Stability in Multiobjective Possibilistic Linear Programs with Weakly Noninteractive Fuzzy Number Coefficients

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Abstract

Stability and sensitivity analysis becomes more and more attractive also in the area of multiple objective mathematical programming (for excellent surveys see e.g. [Gal86] and [Rios90]). Publications on this topic usually investigate the impact of parameter changes (in the righthand side or/and the objective function or/and the 'A-matrix' or/and the domination structure) on the solution in various models of vectormaximization problems, e.g. linear or non-linear, deterministic or stochastic, static or dynamic (see e.g. [Dev79, Rari83, Rios91]). There are only few paper dealing with stability or sensitivity analysis in fuzzy mathematical programming (e.g. [Ham78, Tan86, Ful89, Fed92, Fef92, Dut92]).

In this paper, generalizing the earlier results [Fed92,Fef92] of the second and third co-authors’, we show that the possibility distribution of the objectives of an Multiobjective Possibilistic Linear Program (MPLP) under continuous triangular norms is stable under small changes in the membership function of the continuous fuzzy number parameters.

Keywords: Multiobjective possibilistic linear programming, fuzzy number, possibility theory, stability.

1 Introduction

Sensitivity analysis in FLP problems with crisp coefficients and soft constraints was first considered in [Ham78], where a functional relationship between changes of parameters of the righthand side and those of the optimal value of the primal objective function was derived for almost all conceivable cases.

In [17] a FLP problem was formulated and the value of information was discussed via sensitivity analysis. The stability of fuzzy solution in FLP problems with fuzzy coefficients of symmetric triangular form was shown in [8]. Dutta et al [5] performed sensitivity analysis in fuzzy linear fractional programming problems with crisp coefficients and soft constraints. In this paper, generalizing the authors’ earlier results [6, 7], we show that the possibility distribution of the objectives of an MPLP with continuous fuzzy number coefficients is stable under small changes in the membership function of the parameters.

In this section we recall Buckley’s [2, 3] and Luhandjula’s [12] solution concept for MPLP and set up the notation needed for our main result in the next section.

A fuzzy quantity \( \tilde{a} \) is a fuzzy set of the real line \( \mathbb{R} \). The support of a fuzzy quantity \( \tilde{a} \) (denoted by \( \text{supp} \tilde{a} \)) is the crisp set given by \( \{ t \mid \tilde{a}(t) > 0 \} \).

A fuzzy number \( \tilde{a} \) is a fuzzy quantity with a normalized, upper semicontinuous, fuzzy convex and bounded supported membership function.

A symmetric triangular fuzzy number \( \tilde{a} \) denoted by \( (a, \alpha) \) is defined as \( \tilde{a}(t) = 1 - \frac{|a - t|}{\alpha} \) if \( |a - t| \leq \alpha \) and \( \tilde{a}(t) = 0 \) otherwise, where \( a \in \mathbb{R} \) is the center and \( 2\alpha > 0 \) is the spread of \( \tilde{a} \).

The fuzzy numbers will represent the possibility distributions for fuzzy parameters. A multiobjective possibilistic linear programming is

\[
Z = (\tilde{c}_1 x_1 + \cdots + \tilde{c}_m x_n, \ldots, \tilde{c}_k x_1 + \cdots + \tilde{c}_m x_n) \rightarrow \max
\]

subject to \( \tilde{a}_{i1} x_1 + \cdots + \tilde{a}_{in} x_n \ast \tilde{b}_i, \ i = 1, \ldots, m, \ x \geq 0, \) (1)

where \( \tilde{a}_{ij}, \tilde{b}_i, \) and \( \tilde{c}_{ij} \) are fuzzy numbers, \( x = (x_1, \ldots, x_n) \) is a vector of (non-fuzzy) decision variables, the operations addition and multiplication by a real number of fuzzy numbers are defined by Zadeh’s extension principle [18], and * denotes <, =, =, ≥ or > for each \( i, \ i = 1, \ldots, m \). Even though * may vary from row to row in the constraints, we will rewrite the MPLP (1,2) as

\[
Z = (\tilde{c}_1 x, \ldots, \tilde{c}_k x) \rightarrow \max
\]

subject to \( \tilde{A} x \ast \tilde{b}, \ x \geq 0, \) (4)

where \( \tilde{A} = [\tilde{a}_{ij}] \) is an \( m \times n \) matrix of fuzzy numbers and \( \tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_m) \) is a vector of fuzzy numbers. The fuzzy numbers are the possibility distributions associated with the fuzzy variables and hence place a restriction on the possible values the variable may assume [18]. For example, \( \text{Poss}[\tilde{a}_{ij} = t] = \tilde{a}_{ij}(t) \) is the possibility that \( \tilde{a}_{ij} \) is equal to \( t \).

We will assume that all fuzzy numbers \( \tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_{ij} \) are weakly noninteractive [18]. Weakly-noninteractivity means that there exists a triangular norm \( T \), such that we can find the joint possibility distribution of all the fuzzy variables by calculating the \( T \)-intersection of their possibility distributions. Following Buckley [2, 3] and Luhandjula [12], we define \( \text{Poss}[Z = z] \), the possibility distribution of the objective function \( Z \).

We first specify the possibility that \( x \) satisfies the \( i \)-th constraints. Let

\[
\Pi_T(a_i, b_i) = T(\tilde{a}_{i1}(a_{i1}), \ldots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)),
\]
where \( a_i = (a_{i1}, \ldots, a_{in}) \), which is the joint distribution of \( \tilde{a}_{ij}, j = 1, \ldots, n \), and \( \tilde{b}_i \). Then
\[
\text{Poss}[x \in \mathcal{F}] = \sup_{a_i, b_i} \{ \Pi_T(a_i, b_i) \mid a_{i1}x_1 + \ldots + a_{in}x_n = b_i \},
\]
which is the possibility that \( x \) is feasible with respect to the \( i \)-th constraint. Therefore, for \( x \geq 0 \),
\[
\text{Poss}[x \in \mathcal{F}] = \min_{1 \leq i \leq m} \text{Poss}[x \in \mathcal{F}],
\]
which is the possibility that \( x \) is feasible. We next construct \( \text{Poss}[Z = z|x] \) which is the conditional possibility that \( Z \) equals \( z \) given \( x \). The joint distribution of \( \tilde{c}_{ij}, j = 1, \ldots, n, \) is
\[
\Pi_T(c_i) = T(\tilde{c}_{i1}(c_{i1}), \ldots, \tilde{c}_{in}(c_{in})),
\]
where \( c_i = (c_{i1}, \ldots, c_{in}), l = 1, \ldots, k. \) Therefore,
\[
\text{Poss}[Z = (z_1, \ldots, z_k)|x] = \text{Poss}[\tilde{c}_1x = z_1, \ldots, \tilde{c}_kx = z_k] = \min_{1 \leq i \leq k} \text{Poss}[\tilde{c}_ix = z_i] = \min_{1 \leq i \leq k} \sup_{c_{i1}, \ldots, c_{ik}} \{ \Pi_T(c_i) \mid c_{i1}x_1 + \ldots + c_{in}x_n = z_i \}.
\]
Finally, applying Bellman and Zadeh’s method of fuzzy decision making [Bel72], the possibility distribution of the objective function is defined as follows
\[
\text{Poss}[Z = z] = \sup_{x \geq 0} \min \{ \text{Poss}[Z = z|x], \text{Poss}[x \in \mathcal{F}] \}.
\]
Let \( \tilde{a} \) be a continuous fuzzy number, then for any \( \theta \geq 0 \) we define \( \omega(\theta) \), the modulus of continuity of \( \tilde{a} \) by
\[
\omega(\tilde{a}, \theta) = \max_{|u-v| \leq \theta} |\tilde{a}(u) - \tilde{a}(v)|.
\]
We metricize \( \mathcal{F} \) by the metric [Kal87],
\[
D(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} d([\tilde{a}]^\alpha, [\tilde{b}]^\alpha)
\]
where \( d \) denotes the classical Hausdorff metric in the family of compact subsets of \( IR^2 \), i.e.
\[
d([\tilde{a}]^\alpha, [\tilde{b}]^\alpha) = \max \{|a_1(\alpha) - b_1(\alpha)|, |a_2(\alpha) - b_2(\alpha)|\},
\]
\[
[\tilde{a}]^\alpha = [a_1(\alpha), a_2(\alpha)], \quad [\tilde{b}]^\alpha = [b_1(\alpha), b_2(\alpha)].
\]
Recall that the mapping \( T: [0,1] \times [0,1] \rightarrow [0,1] \) is a triangular norm [Sch63] iff it is commutative, associative, non-decreasing in each argument and \( T(x, 1) = x, \forall x \in [0,1] \).
The following lemma shows that if all the \( \alpha \)-level sets of two continuous fuzzy numbers are close to each other, there can be only a small deviation between their membership grades.

**Lemma 1.1.** [Fed92]. Let \( \delta \geq 0 \) be a real number and let \( \tilde{a}, \tilde{b} \) be fuzzy intervals. If
\[
D(\tilde{a}, \tilde{b}) \leq \delta
\]
then
\[
\sup_{t \in IR} |\tilde{a}(t) - \tilde{b}(t)| \leq \max \{ \omega(\tilde{a}, \delta), \omega(\tilde{b}, \delta) \}.
\]

**Remark 1.1.** It should be noted that if \( \tilde{a} \) or \( \tilde{b} \) are discontinuous fuzzy numbers then there can be a big deviation between their membership grades even if \( D(\tilde{a}, \tilde{b}) \) is arbitrarily small.
2 Stability in multiobjective possibilistic linear programs

An important question [Tan86, Zim87, Ful89, Fed92] is the impact of small perturbations in the membership functions of fuzzy coefficients on the possibility distribution of the objective function.

Suppose that instead of the exact coefficients \( \tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_{lj} \) there is a collection of fuzzy coefficients \( \tilde{a}_{ij}^{\delta}, \tilde{b}_i^{\delta}, \tilde{c}_{lj}^{\delta} \) available with the property

\[
\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}_{ij}^{\delta}) \leq \delta, \quad \max_i D(\tilde{b}_i, \tilde{b}_i^{\delta}) \leq \delta, \quad \max_{lj} D(\tilde{c}_{lj}, \tilde{c}_{lj}^{\delta}) \leq \delta \tag{5}
\]

where \( \delta > 0 \) denotes the maximal error of measurement. Then we have to solve the following problem

\[
Z^\delta = (\tilde{c}_{1}^{\delta}x, \ldots, \tilde{c}_{k}^{\delta}) \rightarrow \max \tag{6}
\]

subject to \( \tilde{A}^{\delta}x \leq \tilde{b}^{\delta}, \quad x \geq 0 \tag{7} \)

The possibility distribution of the objective function, \( Z^\delta \), is computed as

\[
\text{Poss}[Z^\delta = z] = \sup_{x \geq 0} \min \{\text{Poss}[Z^\delta = z|x], \text{Poss}[x \in F^\delta]\}. \]

The next theorem establishes a stability property (with respect to perturbations (5)) of the possibility distribution of the objective function of MPLP (1,2) and (6,7).

**Theorem 2.1.** Let \( \delta \geq 0 \) be a real number and let \( \tilde{a}_{ij}, \tilde{b}_i, \tilde{a}_{ij}^{\delta}, \tilde{b}_i^{\delta}, \tilde{c}_{lj} \) and \( \tilde{c}_{lj}^{\delta} \) be continuous fuzzy numbers. If (5) holds, then

\[
\sup_{z \in \mathbb{R}^k} |\text{Poss}[Z^\delta = z] - \text{Poss}[Z = z]| \leq \omega(T, \Omega(\delta)) \tag{8}
\]

where

\[
\Omega(\delta) = \max_{i,j,l} \max \{\omega(\tilde{a}_{ij}, \delta), \omega(\tilde{a}_{ij}^{\delta}, \delta), \omega(\tilde{b}_i, \delta), \omega(\tilde{b}_i^{\delta}, \delta), \omega(\tilde{c}_{lj}, \delta), \omega(\tilde{c}_{lj}^{\delta}, \delta)\},
\]

i.e., \( \Omega(\delta) \) denotes the maximum of modulus of continuity of all the fuzzy parameters at the point \( \delta \).

**Proof.** It is sufficient to show that

\[
|\text{Poss}[Z^\delta = z] - \text{Poss}[Z = z]| \leq \omega(T, \Omega(\delta))
\]

for any \( z = (z_1, \ldots, z_k) \in \mathbb{R}^k \). Applying Lemma 1.1 we have

\[
|\text{Poss}[Z^\delta = z] - \text{Poss}[Z = z]| \leq \omega(T, \Omega(\delta))
\]

for any \( z = (z_1, \ldots, z_k) \in \mathbb{R}^k \). Applying Lemma 1.1 we have

\[
|\text{Poss}[Z^\delta = z] - \text{Poss}[Z = z]| = \sup_{x \geq 0} \min \{\text{Poss}[Z = z|x], \text{Poss}[x \in F^\delta]\} - \sup_{x \geq 0} \min \{\text{Poss}[Z^\delta = z|x], \text{Poss}[x \in F^\delta]\} \leq
\]

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We illustrate Theorem 2.1 by a very simple example. Consider the following MPLP
respect to perturbations (5).

\[ \max \{ |\text{Poss}[x = z]|, |\text{Poss}[x \in F] - \text{Poss}[x \in F^\delta]| \} \leq \]
\[ \sup_{x \geq 0} \max_{1 \leq i \leq k} \{ |\text{Poss}[\tilde{c}_i x = z]| - \text{Poss}[\tilde{c}_i^\delta x = z]| \}, \]
\[ \sup_{x \geq 0} \max_{1 \leq i \leq k} \{ |\text{Poss}[x \in F_i] - \text{Poss}[x \in F_i^\delta]| \} = \]
\[ \sup_{x \geq 0} \max_{1 \leq i \leq k} \max_{c_{i1}, \ldots, c_{in}} \{ T(\tilde{c}_{i1}(c_{i1}), \ldots, \tilde{c}_{in}(c_{in})) | c_{i1} x + \cdots + c_{in} x_n = z_i \} - \]
\[ \sup_{a_{i1}, \ldots, a_{in}} \max_{1 \leq i \leq m} \{ T(\tilde{a}_{i1}(a_{i1}), \ldots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)) | a_{i1} x_1 + \cdots + a_{in} x_n * b_i \} - \]
\[ \sup_{a_{i1}, \ldots, a_{in}} \max_{1 \leq i \leq m} \{ T(\tilde{a}_{i1}(a_{i1}), \ldots, \tilde{a}_{in}(a_{in}), \tilde{b}_i(b_i)) | a_{i1} x_1 + \cdots + a_{in} x_n * b_i \} \leq \]
\[ \sup_{x \geq 0} \max_{1 \leq i \leq k} \max_{c_{i1}, \ldots, c_{in}} \{ \omega(T, |\tilde{c}_{ij}(c_{ij}) - \tilde{c}_{ij}^\delta(c_{ij})|) | c_{i1} x_1 + \cdots + c_{in} x_n = z_i \}, \]
\[ \max_{1 \leq i \leq m} \max_{a_{i1}, \ldots, a_{in}} \max_{1 \leq j \leq n} \{ \max \{ \omega(T, |\tilde{a}_{ij}(a_{ij}) - \tilde{a}_{ij}^\delta(a_{ij})|), \omega(T, |\tilde{b}_i(b_i) - \tilde{b}_i^\delta(b_i)|) \} \leq \]
\[ \sup_{a_{ij}, b_i, c_{ij}} \max \{ \omega(T, |\tilde{a}_{ij}(a_{ij}) - \tilde{a}_{ij}(a_{ij})|), \omega(T, |\tilde{b}_i(b_i) - \tilde{b}_i(b_i)|), \omega(T, |\tilde{c}_{ij}(c_{ij}) - \tilde{c}_{ij}(c_{ij})|) \} \leq \]
\[ \max_{i,j,l} \{ \omega(T, \omega(\tilde{a}_{ij}, \delta)), \omega(T, \omega(\tilde{a}_{ij}, \delta)), \omega(T, \omega(\tilde{b}_i, \delta)), \omega(T, \omega(\tilde{b}_i, \delta)), \omega(T, \omega(\tilde{c}_{ij}, \delta)), \omega(T, \omega(\tilde{c}_{ij}, \delta)) \} \leq \omega(T, \Omega(\delta)). \]

Which ends the proof. \( \square \)

**Remark 2.1.** From \( \lim_{\delta \to 0} \Omega(\delta) = 0 \) and (8) it follows that
\[ \sup_z |\text{Poss}[Z^\delta = z] - \text{Poss}[Z = z]| \to 0 \text{ as } \delta \to 0 \]
which means the stability of the possibility distribution of the objective function with respect to perturbations (5).

We illustrate Theorem 2.1 by a very simple example. Consider the following MPLP
\[ (\tilde{c}_1 x, \tilde{c}_2 x) \to \max \]
subject to \( \tilde{a} x \leq \tilde{b}, \ x \geq 0, \)
where \( \tilde{c}_i = (c_i, \gamma_i), i = 1, 2, \tilde{a} = (a, \alpha), \tilde{b} = (b, \beta) \) and \( T(u, v) = \max\{0, u + v - 1\}, \)
(Lukasiewicz triangular norm).
Suppose now that instead of the exact centers $\gamma_i, i = 1, 2, \alpha$ and $\beta$ we have to work with their approximations $\gamma_i', i = 1, 2, \alpha'$ and $\beta'$, such that

$$|\alpha - \alpha'| \leq \delta, \quad |\beta - \beta'| \leq \delta, \quad \max_{i=1,2} |\gamma_i - \gamma_i'| \leq \delta,$$

where $\delta > 0$ is the maximal error of measurement.

Then applying Theorem 2.1. we have the following estimation for the distance between the possibility distributions of the objective functions of the exact and the perturbed MPLP:

$$\sup_z |\text{Poss}[Z^\delta = z] - \text{Poss}[Z = z]| \leq 2\delta \max \left\{ \frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\alpha'}, \frac{1}{\beta'} \right\}.$$ 

References


