}{q - p} \cdot \frac{2}{n} \cdot (w_q - w_p) = 0.
\]
That is the \( j \)th optimal weight should satisfy the equation
\[
w_j^* = \frac{q - j}{q - p} \cdot w_p + \frac{j - p}{q - p} \cdot w_q, \quad j \in I_{\{p,q\}}. \quad (6)
\]
From representation (4) we get
\[ \sum_{i=p}^{q} w_i^* = 1, \]
That is,
\[ \sum_{i=p}^{q} \left( \frac{q-i}{q-p} \cdot w_p + \frac{i-p}{q-p} \cdot w_q \right) = 1, \]
i.e.
\[ w_p + w_q = \frac{2}{q-p+1}. \]
From the constraint orness(w) = \( \alpha \) we find
\[ \sum_{i=p}^{q} \frac{n-i}{n-1} \cdot w_i = \sum_{i=p}^{q} \frac{n-i}{n-1} \cdot \frac{q-i}{q-p} \cdot w_p + \sum_{i=p}^{q} \frac{n-i}{n-1} \cdot \frac{i-p}{q-p} \cdot w_q = \alpha \]
that is,
\[ w_p^* = \frac{2(2q+p-2) - 6(n-1)(1-\alpha)}{(q-p+1)(q-p+2)}, \]  
(7)
and
\[ w_q^* = \frac{2}{q-p+1} - w_p^* = \frac{6(n-1)(1-\alpha) - 2(q+2p-4)}{(q-p+1)(q-p+2)}. \]  
(8)
The optimal weighting vector
\[ W^* = (0, \ldots, 0, w_p^*, \ldots, w_q^*, 0, \ldots, 0)^T \]
is feasible if and only if \( w_p^*, w_q^* \in [0, 1] \), because according to (6) any other \( w_j^* \), \( j \in I_{\{p,q\}} \) is computed as their convex linear combination.
Using formulas (7) and (8) we find
\[ w_p^*, w_q^* \in [0, 1] \iff \alpha \in \left[ 1 - \frac{1}{3} \cdot \frac{2q+p-2}{n-1}, 1 - \frac{1}{3} \cdot \frac{q+2p-4}{n-1} \right] \]
The following (disjunctive) partition of the unit interval (0, 1) will be crucial in finding an optimal solution to problem (2):
\[ (0, 1) = \bigcup_{r=2}^{n-1} J_{r,n} \cup J_{1,n} \cup \bigcup_{s=2}^{n-1} J_{1,s}. \]  
(9)
where
\[ J_{r,n} = \left( 1 - \frac{1}{3} \cdot \frac{2n+r-2}{n-1}, 1 - \frac{1}{3} \cdot \frac{2n+r-3}{n-1} \right), r = 2, \ldots, n-1, \]
\[ J_{1,n} = \left( 1 - \frac{1}{3} \cdot \frac{2n-1}{n-1}, 1 - \frac{1}{3} \cdot \frac{n-2}{n-1} \right), \]
\[ J_{1,s} = \left[ 1 - \frac{1}{3} \cdot \frac{s-1}{n-1}, 1 - \frac{1}{3} \cdot \frac{s-2}{n-1} \right), s = 2, \ldots, n-1. \]
Consider again problem (2) and suppose that $\alpha \in J_{r,s}$ for some $r$ and $s$ from partition (9). Such $r$ and $s$ always exist for any $\alpha \in (0,1)$, furthermore, $r = 1$ or $s = n$ should hold.

Then

$$W^* = (0, \ldots, 0, w^*_r, \ldots, w^*_s, 0 \ldots, 0)^T$$

(10)

where

$$w^*_j = 0, \text{ if } j \notin I_{(r,s)},$$

$$w^*_r = \frac{2(2s + r - 2) - 6(n - 1)(1 - \alpha)}{(s - r + 1)(s - r + 2)},$$

$$w^*_s = \frac{6(n - 1)(1 - \alpha) - 2(s + 2r - 4)}{(s - r + 1)(s - r + 2)}$$

(11)

$$w^*_j = \frac{s - j}{s - r} \cdot w_r + \frac{j - r}{s - r} \cdot w_s, \text{ if } j \in I_{(r+1,s-1)}.$$

and $I_{(r+1,s-1)} = \{r + 1, \ldots, s - 1\}$. Furthermore, from the construction of $W^*$ it is clear that

$$\sum_{i=1}^{n} w^*_i - \sum_{i=r}^{s} w^*_i = 1, \quad w^*_i \geq 0, \quad i = 1, 2, \ldots, n,$$

and $\text{orness}(W^*) = \alpha$, that is, $W^*$ is feasible for problem (2).

We will show now that $W^*$, defined by (10), satisfies the Kuhn-Tucker second-order sufficiency conditions for optimality ([2], page 58). Namely,

(i) There exist $\lambda^*_1, \lambda^*_2 \in \mathbb{R}$ and $\mu^*_1 \geq 0, \ldots, \mu^*_n \geq 0$ such that,

$$\left. \frac{\partial}{\partial w_k} \left( D^2(W) + \lambda^*_1 \left[ \sum_{i=1}^{n} w_i - 1 \right] + \lambda^*_2 \left[ \sum_{i=1}^{n-1} \frac{n-i}{n-1} \cdot w_i - \alpha \right] \right) \right|_{W=W^*} = 0$$

for $1 \leq k \leq n$ and $\mu^*_j w^*_j = 0, j = 1, \ldots, n$.

(ii) $W^*$ is a regular point of the constraints,

(iii) The Hessian matrix,

$$\left. \frac{\partial^2}{\partial W^2} \left( D^2(W) + \lambda^*_1 \left[ \sum_{i=1}^{n} w_i - 1 \right] + \lambda^*_2 \left[ \sum_{i=1}^{n-1} \frac{n-i}{n-1} \cdot w_i - \alpha \right] \right) \right|_{W=W^*}$$
is positive definite on
\[ \tilde{X} = \left\{ y \mid h_1 y^T = 0, h_2 y^T = 0 \text{ and } g_j y^T = 0 \text{ for all } j \text{ with } \mu_j > 0 \right\}, \tag{12} \]
where
\[ h_1 = \left( \frac{n-1}{n-1}, \frac{n-2}{n-1}, \ldots, \frac{1}{n-1}, 0 \right)^T, \tag{13} \]
and
\[ h_2 = (1, 1, \ldots, 1)^T. \tag{14} \]
are the gradients of linear equality constraints, and
\[ g_j = (0, 0, \ldots, 0, 2^{j-1}, 0, 0, \ldots, 0)^T \tag{15} \]
is the gradient of the \( j \)th linear inequality constraint of problem (2).

**Proof.** (i) According to (5) we get
\[ \lambda^*_1 = \frac{2}{n} \left[ \frac{n-s}{s-r} \cdot w^*_r - \frac{n-r}{s-r} \cdot w^*_s \right] \quad \text{and} \quad \lambda^*_2 = \frac{n-1}{n-1} \cdot \frac{2}{n} \cdot (w^*_s - w^*_r) \]
and
\[ \frac{2}{n} \cdot w^*_k + \lambda^*_1 + \frac{n-k}{n-1} \cdot \lambda^*_2 - \mu_k = 0. \]
for \( k = 1, \ldots, n \). If \( k \in I_{\{r,s\}} \) then
\[ \mu^*_k = \frac{2}{n} \left[ \frac{s-k}{s-r} \cdot w^*_r + \frac{k-r}{s-r} \cdot w^*_s \right] + \frac{2}{n} \left[ \frac{n-s}{s-r} \cdot w^*_r - \frac{n-r}{s-r} \cdot w^*_s \right] \]
\[ + \frac{n-k}{n-1} \cdot \frac{2}{n} \cdot (w^*_s - w^*_r) \]
\[ = \frac{2}{n} \cdot \frac{1}{s-r} \left[ (s-k+n-s-n+k)w^*_r + (k-r-n+r+n-k)w^*_s \right] \]
\[ = 0. \]
If \( k \notin I_{\{r,s\}} \) then \( w^*_k = 0 \). Then from the equality
\[ \lambda^*_1 + \frac{n-k}{n-1} \cdot \lambda^*_2 - \mu_k = 0, \]
we find
\[ \mu^*_k = \lambda^*_1 + \frac{n-k}{n-1} \cdot \lambda^*_2 \]
\[ = \frac{2}{n} \left[ \frac{n-s}{s-r} \cdot w^*_r - \frac{n-r}{s-r} \cdot w^*_s \right] + \frac{n-k}{n-1} \cdot \frac{2}{n} \cdot (w^*_s - w^*_r) \]
\[ = \frac{2}{n} \cdot \frac{1}{s-r} \left[ (k-s)w^*_r + (r-k)w^*_s \right]. \]
We need to show that $\mu^*_k \geq 0$ for $k \notin I_{\{r,s\}}$. That is,

$$(k-s)w^*_r + (r-k)w^*_s = (k-s) \cdot \frac{2(2s+r-2) - 6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)} + (r-k) \cdot \frac{6(n-1)(1-\alpha) - 2(s+2r-4)}{(s-r+1)(s-r+2)} \geq 0. \tag{16}$$

If $r = 1$ and $s = n$ then we get that $\mu^*_k = 0$ for $k = 1, \ldots, n$. Suppose now that $r = 1$ and $s < n$. In this case the inequality $k > s > 1$ should hold and (16) leads to the following requirement for $\alpha$,

$$\alpha \geq 1 - \frac{(s-1)(3k-2s-2)}{3(n-1)(2k-s-1)}. \tag{16}$$

On the other hand, from $\alpha \in J_{1,s}$ and $s < n$ we have

$$\alpha \in \left[ 1 - \frac{1}{3} \cdot \frac{s-1}{n-1}, 1 - \frac{1}{3} \cdot \frac{s-2}{n-1} \right],$$

and, therefore,

$$\alpha \geq 1 - \frac{1}{3} \cdot \frac{s-1}{n-1}. \tag{16}$$

Finally, from the inequality

$$1 - \frac{1}{3} \cdot \frac{s-1}{n-1} \geq 1 - \frac{(s-1)(3k-2s-2)}{3(n-1)(2k-s-1)}$$

we get that (16) holds. The proof of the remaining case ($r > 1$ and $s = n$) is carried out analogously.

(ii) The gradient vectors of linear equality and inequality constraints are computed by (13), (14) and (15), respectively. If $r = 1$ and $s = n$ then $w^*_j \neq 0$ for all $j = 1, \ldots, n$. Then it is easy to see that $h_1$ and $h_2$ are linearly independent. If $r = 1$ and $s < n$ then $w^*_j = 0$ for $j = s+1, \ldots, n$, and in this case

$$g_j = (0,0,\ldots,0, -1,0,0,\ldots,0)^T,$$

for $j = s+1, \ldots, n$. Consider the matrix

$$G = [h^T_1, h^T_2, g^T_{s+1}, \ldots, g^T_n] \in \mathbb{R}^{n \times (n-s+2)}.$$

Then the determinant of the lower-left submatrix of dimension $(n-s+2) \times (n-s+2)$ is
of $G$ is
\[
\begin{vmatrix}
\frac{n-s+1}{n-1} & 1 & 0 & \ldots & 0 \\
\frac{n-s}{n-1} & 1 & 0 & \ldots & 0 \\
\frac{n-s-1}{n-1} & 1 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \ldots & -1
\end{vmatrix}
\begin{vmatrix}
\frac{n-s+1}{n-1} & 1 \\
\frac{n-s}{n-1} & 1 \\
\frac{n-s-1}{n-1} & 1 \\
\vdots & \vdots \\
0 & 1
\end{vmatrix}
= (-1)^{n-s}
\begin{vmatrix}
\frac{n-s+1}{n-1} & 1 \\
\frac{n-s}{n-1} & 1 \\
\frac{n-s-1}{n-1} & 1 \\
\vdots & \vdots \\
0 & 1
\end{vmatrix}
= \frac{1}{n-1}(-1)^{n-s}
\]
which means that the columns of $G$ are linearly independent, and therefore, the system
\[
\{h_1, h_2, g_{s+1}, \ldots, g_n\},
\]
is linearly independent.

If $r > 1$ and $n$ then $w_j^* = 0$ for $j = 1, \ldots, r-1$, and in this case
\[
g_j = (0, 0, \ldots, 0, \underbrace{-1}_j, 0, 0, \ldots, 0)^T,
\]
for $j = 1, \ldots, r-1$. Consider the matrix
\[
F = [h_1^T, h_2^T, g_1^T, \ldots, g_{r-1}^T] \in \mathbb{R}^{n \times (r+1)}.
\]
Then the determinant of the upper-left submatrix of dimension $(r+1) \times (r+1)$ of $F$ is
\[
\begin{vmatrix}
\frac{n-1}{n-1} & 1 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{n-r+1}{n-1} & 1 & 0 & \ldots & -1 \\
\frac{n-r}{n-1} & 1 & 0 & \ldots & 0 \\
\frac{n-r-1}{n-1} & 1 & 0 & \ldots & 0
\end{vmatrix}
\begin{vmatrix}
\frac{n-r}{n-1} & 1 \\
\frac{n-r-1}{n-1} & 1
\end{vmatrix}
= (-1)^{r-1}
\begin{vmatrix}
\frac{n-r}{n-1} & 1 \\
\frac{n-r-1}{n-1} & 1
\end{vmatrix}
= \frac{1}{n-1}(-1)^{r-1}
\]
which means that the columns of $F$ are linearly independent, and therefore, the system
\[
\{h_1, h_2, g_1, \ldots, g_{r-1}\},
\]
is linearly independent. So $W^*$ is a regular point for problem (2).

(iii) Let us introduce the notation
\[
K(W) = D^2(W) + \lambda_1^* \left[ \sum_{i=1}^{n} w_i - 1 \right] + \lambda_2^* \left[ \sum_{i=1}^{n-1} \frac{n-i}{n-1} \cdot w_i - \alpha \right] + \sum_{j=1}^{n} a_j^*(-w_j).
\]
The Hessian matrix of $K$ at $W^*$ is
\[
\frac{\partial^2}{\partial w_k \partial w_j} K(W) \bigg|_{W = W^*} = \frac{\partial^2}{\partial w_k \partial w_j} D^2(W) \bigg|_{W = W^*} = \frac{2}{n} \delta_{kj},
\]
where
\[
\delta_{jk} = \begin{cases} 
1 & \text{if } j = k \\
0 & \text{otherwise.}
\end{cases}
\]
That is,
\[
K''(W^*) = \frac{2}{n} \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}
\]
which is a positive definite matrix on $\mathbb{R}^n$.

So, the objective function $D^2(W)$ has a local minimum at point $W = W^*$ on
\[
X = \left\{ W \in \mathbb{R}^n \mid W \geq 0, \sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{n} \frac{n - i}{n - 1} w_i = \alpha \right\}
\]
where $X$ is the set of feasible solutions of problem (2). Taking into consideration that $D^2 : \mathbb{R}^n \rightarrow \mathbb{R}$ is a strictly convex, bounded and continuous function, and $X$ is a convex and compact subset of $\mathbb{R}^n$, we can conclude that $D^2$ attains its (unique) global minimum on $X$ at point $W^*$.

3 Example

In this Section we will determine the minimal variability five-dimensional weighting vector under orness levels $\alpha = 0, 0.1, \ldots, 0.9$ and 1.0. First, we construct the corresponding partition as
\[
(0, 1) = \bigcup_{r=2}^{4} J_{r,5} \cup J_{1,5} \cup \bigcup_{s=2}^{4} J_{1,s}.
\]
where
\[
J_{r,5} = \left( \frac{1}{3} \cdot \frac{5 - r}{5 - 1}, \frac{4 - r}{12}, \frac{5 - r}{12} \right).
\]
for $r = 2, 3, 4$ and
\[
J_{1,5} = \left( \frac{1}{3} \cdot \frac{5 - 2}{5 - 1}, \frac{3}{5 - 1}, \frac{10 - 1}{5 - 1} \right) = \left( \frac{3}{12}, \frac{9}{12} \right),
\]
and
\[
J_{1,s} = \left[ \frac{1}{3} - \frac{s - 1}{3} \cdot \frac{5 - 1}{5 - 1}, \frac{1}{3} - \frac{s - 2}{3} \cdot \frac{5 - 1}{5 - 1} \right] = \left[ \frac{13 - s}{12}, \frac{14 - s}{12} \right).
\]
for $s = 2, 3, 4$, and, therefore we get,

$$(0, 1) = \left[ \begin{array}{c} 0, \frac{1}{12} \\ \left[ \begin{array}{cc} 1 & 2 \\ \frac{12}{12} & \frac{2}{12} \\ \frac{12}{12} & \frac{12}{12} \end{array} \right] \right] \cup \left[ \begin{array}{c} 2, \frac{1}{12} \\ \left[ \begin{array}{cc} 2 & 3 \\ \frac{12}{12} & \frac{3}{12} \end{array} \right] \right] \cup \left[ \begin{array}{c} 3, \frac{9}{12} \\ \left[ \begin{array}{c} 9, \frac{10}{12} \\ \frac{12}{12} \end{array} \right] \right] \cup \left[ \begin{array}{c} 9, \frac{10}{12} \\ \left[ \begin{array}{c} 10, \frac{11}{12} \\ \frac{12}{12} \end{array} \right] \right] \cup \left[ \begin{array}{c} 10, \frac{11}{12} \\ \left[ \begin{array}{c} 11, \frac{12}{12} \\ \frac{12}{12} \end{array} \right] \right] \cup \left[ \begin{array}{c} 11, \frac{12}{12} \\ \left[ \begin{array}{c} 12, \frac{12}{12} \\ \frac{12}{12} \end{array} \right] \right]. \right.$$

Without loss of generality we can assume that $\alpha < 0.5$, because if a weighting vector $W$ is optimal for problem (2) under some given degree of orness, $\alpha < 0.5$, then its reverse, denoted by $W^R$, and defined as

$$w_i^R = w_{n-i+1}$$

is also optimal for problem (2) under degree of orness $(1 - \alpha)$. Really, as was shown by Yager [7], we find that

$$D^2(W^R) = D^2(W) \text{ and } \text{orness}(W^R) = 1 - \text{orness}(W).$$

Therefore, for any $\alpha > 0.5$, we can solve problem (2) by solving it for level of orness $(1 - \alpha)$ and then taking the reverse of that solution.

Then we obtain the optimal weights from (11) as follows

- If $\alpha = 0$ then $W^*(\alpha) = W^*(0) = (0, 0, \ldots, 0, 1)^T$ and, therefore,

  $$W^*(1) = (W^*(0))^R = (1, 0, \ldots, 0, 0)^T.$$

- If $\alpha = 0.1$ then

  $$\alpha \in J_{3.5} = \left[ \begin{array}{c} 1 \\ \frac{2}{12} \end{array} \right],$$

and the associated minimal variability weights are

$$w_1^*(0.1) = 0,$$
$$w_2^*(0.1) = 0,$$
$$w_3^*(0.1) = \frac{2(10 + 3 - 2) - 6(5 - 1)(1 - 0.1)}{(5 - 3 + 1)(5 - 3 + 2)} = \frac{0.4}{12} = 0.0333,$$
$$w_4^*(0.1) = \frac{2}{5 - 3 + 1} - w_3^*(0.1) = 0.6334,$$
$$w_5^*(0.1) = \frac{1}{2} \cdot w_1^*(0.1) + \frac{1}{2} \cdot w_5^*(0.1) = 0.3333,$$

So,

$$W^*(\alpha) = W^*(0.1) = (0, 0, 0.033, 0.333, 0.633)^T,$$

and, consequently,

$$W^*(0.9) = (W^*(0.1))^R = (0.633, 0.333, 0.333, 0.0, 0)^T.$$

with variance $D^2(W^*(0.1)) = 0.0625$. 


• if $\alpha = 0.2$ then
  
  $\alpha \in J_{2.5} = \left(\begin{array}{cc} 2 & 3 \\ 12 & 12 \end{array}\right)$

  and in a similar manner we find that the associated minimal variability weighting vector is
  
  $W^*(0.2) = (0.0, 0.04, 0.18, 0.32, 0.46)^T,$

  and, therefore,
  
  $W^*(0.8) = (0.46, 0.32, 0.18, 0.04, 0.0)^T,$

  with variance $D^2(W^*(0.2)) = 0.0296.$

• if $\alpha = 0.3$ then
  
  $\alpha \in J_{1.5} = \left(\begin{array}{cc} 3 & 9 \\ 12 & 12 \end{array}\right)$

  and in a similar manner we find that the associated minimal variability weighting vector is
  
  $W^*(0.3) = (0.04, 0.12, 0.20, 0.28, 0.36)^T,$

  and, therefore,
  
  $W^*(0.7) = (0.36, 0.28, 0.20, 0.12, 0.04)^T,$

  with variance $D^2(W^*(0.3)) = 0.0128.$

• if $\alpha = 0.4$ then
  
  $\alpha \in J_{1.5} = \left(\begin{array}{cc} 3 & 9 \\ 12 & 12 \end{array}\right)$

  and in a similar manner we find that the associated minimal variability weighting vector is
  
  $W^*(0.4) = (0.12, 0.16, 0.20, 0.24, 0.28)^T,$

  and, therefore,
  
  $W^*(0.6) = (0.28, 0.24, 0.20, 0.16, 0.12)^T,$

  with variance $D^2(W^*(0.4)) = 0.0032.$

• if $\alpha = 0.5$ then
  
  $W^*(0.5) = (0.2, 0.2, 0.2, 0.2, 0.2)^T.$

  with variance $D^2(W^*(0.5)) = 0.$
4 Summary

We have extended the power of decision making with OWA operators by introducing considerations of minimizing variability into the process of selecting the optimal alternative. Of particular significance in this work is the development of a methodology to calculate the minimal variability weights in an analytic way.

References


Turku Centre for Computer Science
Lemminkäisenkatu 14
FIN-20520 Turku
Finland
http://www.tucs.abo.fi

University of Turku
• Department of Mathematical Sciences

Åbo Akademi University
• Department of Computer Science
• Institute for Advanced Management Systems Research

Turku School of Economics and Business Administration
• Institute of Information Systems Science