On the metric dimension of Grassmann graphs

Robert F. Bailey and Karen Meagher

October 22, 2010

Abstract

The metric dimension of a graph $\Gamma$ is the least number of vertices in a set with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex. We consider the Grassmann graph $G_q(n,k)$ (whose vertices are the $k$-subspaces of $\mathbb{F}_q^n$, and are adjacent if they intersect in a $(k-1)$-subspace) for $k \geq 2$, and find a constructive upper bound on its metric dimension. Our bound is equal to the number of 1-dimensional subspaces of $\mathbb{F}_q^n$.

Keywords: Grassmann graph, metric dimension

MSC2010 classification: 05C12 (primary), 05E30 (secondary)

1 Introduction

In this paper, we are concerned with finding an upper bound on the metric dimension of Grassmann graphs. We use the notation of [2]. To define the metric dimension of a graph, we need the following concept.

Definition 1. A resolving set for a graph $\Gamma = (V,E)$ is a set of vertices $S = \{v_1, \ldots, v_k\}$ such that for all $w \in V$, the list of distances $D(w|S) = (d(w,v_1), \ldots, d(w,v_k))$ uniquely determines $w$.

That is, $S$ is a resolving set for $\Gamma$ if for any pair of vertices $u,w$, $D(u|S) = D(w|S)$ if and only if $u = w$. We call the list of distances $D(w|S)$ the code of the vertex $w$, so $S$ is a resolving set if and only if each vertex has a different code.

*Department of Mathematics and Statistics, University of Regina, 3737 Wascana Parkway, Regina, SK, S4S 0A2, Canada. E-mail: robert.bailey@uregina.ca, karen.meagher@uregina.ca
Definition 2. The metric dimension of $\Gamma$, denoted $\mu(\Gamma)$, is the smallest size of a resolving set for $\Gamma$.

Metric dimension was first introduced in the 1970s, independently by Harary and Melter [7] and by Slater [9]. In recent years, a considerable literature has developed (see [2] for details). A particularly interesting case is that of distance-regular graphs (we refer the reader to the book by Brouwer, Cohen and Neumaier [4] for background on this topic). In fact, in the case of primitive distance-regular graphs, bounds on a parameter equivalent to the metric dimension were obtained in 1981 by Babai [1] (where they were used to obtain combinatorial bounds on the possible orders of primitive permutation groups: see [2] for further details).

Many families of distance-regular graphs are so-called “graphs with classical parameters” (see [4], Chapter 9): these include the well-known Hamming graphs and Johnson graphs, for which metric dimension has already been studied. In the case of Hamming graphs, various results on the metric dimension have been obtained (although not always phrased in these terms: see [2]). For the Johnson graphs $J(n,k)$, where the vertices are $k$-subsets of an $n$-set, and two $k$-subsets are adjacent if they intersect in a $(k-1)$-set, the exact value is known for $k = 2$: this was obtained by Cameron and the first author in [2], while for $k \geq 3$ bounds were obtained by Cáceres et al. in [5]. The obvious next case to consider is that of the Grassmann graphs. Throughout this paper, $V(n,q)$ denotes the $n$-dimensional vector space over the finite field $\mathbb{F}_q$.

Definition 3. The Grassmann graph $G_q(n,k)$ has as its vertex set the set of all $k$-dimensional subspaces of $V(n,q)$, and two vertices are adjacent if the corresponding subspaces intersect in a subspace of dimension $k-1$.

Note that if $k = 1$, we have a complete graph, so we shall assume that $k \geq 2$. Also, it is not difficult to show that $G_q(n,k) \cong G_q(n,n-k)$, so it suffices to consider $k \leq n/2$. The number of vertices is simply the Gaussian binomial coefficient:

$$\binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{(q^n - q^i)}{(q^k - q^i)}.$$  

The Grassmann graph is a distance-regular graph of diameter $k$: the distance between two vertices $X$ and $Y$ is $k - \dim(X \cap Y)$. It is considered to be the “$q$-analogue” of the Johnson graph $J(n,k)$.

Having defined the Grassmann graphs, we are now ready to state our main result.

Theorem 4. Let $G_q(n,k)$ be a Grassmann graph, where $2 \leq k \leq n/2$. Then the metric dimension of $G_q(n,k)$ is at most $\binom{n}{1}_q$.

This bound, and its proof, is inspired by a similar result for the Johnson graph $J(n,k)$ obtained by Cáceres et al. in [5].
2 Proof of the bound

We will prove Theorem 4 in two cases: the first, when \( n \) is divisible by \( k + 1 \), is considered in Proposition 6; the second, when \( n \) is not divisible by \( k + 1 \), is considered in Proposition 8. We could have combined the two separate cases into one proof, but in doing so risks losing the intuition of how the constructions arise.

Our constructions of resolving sets require some notions from finite geometry.

Definition 5. A \( t \)-spread of \( V(n, q) \) is collection of \( t \)-subspaces \( \{W_1, \ldots, W_m\} \) with the following properties:

(i) any non-zero vector \( x \in V(n, q) \) belongs to exactly one \( W_i \);

(ii) if \( i \neq j \), then \( W_i \cap W_j = \{0\} \).

A classical result (see Dembowski [6], page 29) shows that a \( t \)-spread of \( V(n, q) \) exists if and only if \( t \) is a divisor of \( n \). The number of subspaces in a \( t \)-spread is necessarily \( (q^n - 1)/(q^t - 1) \).

Proposition 6. Suppose \( k + 1 \) divides \( n \). Then the metric dimension of \( G_q(n,k) \) is at most \( \left\lceil \frac{n}{k+1} \right\rceil q \).

Proof. We obtain this upper bound by giving an explicit construction of a resolving set of that size. Suppose \( S = \{W_1, \ldots, W_m\} \) is a \( (k + 1) \)-spread of \( V(n, q) \). For each \( (k + 1) \)-space in \( S \), we take the collection of all \( k \)-subspaces; denote the union of all of these collections by \( \mathcal{M} \). That is,

\[
\mathcal{M} = \bigcup_{i=1}^{m} \{ U \leq W_i : \dim(U) = k \}.
\]

We will show that \( \mathcal{M} \) is a resolving set, by demonstrating that whenever \( A \) and \( B \) are two distinct \( k \)-subspaces of \( V(n, q) \), there exists some \( U \in \mathcal{M} \) with \( \dim(A \cap U) \neq \dim(B \cap U) \).

We consider how \( A \) and \( B \) intersect with the members \( W_1, \ldots, W_m \) of the \( (k + 1) \)-spread \( S \).

Case 1. If either of \( A \) or \( B \) (say \( A \)) is contained within some member of \( S \), then clearly \( A \in \mathcal{M} \), and we are done (as \( \dim(A \cap B) \neq k \)).

So we suppose not. For each \( i \), let \( A_i = A \cap W_i \) and \( B_i = B \cap W_i \). There are now two possible cases.

Case 2. Suppose there exists some \( i \) where \( \dim(B_i) < \dim(A_i) \). Since \( \dim(A_i) < k \), then there must exist some \( k \)-subspace \( U \) of \( W_i \) which contains \( A_i \) as a subspace. Therefore \( \dim(A_i \cap U) = \dim(A_i) \), while \( \dim(B_i \cap U) \leq \dim(B_i) < \dim(A_i) \). In particular, we have \( \dim(A_i \cap U) \neq \dim(B_i \cap U) \), and thus \( \dim(A \cap U) \neq \dim(B \cap U) \), where \( U \in \mathcal{M} \).
Case 3. The remaining scenario is that \( \dim(A_i) = \dim(B_i) \) for all \( i \). However, there must exist some \( i \) where \( A_i \neq B_i \) (and where both are non-zero); otherwise we would have \( A = B \).

For that particular value of \( i \), suppose \( \dim(A_i) = \dim(B_i) = d < k \).

Now, for any \( k \)-subspace \( U \) of \( W_i \), we must have \( \dim(A_i \cap U) \geq d - 1 \) and \( \dim(B_i \cap U) \geq d - 1 \) (this follows from the identity \( \dim(A_i \cap U) = \dim(A_i) + \dim(U) - \dim(A_i + U) \), and that \( \dim(A_i + U) \leq \dim(W_i) = k + 1 \); likewise for \( \dim(B_i \cap U) \)). Thus, for there to be a \( k \)-subspace \( U \) with \( \dim(A_i \cap U) \neq \dim(B_i \cap U) \), we must have (without loss of generality) that \( A_i \) is a subspace of \( U \), and \( B_i \) is not. We aim to construct such a \( U \).

Take a basis \( \{a_1, \ldots, a_d\} \) for \( A_i \). Since \( A_i \neq B_i \), there exists a vector \( x \in B_i \setminus A_i \). Because \( x \not\in A_i \), we have that the set \( \{a_1, \ldots, a_d, x\} \) is linearly independent. Extend this to a basis \( \{a_1, \ldots, a_d, w_1, \ldots, w_{k-d}\} \) for \( W_i \). Now let \( U \) be the \( k \)-subspace spanned by \( \{a_1, \ldots, a_d, w_1, \ldots, w_{k-d}\} \): by construction, \( A_i \) is a subspace of \( U \), but \( B_i \) is not, since \( x \in B_i \setminus U \). In particular, we see that \( d = \dim(A_i \cap U) \neq \dim(B_i \cap U) = d - 1 \), and thus \( \dim(A \cap U) \neq \dim(B \cap U) \), where \( U \in \mathcal{M} \).

In all cases, we have \( U \in \mathcal{M} \) with \( \dim(A \cap U) \neq \dim(B \cap U) \), so therefore \( \mathcal{M} \) is a resolving set. Finally, to obtain the bound, we observe that

\[
\mu(G_q(n,k)) \leq |\mathcal{M}| = \frac{q^n - 1}{q^{k+1} - 1} \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q = \frac{q^n - 1}{q^{k+1} - 1} \frac{q^{k+1} - 1}{q - 1} = \begin{bmatrix} n \\ 1 \end{bmatrix}_q.
\]

\( \square \)

Now it remains to consider the case when \( k + 1 \) does not divide \( n \). In that situation, there is no \( (k+1) \)-spread; however, a result of Beutelspacher \cite{Beutelspacher} provides an alternative. Following his notation, where \( T \) is a set of positive integers, a \( T \)-partition of \( V(n,q) \) is a partition of the non-zero vectors of \( V(n,q) \) into subspaces whose dimensions form the set \( T \). Thus if \( T = \{t\} \), then a \( T \)-partition of \( V(n,q) \) is simply a \( t \)-spread of \( V(n,q) \). The following lemma is an immediate consequence of Lemma 3 in Beutelspacher’s 1978 paper \cite{Beutelspacher}.

**Lemma 7** (Beutelspacher \cite{Beutelspacher}). Suppose \( n = r(k+1) + t \), where \( 0 < t < k+1 \). Then there exists a \( \{k+1,t\} \)-partition of \( V(n,q) \).

Beutelspacher’s construction works as follows: write \( s = r(k+1) \) (so that \( n = s + t \)) and take a \( (k+1) \)-spread of \( V(s,q) \). The remaining \( q^s - q^t \) vectors in \( V(n,q) \setminus V(s,q) \) are then partitioned into the non-zero vectors of \( t \)-subspaces (of which there are necessarily \( q^t \)).

**Proposition 8.** Suppose \( n = s + t \), where \( (k+1) \) divides \( s \) and \( 0 < t < k+1 \). Then the metric dimension of \( G_q(n,k) \) is at most \( \left\lceil \frac{n}{1} \right\rceil \).

\([n]_q\)
Proof. As with Proposition 5 we give an explicit construction of a resolving set.

Suppose \( \mathfrak{P} \) is a \( \{k+1, t\} \)-partition of \( V(n, q) \), obtained as described above. Write \( \mathfrak{P} = \mathcal{S} \cup \mathcal{T} \), where \( \mathcal{S} = \{ W_1, \ldots, W_m \} \) is a \( (k+1) \)-spread of \( V(s, q) \), and \( \mathcal{T} = \{ X_1, \ldots, X_t \} \) consists of \( k \)-subspaces covering the remaining vectors. For each member of \( \mathcal{S} \), we take all the \( k \)-subspaces (as in Proposition 5), while for each member of \( \mathcal{T} \), we take all the \( (t-1) \)-subspaces, and extend each to a \( k \)-subspace using a fixed \( (k-t+1) \)-dimensional subspace \( Z \) of \( V(s, q) \). Let \( \mathcal{M} \) denote this collection: that is,

\[
\mathcal{M} = \bigcup_{i=1}^{m} \{ U \leq W_i : \dim(U) = k \} \cup \bigcup_{j=1}^{t} \{ Z \oplus Y : Y \leq X_j, \dim(Y) = t-1 \}.
\]

We will show that \( \mathcal{M} \) is a resolving set, in that whenever \( A \) and \( B \) are two distinct \( k \)-subspaces of \( V(n, q) \), there exists some \( U \in \mathcal{M} \) with \( \dim(A \cap U) \neq \dim(B \cap U) \).

If \( A \) or \( B \) are entirely contained within \( V(s, q) \), then we are done; by the arguments used in the proof of Proposition 6 \( \mathcal{M} \) contains a resolving set for the subgraph \( G_q(s, k) \). Also, if \( A \cap V(s, q) \neq B \cap V(s, q) \), the proof of Proposition 5 shows that a vertex of the resolving set for \( G_q(s, k) \) is able to distinguish \( A \) and \( B \). Thus the only situation remaining is when both \( A \) and \( B \) intersect non-trivially with \( V(n, q) \setminus V(s, q) \), and where \( A \cap V(s, q) = B \cap V(s, q) \).

If there exists an \( X_i \) where \( \dim(A \cap X_i) < \dim(B \cap X_i) \), then (by a similar argument to Case 2 in the proof of Proposition 5) there must exist a \( (t-1) \)-subspace \( Y \) with \( \dim(A \cap Y) \neq \dim(B \cap Y) \). Since \( A \) and \( B \) agree on \( V(s, q) \), we have \( A \cap Z = B \cap Z \); therefore if \( U = Z \oplus Y \), it follows that \( \dim(A \cap U) \neq \dim(B \cap U) \). By construction, \( U \in \mathcal{M} \).

So the remaining possibility is that \( \dim(A \cap X_i) = \dim(B \cap X_i) \) for all indices \( i \). However, since \( A \neq B \), there must exist an index where, although the dimensions are equal, \( A \cap X_i = B \cap X_i \). By the argument in Case 3 of the proof of Proposition 6 there exists a \( (t-1) \)-subspace \( Y \) of \( X_i \) with \( \dim(A \cap Y) \neq \dim(B \cap Y) \), and thus by letting \( U = Z \oplus Y \), it follows once more that \( \dim(A \cap U) \neq \dim(B \cap U) \). Again, by construction, \( U \in \mathcal{M} \).

Finally, we obtain the bound by observing that

\[
|\mathcal{M}| = \left[ \frac{n}{1} \right] q + q^t \left[ \frac{t}{1} \right] q \\
= \frac{q^t - 1}{q - 1} + q^t \frac{q^t - 1}{q - 1} \\
= \frac{q^{t+1} - 1}{q - 1} \\
= \left[ \frac{n}{1} \right] q.
\]

\[\Box\]
3 Discussion

A natural question is to compare our result with the previously-known bounds due to Babai [1] (see [2] for an interpretation of these in terms of metric dimension of distance-regular graphs). For the case of the Grassmann graphs, Babai’s most general bound (see [1], Theorem 2.1; see also [2], Theorem 3.14) yields

$$
\mu(G_q(n,k)) < 4 \sqrt{\binom{n}{k}_q \log \binom{n}{k}_q}
$$

(1)

while his stronger bound (see [1], Theorem 2.4; see also [2], Theorem 3.21) yields

$$
\mu(G_q(n,k)) < 2k \frac{\binom{n}{k}_q}{\binom{n}{k}_q - M} \log \binom{n}{k}_q
$$

(2)

where

$$
M = \max_{0 \leq j \leq k} q^j \left[ \frac{n-k}{j}_q \frac{k}{j}_q \right].
$$

These bounds are difficult to evaluate exactly, so we conducted some experiments using MAPLE to compare these bounds with the one obtained in Theorem 4. Our experiments indicate that for $k > 2$, our constructive bound of $\binom{n}{1}_q$ is an improvement on Babai’s weaker bound. When $k = 2$ and $q$ is large, Babai gives a better bound. They also suggest that Babai’s stronger bound is, for fixed $q$ and $n$, is decreasing in $k$ (in comparison, our bound is independent of $k$ and thus stays fixed), and gives a better bound for larger values of $k$. However, it should be mentioned that our bound is obtained from an explicit construction of a resolving set, whereas Babai’s results are obtained using a result of Lovász on fractional covers in hypergraphs [8], and is not so explicit.

Finally, we remark that it is likely that one can refine the constructions in Propositions 6 and 8 (in the manner of the results for Johnson graphs in [5]) to obtain a tighter bound on the metric dimension. However, unlike the case of the Johnson graphs, we believe it is unlikely that such refinements will affect the order of magnitude of the bound too much, or that they would have such a tidy form.

Acknowledgements

The authors would like to thank J. Cáceres for communicating [5], and P. J. Cameron for pointing out an embarrassing error in an earlier version of this paper. The first author is a PIMS Postdoctoral Fellow at the University of Regina.
References


