ADJUNCTION CONTEXTS AND REGULAR QUASI-MONADS

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Abstract. Generalising the unit element in a ring, one may consider (central) idempotents in the ring. Similarly, the unitality condition required for a monad (F, μ, η) on any category was released (by G. Böhm et al.) to define *pre-monads* by imposing weaker requirements on η. Doing so, the adjointness of the free functor from $\mathbb{A}$ to the category of unital F-modules $\mathbb{A}_F$ and the forgetful functor is lost. In this paper we establish, for a premonad (F, μ, η), a weakened form of adjointness between the free functor from $\mathbb{A}$ to the category $\underline{\mathbb{A}}_F$ of *regular quasi-F-modules* with the forgetful functor.

For this we consider, for functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between any categories $\mathbb{A}$ and $\mathbb{B}$, an *adjunction context* given by maps

$$\begin{tikzcd}\mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[r, shift left, "\alpha"] & \mathrm{Mor}_{\mathbb{A}}(A, R(B)), \arrow[l, shift left, "\beta"]\end{tikzcd}$$

natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$. We call this a *regular adjunction context* if both α and β are regular, that is $\alpha = \alpha \circ \beta \circ \alpha$ and $\beta = \beta \circ \alpha \circ \beta$.

From this configuration we derive the notion of a *regular quasi-monad* and a *regular quasi-comonad* leading to *pre-units* and *pre-monads* (as considered by G. Böhm, J.N. Alonso Álvarez, and others). The notions allow to study the lifting of functors between categories to the corresponding categories of regular quasi-modules. Hereby also the notion of a *wreath product* between a monad F and an endofunctors T (in the sense of Lack and Street) can be extended to regular quasi-monads.

Along the way, the corresponding notions for *quasi-comonads* are formulated. The entwinings of regular quasi-monads and quasi-comonads considered in the final section provide the techniques to handle *weak bialgebras* and *weak Hopf algebras* on arbitrary categories but this aspect is not exploited in the present paper.

MSC: 18A40, 18C20, 16T15.

Keywords: Adjoint functors; weak (co)monads; adjunction contexts; regular quasi-(co)monads; quasi-(co)modules; lifting of functors; distributive laws.

Contents

1. Introduction

Among other needs, the investigation of weak Hopf-algebras (e.g. Böhm et al. [6], [3]) motivated the study of generalised forms of monads by weakening the unitality condition. This lead to *weak entwining structures* studied by Caenepeel and De Groot in [8] which were put in a more general context by Alonso Álvarez et al. [1] and eventually were interpreted in 2-categories in Böhm [4]. We do approach the questions behind from a different perspective thus attempting to gain a deeper understanding of these structures.

For functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between categories $\mathbb{A}$ and $\mathbb{B}$, we consider maps, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\mathrm{Mor}_{\mathbb{B}}(L(A), B) \overset{\alpha}{\underset{\beta}{\rightleftarrows}} \mathrm{Mor}_{\mathbb{A}}(A, R(B)),$$

requiring that α or β are *regular*, that is,

$$\alpha = \alpha \circ \beta \circ \alpha \quad \text{or} \quad \beta = \beta \circ \alpha \circ \beta.$$

Clearly this describes an adjunction provided α and β are inverse to each other. Thus our setting extends the theory of adjunctions and triples (as considered by Eilenberg an Moore in [9]) to more general pairs of functors.

In Section 3, a triple (F, μ, η) is named a *quasi-monad* on $\mathbb{A}$ provided $F : \mathbb{A} \to \mathbb{A}$ is an endofunctor with natural transformations $\mu : FF \to F$ and $\eta : I_{\mathbb{A}} \to F$ (*quasi-unit*) and the sole condition that μ is associative. *Quasi-F-modules* are defined by morphisms $\varrho : F(A) \to A$ which are compatible with the product μ of F, and the category of all quasi-F-modules is denoted by $\underset{\to}{\mathbb{A}}_F$. For these data the free and forgetful functors,

$$\phi_F : \mathbb{A} \to \underset{\to}{\mathbb{A}}_F \quad \text{and} \quad U_F : \underset{\to}{\mathbb{A}}_F \to \mathbb{A},$$

give rise to an adjunction context and the properties of the resulting α's and β's lead to the definition of η, μ, and (F, μ, η) to be regular, and eventually to the category $\underline{\mathbb{A}}_F$ of *regular quasi-F-modules*. For a regular quasi-monad (F, μ, η), the relation between $\mathbb{A}$ and $\underline{\mathbb{A}}_F$ yields a *regular* adjunction context and leads to a generalisation of *pre-units* and *pre-monads* (as considered by Alonso Álvarez, Böhm and others). Dual to the quasi-monads, in Section 4, *quasi-comonads* are introduced and the basic relationships are outlined. Examples for these are *weak corings* (from [19]) and *pre-A-corings* from [7] (see 4.15).

The notions allow to study the lifting of functors between categories to the corresponding categories of regular quasi-modules and this is done in Section 5. They are described by generalising Beck's distributive laws (see [2]), also called *entwinings*, and it turns out that most of the diagrams are the same as for the lifting to (proper) modules but to compensate the missing unitality extra conditions are imposed on the entwining (e.g. Proposition 5.2). Again we have a dual theory for quasi-comonads and this is the subject of Section 6.

Lifting an endofunctors T of $\mathbb{A}$ to an endofunctor $\overline{T}$ of $\underline{\mathbb{A}}_F$ leads to the question when $\overline{T}$ is a (regular) quasi-monad and in Section 7 we provide conditions to make this happen. Then TF allows for the structure of a regular quasi-monad (see 7.6). Hereby also the notion of a *wreath product* between a monad F and an endofunctors T (in the sense of Lack and Street [14]) can be extended to regular quasi-monads (see 7.7, 7.8). The corresponding questions for quasi-comonads are handled in Section 8.

The final Section 9 is concerned with a regular quasi-monad (F, μ, η) and a regular quasi-comonad (G, δ, ε) on any category $\mathbb{A}$ and the interplay between the respective lifting properties. Hereby properties of the lifting $\overline{G}$ to $\underline{\mathbb{A}}_F$ and the lifting $\widehat{F}$ to $\underline{\mathbb{A}}^G$ are investigated (see Theorems 9.9 and 9.10).

In case $F = G$ the results in the last section provide the basics for a theory of *weak bimonads* and *Hopf bimonads* on arbitrary categories. We will not persue the resulting questions here.

2. Adjunction contexts

Throughout $\mathbb{A}$ and $\mathbb{B}$ will denote arbitrary categories. By I_A, A or just by I, we denote the identity morphism of an object $A \in \mathbb{A}$, I_F or F stands for the identity on the funtor F, and $I_{\mathbb{A}}$ means the identity functor of a category $\mathbb{A}$. Recall that any covariant functor $F : \mathbb{A} \to \mathbb{B}$ induces a map

$$F_{A,A'} : \mathrm{Mor}_{\mathbb{A}}(A, A') \to \mathrm{Mor}_{\mathbb{B}}(F(A), F(A'))$$

which is natural in $A, A' \in \mathbb{A}$.

2.1. Regular morphism. Let A, A' be any objects in a category $\mathbb{A}$. Then a morphism $f : A \to A'$ is called *regular* provided there is a morphism $g : A' \to A$ with $fgf = f$. Clearly, in this case $gf : A \to A$ and $fg : A' \to A'$ are idempotent endomorphisms.

Such a morphism g is not necessarily unique. In particular, for gfg we also have $f(gfg)f = fgf = f$, and the identity $(gfg)f(gfg) = gfg$ shows that gfg is again a regular morphism.

We call (f, g) a *regular pair* of morphisms provided $fgf = f$ and $g = gfg$.

If idempotents split in $\mathbb{A}$, then every idempotent morphism $e : A \to A$ determines a subobject of A, we denote it by eA.

If f is regular with $fgf = f$, then the restriction of fg is the identity morphism on fgA' and gf is the identity on gfA.

Examples for regular morphisms are retractions, coretractions, and isomorphisms. For modules M, N over any ring, a morphism $f : M \to N$ is regular if and only if the image and the kernel of f are direct summands in N and M, respectively.

This notion of regularity is derived from von Neumann regularity of rings. For modules (and in preadditive categories) it was considered by Nicholson, Kasch, Mader and others (see [13]).

We use the terminology also for natural transformations and functors with obvious interpretations.

2.2. Adjunction context. Let $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ be covariant functors. Assume there are morphisms, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$,

$$\begin{aligned} \alpha_{A,B} &: \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)), \\ \beta_{A,B} &: \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \to \mathrm{Mor}_{\mathbb{B}}(L(A), B). \end{aligned}$$

These maps correspond to natural transformations α and β between the obvious functors $\mathbb{A}^{op} \times \mathbb{B} \to \mathsf{Set}$. The quadruple (L, R, α, β) is called an *adjunction context.*

2.3. Quasi-unit and quasi-counit. Given an adjunction context (L, R, α, β), the morphisms, for $A \in \mathbb{A}$, $B \in \mathbb{B}$,

$$\eta_A := \alpha_{A,L(A)}(I) : A \to RL(A) \quad \text{and} \quad \varepsilon_B := \beta_{R(B),B}(I) : LR(B) \to B$$

yield natural transformations

$$\eta : I_{\mathbb{A}} \to RL, \quad \varepsilon : LR \to I_{\mathbb{B}},$$

called *quasi-unit* and *quasi-counit* of (L, R, α, β), respectively.

By naturality, for $f : L(A) \to B$ and $g : A \to R(B)$, there are commutative diagrams

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{B}}(L(A), L(A)) \arrow[r, "\alpha_{A,L(A)}"] \arrow[d, "\mathrm{Mor}_{\mathbb{B}}(L(A),f)"'] & \mathrm{Mor}_{\mathbb{A}}(A, RL(A)) \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(A,R(f))"] \\
\mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[r, "\alpha_{A,B}"] & \mathrm{Mor}_{\mathbb{A}}(A, R(B)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(R(B), R(B))) \arrow[r, "\beta_{R(B),B}"] \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(g,R(B))"'] & \mathrm{Mor}_{\mathbb{B}}(LR(B), B) \arrow[d, "\mathrm{Mor}_{\mathbb{B}}(L(g),B)"] \\
\mathrm{Mor}_{\mathbb{A}}(A, R(B)) \arrow[r, "\beta_{A,B}"] & \mathrm{Mor}_{\mathbb{B}}(L(A), B),
\end{tikzcd}$$

which show that the transformations α and β are given by

$$\begin{aligned}
\alpha_{A,B} : \quad & L(A) \xrightarrow{f} B \quad \longmapsto \quad A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B), \\
\beta_{A,B} : \quad & A \xrightarrow{g} R(B) \quad \longmapsto \quad L(A) \xrightarrow{L(g)} LR(B) \xrightarrow{\varepsilon_B} B.
\end{aligned}$$

Naturality of ε and η induces an associative product on RL and a coassociative coproduct on LR,

$$R\varepsilon L : RLRL \to RL, \qquad L\eta L : LR \to LRLR.$$

2.4. Natural endomorphisms. *With the notions from 2.3, consider the natural transformations*

$$\begin{aligned}
\vartheta : \ & RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL, \quad & \underline{\vartheta} : \ & RL \xrightarrow{\eta RL} RLRL \xrightarrow{R\varepsilon L} RL, \\
\gamma : \ & LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR, \quad & \underline{\gamma} : \ & LR \xrightarrow{L\eta R} LRLR \xrightarrow{\varepsilon LR} LR.
\end{aligned}$$

(1) ϑ *respects left* RL*-action and* $\underline{\vartheta}$ *respects right* RL*-action, that is,*

$$R\varepsilon L \circ RL\vartheta = \vartheta \circ R\varepsilon L, \quad R\varepsilon L \circ \underline{\vartheta} RL = \underline{\vartheta} \circ R\varepsilon L.$$

(2) $\underline{\vartheta} \circ \vartheta = \vartheta \circ \underline{\vartheta}$.

(3) γ *respects left* LR*-coaction and* $\underline{\gamma}$ *respects right* LR*-coaction, that is,*

$$LR\gamma \circ L\eta R = L\eta R \circ \gamma, \quad \underline{\gamma} LR \circ L\eta R = L\eta R \circ \underline{\gamma}.$$

(4) $\underline{\gamma} \circ \gamma = \gamma \circ \underline{\gamma}$.

Proof. In the diagram

$$\begin{tikzcd}
RL \arrow[r, "RL\eta"] \arrow[d, "\eta RL"'] & RLRL \arrow[r, "R\varepsilon L"] \arrow[d, "\eta RLRL"] & RL \arrow[d, "\eta RL"] \\
RLRL \arrow[r, "RLRL\eta"] \arrow[d, "R\varepsilon L"'] & RLRLRL \arrow[r, "RLR\varepsilon L"] \arrow[d, "R\varepsilon LRL"] & RLRL \arrow[d, "R\varepsilon L"] \\
RL \arrow[r, "RL\eta"] & RLRL \arrow[r, "R\varepsilon L"] & RL
\end{tikzcd}$$

all partial rectangles are commutative by naturality.

The lower part shows that ϑ respects left RL-action and the right part shows that $\underline{\vartheta}$ respects right RL-action. The outer rectangle shows that ϑ and $\underline{\vartheta}$ commute.

Dual to the above, we have the commutative diagram

$$\begin{tikzcd}
LR \arrow[r, "L\eta R"] \arrow[d, "L\eta R"'] & LRLR \arrow[r, "\varepsilon LR"] \arrow[d, "LRL\eta R"] & LR \arrow[d, "L\eta R"] \\
LRLR \arrow[r, "L\eta RLR"] \arrow[d, "LR\varepsilon"'] & LRLRLR \arrow[r, "\varepsilon LRLR"] \arrow[d, "LRLR\varepsilon"] & LRLR \arrow[d, "LR\varepsilon"] \\
LR \arrow[r, "L\eta R"] & LRLR \arrow[r, "\varepsilon LR"] & LR.
\end{tikzcd}$$

From this the assertions (3) and (4) are derived. □

For later use we record some elementary computations.

2.5. Composing α and β. Let (L, R, α, β) be an adjunction context with quasi-unit η and quasi-counit ε. The descriptions of α and β in 2.3 yield, for the identity transformations $I_L : L \to L$, $I_R : R \to R$,

$$\begin{aligned}
\alpha(I_L) &= I_{\mathbb{A}} \xrightarrow{\eta} RL, \\
\beta \circ \alpha(I_L) &= L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L, \\
\alpha \circ \beta \circ \alpha(I_L) &= I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL, \\
\beta(I_R) &= LR \xrightarrow{\varepsilon} I_{\mathbb{B}}, \\
\alpha \circ \beta(I_R) &= R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R, \\
\beta \circ \alpha \circ \beta(I_R) &= LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR \xrightarrow{\varepsilon} I_{\mathbb{B}}.
\end{aligned}$$

As special cases of this setting we observe:

2.6. Adjoint pair of functors. Let (L, R, α, β) be an adjunction context with quasi-unit η and quasi-counit ε (see 2.2, 2.4).

(1) $\beta \circ \alpha = I_L$ if and only if $\varepsilon L \circ L\eta = I_L$.

(2) $\alpha \circ \beta = I_R$ if and only if $R\varepsilon \circ \eta R = I_R$.

(3) (L, R, α, β) is an adjunction if and only if $\beta \circ \alpha = I$ and $\alpha \circ \beta = I$ and this implies

$$R\varepsilon L \circ RL\eta = I_{RL} = R\varepsilon L \circ \eta RL, \quad LR\varepsilon \circ L\eta R = I_{LR} = \varepsilon LR \circ L\eta R.$$

We generalise adjoint pairs of functors by modifying the conditions on α and β.

2.7. α regular. *Let (L, R, α, β) be an adjunction context (see 2.2).*

(1) *The following are equivalent:*

  (a) $\alpha \circ \beta \circ \alpha = \alpha$*;*

  (b) η *induces commutativity of the diagram*

$$\begin{tikzcd}
I_{\mathbb{A}} \arrow[r, "\eta"] \arrow[d, "\eta"'] & RL \arrow[d, "\eta RL"] \\
RL & RLRL. \arrow[l, "R\varepsilon L"]
\end{tikzcd}$$

If these conditions hold, we say that α is *regular*, and then

(i) $\beta \circ \alpha(I_L) = L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L$ *is idempotent;*

(ii) ϑ *and* $\underline{\vartheta}$ *are idempotent and* $\vartheta \circ \eta = \eta = \underline{\vartheta} \circ \eta$.

(2) *The following are equivalent:*

(a) $R_{-,-} \circ \beta \circ \alpha = \alpha \circ \beta \circ R_{-,-}$, *that is commutativity of the diagram*

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{B}}(L(A),B) \arrow[r, "\alpha_{A,B}"] \arrow[d, "R_{L(A),B}"'] & \mathrm{Mor}_{\mathbb{A}}(A,R(B)) \arrow[r, "\beta_{A,B}"] & \mathrm{Mor}_{\mathbb{B}}(L(A),B) \arrow[d, "R_{L(A),B}"] \\
\mathrm{Mor}_{\mathbb{A}}(RL(A),R(B)) \arrow[r, "\beta_{RL(A),B}"] & \mathrm{Mor}_{\mathbb{B}}(LRL(A),B) \arrow[r, "\alpha_{RL(A),B}"] & \mathrm{Mor}_{\mathbb{A}}(RL(A),R(B));
\end{tikzcd}$$

(b) $\vartheta = \underline{\vartheta}$, *that is, commutativity of the diagram*

$$\begin{tikzcd}
RL \arrow[r, "RL\eta"] \arrow[d, "\eta RL"'] & RLRL \arrow[d, "R\varepsilon L"] \\
RLRL \arrow[r, "R\varepsilon L"'] & RL.
\end{tikzcd}$$

If these conditions are satisfied we say that α is *symmetric.*

(3) *If* α *is regular and symmetric, then* ϑ *respects the product of* RL *(in fact, is a quasi-monad morphism, see 3.2).*

Proof. (1) (a)⇔(b) This follows from the list in 2.5.

(i) can be seen from the commutative diagram

$$\begin{tikzcd}
L \arrow[r, "L\eta"] \arrow[drr, "L\eta"'] & LRL \arrow[r, "\varepsilon L"] \arrow[d, "LRL\eta"'] & L \arrow[d, "L\eta"] \\
 & LRLRL \arrow[r, "\varepsilon LRL"] \arrow[d, "LR\varepsilon L"] & LRL \arrow[d, "\varepsilon L"] \\
 & LRL \arrow[r, "\varepsilon L"'] & L.
\end{tikzcd}$$

(ii) The idempotency of ϑ follows from (i).
The idempotency of $\underline{\vartheta}$ follows from the commutative diagram

$$\begin{tikzcd}
RL \arrow[d, "\eta RL"'] \arrow[drr, "\eta RL"] & & \\
RLRL \arrow[r, "\eta RLRL"] \arrow[d, "R\varepsilon L"'] & RLRLRL \arrow[r, "R\varepsilon LRL"'] \arrow[d, "RLR\varepsilon L"'] & RLRL \arrow[d, "R\varepsilon L"] \\
RL \arrow[r, "\eta RL"'] & RLRL \arrow[r, "R\varepsilon L"'] & RL.
\end{tikzcd}$$

(2) (a)⇒(b) Applying R to $\beta \circ \alpha(I_L)$ (see 2.5) yields

$$RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL,$$

and $\alpha \circ \beta(I_{RL})$ produces the sequence

$$RL \xrightarrow{\eta RL} RLRL \xrightarrow{R\varepsilon L} RL.$$

(b)⇒(a) follows from the fact that α is defined by η.

(3) If α is symmetric, ϑ respects left and right action of RL, that is, we have the commutative diagram

$$\begin{tikzcd}
RLRL \arrow[r, "RL\vartheta"] \arrow[d, "R\varepsilon L"'] & RLRL \arrow[r, "\vartheta RL"] \arrow[d, "R\varepsilon L"'] & RLRL \arrow[d, "R\varepsilon L"'] \\
RL \arrow[r, "\vartheta"] & RL \arrow[r, "\vartheta"] & RL.
\end{tikzcd}$$

Now, by regularity of α, ϑ is idempotent and hence the diagram tells us

$$R\varepsilon L \circ \vartheta\vartheta = \vartheta \circ R\varepsilon L,$$

that is, ϑ respects the product on RL. □

For regular α we have the following criterion for symmetry:

2.8. Proposition. *Let* (L, R, α, β) *be an adjunction context with* α *regular. Then the following are equivalent:*

(a) α *is symmetric (i.e.* $\vartheta = \underline{\vartheta}$*, see 2.4);*

(b) ϑ *and* $\underline{\vartheta}$ *both respect left and right* RL*-action.*

Proof. (a)⇒(b) is obvious.

(b)⇒(a) Assume ϑ to respect right RL-action, that is, commutativity of the rectangle in the diagram

$$\begin{tikzcd}
RL \arrow[r, "\eta RL"] & RLRL \arrow[r, "RL\eta RL"] \arrow[d, "R\varepsilon L"'] & RLRLRL \arrow[r, "R\varepsilon LRL"] & RLRL \arrow[d, "R\varepsilon"] \\
 & RL \arrow[r, "RL\eta"] & RLRL \arrow[r, "R\varepsilon L"] & RL.
\end{tikzcd}$$

Since α is regular, the top sequence yields ηRL. Thus the diagram shows the equality $\vartheta \circ \underline{\vartheta} = \underline{\vartheta}$.

If $\underline{\vartheta}$ respects left RL-action we obtain a similar diagram leading to $\underline{\vartheta} \circ \vartheta = \vartheta$. Since ϑ and $\underline{\vartheta}$ commute we conclude $\vartheta = \underline{\vartheta}$. □

2.9. β **regular.** *Let* (L, R, α, β) *be an adjunction context (see 2.2).*

(1) *The following are equivalent:*

  (a) $\beta \circ \alpha \circ \beta = \beta$*;*

  (b) ε *induces commutativity of the diagram*

$$\begin{tikzcd}
LR \arrow[r, "\varepsilon"] \arrow[d, "L\eta R"'] & I_{\mathbb{B}} \\
LRLR \arrow[r, "\varepsilon LR"'] & LR. \arrow[u, "\varepsilon"']
\end{tikzcd}$$

If these conditions hold, we say that β is *regular,* and then

  (i) $\alpha \circ \beta(I_R) = R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R$ *is idempotent.*

(ii) *γ and $\underline{\gamma}$ (see 2.4) are idempotent and $\varepsilon \circ \gamma = \varepsilon = \varepsilon \circ \underline{\gamma}$.*

(2) *The following are equivalent:*

(a) *$L_{-,-} \circ \alpha \circ \beta = \beta \circ \alpha \circ L_{-,-}$, that is, commutativity of the diagram*

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{A}}(A, R(B)) \arrow[r, "\beta_{A,B}"] \arrow[d, "L_{A,R(B)}"'] & \mathrm{Mor}_{\mathbb{B}}(L(A), B) \arrow[r, "\alpha_{A,B}"] & \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \arrow[d, "L_{A,R(B)}"] \\
\mathrm{Mor}_{\mathbb{B}}(L(A), LR(B)) \arrow[r, "\alpha_{A,R(B)}"] & \mathrm{Mor}_{\mathbb{A}}(A, RLR(B)) \arrow[r, "\beta_{A,LR(B)}"] & \mathrm{Mor}_{\mathbb{B}}(L(A), LR(B));
\end{tikzcd}$$

(b) *$\gamma = \underline{\gamma}$, that is, commutativity of the diagram*

$$\begin{tikzcd}
LR \arrow[r, "L\eta R"] \arrow[d, "L\eta R"'] & LRLR \arrow[d, "LR\varepsilon"] \\
LRLR \arrow[r, "\varepsilon LR"'] & LR.
\end{tikzcd}$$

If these conditions hold we say that β is *symmetric.*

(1) *If β is regular and symmetric, then γ respects the coproduct of LR (in fact, is a quasi-comonad morphism, see 4.2).*

Proof. (dual to 2.7) (1) (a)⇔(b) follows from the list in 2.5.

(i) can be seen from the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\eta R"] \arrow[d, "\eta R"'] & RLR \arrow[r, "R\varepsilon"] \arrow[d, "\eta RLR"] & R \arrow[d, "\eta R"] \\
RLR \arrow[r, "RL\eta R"] \arrow[drr, "R\varepsilon"'] & RLRLR \arrow[r, "LRL\varepsilon"] & RLR \arrow[d, "R\varepsilon"] \\
 & & R.
\end{tikzcd}$$

(ii) $\gamma = L(k)$ and hence is idempotent by (i).

The idempotency of $\underline{\gamma}$ is seen from the commutative diagram

$$\begin{tikzcd}
LR \arrow[r, "L\eta R"] \arrow[d, "L\eta R"'] & LRLR \arrow[d, "L\eta LRL"'] \arrow[ddr, "\varepsilon LR"] & \\
LRLR \arrow[r, "LRL\eta R"] \arrow[d, "\varepsilon LR"'] & LRLRLR \arrow[d, "\varepsilon LRLR"'] & \\
LR \arrow[r, "L\eta R"'] & LRLR \arrow[r, "\varepsilon LR"'] & LR.
\end{tikzcd}$$

(3) By symmetry of β, γ respects left and right coactions of LR, so we have the commutative diagram

$$\begin{tikzcd}
LR \arrow[r, "\gamma"] \arrow[d, "L\eta R"'] & LR \arrow[r, "\gamma"] \arrow[d, "L\eta R"'] & LR \arrow[d, "L\eta R"'] \\
LRLR \arrow[r, "LR\gamma"] & LRLR \arrow[r, "\gamma LR"] & LRLR.
\end{tikzcd}$$

By regularity of β, γ is idempotent and hence we see from the diagram

$$L\eta R \circ \gamma = \gamma\gamma \circ L\eta R,$$

that is, γ respects the coproduct of LR. □

If γ is regular, we have the following criterion for symmetry:

2.10. Proposition. *Let (L, R, α, β) be an adjunction context with β regular. Then the following are equivalent:*

(a) *β is symmetric (i.e. $\gamma = \underline{\gamma}$, see 2.4);*

(b) *γ and $\underline{\gamma}$ both respect left and right LR-coaction.*

Proof. The statements and the proofs are dual to 2.8. □

2.11. Definition. We call an adjunction context (L, R, α, β) *regular* if both α and β are regular and call it *symmetric* if they are both symmetric (see 2.7, 2.9).

Any adjunction context with one of the maps regular can be transferred to a regular context.

2.12. Proposition. *Let (L, R, α, β) be an adjunction context.*

(1) *If α is regular, then, for $\beta' = \beta \circ \alpha \circ \beta$, (L, R, α, β') is a regular adjunction context. For $A \in \mathbb{A}$ and $B \in \mathbb{B}$,*

$$\begin{array}{rcl} \beta' : \mathrm{Mor}_{\mathbb{A}}(A, R(B)) & \to & \mathrm{Mor}_{\mathbb{B}}(L(A), B), \\ R \xrightarrow{I_R} R & \mapsto & LR \xrightarrow{L\eta R} LRLR \xrightarrow{LR\varepsilon} LR \xrightarrow{\varepsilon} I_{\mathbb{B}}. \end{array}$$

(2) *If β is regular, then, for $\alpha' = \alpha \circ \beta \circ \alpha$, (L, R, α', β) is a regular adjunction context. For $A \in \mathbb{A}$ and $B \in \mathbb{B}$,*

$$\begin{array}{rcl} \alpha' : \mathrm{Mor}_{\mathbb{B}}(L(A), B) & \to & \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \\ L \xrightarrow{I_L} L & \mapsto & I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{RL\eta} RLRL \xrightarrow{R\varepsilon L} RL. \end{array}$$

Proof. The assertions are easily verified. The values of the maps β' and α' can be seen from the list in 2.5. □

For an adjoint pair (L, R) of functors, there are well-known bijections between the classes of natural transformations $\mathrm{Nat}(L, L)$, $\mathrm{Nat}(R, R)$, $\mathrm{Nat}(I_{\mathbb{A}}, RL)$ and $\mathrm{Nat}(LR, I_{\mathbb{B}})$. The maps providing these connections can also be defined for any adjunction context but they do not lead to bijections. We pick out two pairs of them.

2.13. Related natural transformations. *Let (L, R, α, β) be a regular adjunction context. Then we get the following pairs of regular maps:*

(i) $\mathrm{Nat}(L, L) \to \mathrm{Nat}(R, R)$, $\quad s \mapsto R \xrightarrow{\eta R} RLR \xrightarrow{RsR} RLR \xrightarrow{R\varepsilon} R$,
$\mathrm{Nat}(R, R) \to \mathrm{Nat}(L, L)$, $\quad t \mapsto L \xrightarrow{L\eta} LRL \xrightarrow{LtL} LRL \xrightarrow{\varepsilon L} L$.

(ii) $\mathrm{Nat}(I_{\mathbb{A}}, RL) \to \mathrm{Nat}(R, R)$, $\quad h \mapsto R \xrightarrow{hR} RLR \xrightarrow{R\varepsilon} R$,
$\mathrm{Nat}(R, R) \to \mathrm{Nat}(I_{\mathbb{A}}, RL)$, $\quad k \mapsto I_{\mathbb{A}} \xrightarrow{\eta} RL \xrightarrow{kL} RL$.

Proof. The assertions can be shown by straightforward computations. □

2.14. Special cases. Let (L, R, α, β) be an adjunction context.

(i) If $\beta \circ \alpha = I$, then $\beta \circ \alpha \circ \beta = \beta$ and $\alpha \circ \beta \circ \alpha = \alpha$, that is, α and β are regular.

(ii) Similarly, $\alpha \circ \beta = I$ implies that α and β are regular. This case is considered in Medvedev [15] and L, R are then called *semiadjoint functors*.

(iii) In [17, 3.1], (L, R) is said to be a *rational pairing* if $\beta_{A,B} : \mathrm{Mor}_{\mathbb{A}}(A, R(B)) \to \mathrm{Mor}_{\mathbb{B}}(L(A), B)$ is injective for all $A \in \mathbb{A}$, $b \in \mathbb{B}$. If, in addition, β is regular, then clearly $\alpha \circ \beta = I$.

For categories and natural transformations allowing certain constructions, we can relate regular adjunction contexts with proper adjunctions. Note that the conditions employed are satisfied provided idempotents split in the respective categories.

2.15. Relation to semiadjoint functors. *Let (L, R, α, β) be an adjunction context with quasi-unit η and quasi-counit ε.*

(1) *Let α be regular and suppose that the idempotent natural transformation $h : L \xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L$ splits, that is, there are a functor $\widehat{L} : \mathbb{A} \to \mathbb{B}$ and natural transformations*

$$\widehat{p} : L \to \widehat{L}, \quad \widehat{i} : \widehat{L} \to L \quad \text{with} \quad \widehat{i} \circ \widehat{p} = h \quad \text{and} \quad \widehat{p} \circ \widehat{i} = I_{\widehat{L}}.$$

Then the natural transformations

$$\widehat{\eta} : \ I_{\mathbb{A}} \xrightarrow{\ \eta\ } RL \xrightarrow{\ R\widehat{p}\ } R\widehat{L}\ , \quad \widehat{\varepsilon} : \ \widehat{L}R \xrightarrow{\ \widehat{i}R\ } LR \xrightarrow{\ \varepsilon\ } I_{\mathbb{B}}$$

as quasi-unit and quasi-counit, define an adjunction context $(\widehat{L}, R, \widehat{\alpha}, \widehat{\beta})$ with $\widehat{\beta} \circ \widehat{\alpha} = I_{\widehat{L}}$, where for $A \in \mathbb{A}$ and $B \in \mathbb{B}$, the maps are given by

$$\begin{aligned}
\widehat{\alpha}_{A,B} : \quad & \widehat{L}(A) \xrightarrow{f} B & \longmapsto \quad & A \xrightarrow{\widehat{\eta}_A} R\widehat{L}(A) \xrightarrow{R(f)} R(B), \\
\widehat{\beta}_{A,B} : \quad & A \xrightarrow{g} R(B) & \longmapsto \quad & \widehat{L}(A) \xrightarrow{\widehat{L}(g)} \widehat{L}R(B) \xrightarrow{\widehat{\varepsilon}_B} B.
\end{aligned}$$

If α is symmetric then so is $\widehat{\alpha}$.

(2) *Let β be regular and suppose that the idempotent natural transformation $k : R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R$ splits, that is, there are a functor $\widetilde{R} : \mathbb{A} \to \mathbb{B}$ and natural transformations*

$$\widetilde{p} : R \to \widetilde{R}, \quad \widetilde{i} : \widetilde{R} \to R \quad \text{with} \quad \widetilde{i} \circ \widetilde{p} = k \text{ and } \widetilde{p} \circ \widetilde{i} = I_{\widetilde{R}}.$$

Then the natural transformations

$$\widetilde{\eta} : \ I_{\mathbb{A}} \xrightarrow{\ \eta\ } RL \xrightarrow{\ \widetilde{p}L\ } \widetilde{R}L\ , \quad \widetilde{\varepsilon} : \ L\widetilde{R} \xrightarrow{\ L\widetilde{i}\ } LR \xrightarrow{\ \varepsilon\ } I_{\mathbb{B}}$$

as quasi-unit and quasi-counit, define an adjunction context $(L, \widetilde{R}, \widetilde{\alpha}, \widetilde{\beta})$ with $\widetilde{\alpha} \circ \widetilde{\beta} = I_{\widetilde{R}}$, where for $A \in \mathbb{A}$ and $B \in \mathbb{B}$, the maps are given by

$$\begin{aligned}
\widetilde{\alpha}_{A,B} : \quad & L(A) \xrightarrow{f} B & \longmapsto \quad & A \xrightarrow{\widetilde{\eta}_A} \widetilde{R}L(A) \xrightarrow{\widetilde{R}(f)} \widetilde{R}(B), \\
\widetilde{\beta}_{A,B} : \quad & A \xrightarrow{g} \widetilde{R}(B) & \longmapsto \quad & L(A) \xrightarrow{L(g)} L\widetilde{R}(B) \xrightarrow{\widetilde{\varepsilon}_B} B.
\end{aligned}$$

If β is symmetric then so is $\widetilde{\beta}$.

Proof. (1) In view of the properties of $\widehat{i}$ and $\widehat{p}$, the commutative diagram

$$\begin{tikzcd}
\widehat{L} \arrow[r, "\widehat{L}\eta"] \arrow[d, "\widehat{i}"'] & \widehat{L}RL \arrow[r, "\widehat{L}R\widehat{p}"] \arrow[d, "\widehat{i}RL"'] & \widehat{L}R\widehat{L} \arrow[d, "\widehat{i}R\widehat{L}"] \\
L \arrow[r, "L\eta"] \arrow[d, "\widehat{p}"'] & LRL \arrow[r, "LR\widehat{p}"] \arrow[d, "\varepsilon L"'] & LR\widehat{L} \arrow[d, "\varepsilon\widehat{L}"] \\
\widehat{L} \arrow[r, "\widehat{i}"] & L \arrow[r, "\widehat{p}"] & \widehat{L}
\end{tikzcd}$$

implies $\widehat{\varepsilon}\widehat{L} \circ \widehat{L}\widehat{\eta} = I_{\widehat{L}}$.

An easy computation shows that the symmetry of α implies that of $\widehat{\alpha}$.

(2) In view of the properties of $\widetilde{i}$ and $\widetilde{p}$, the commutative diagram (dual to that in (1))

$$\begin{tikzcd}
\widetilde{R} \arrow[r, "\eta\widetilde{R}"] \arrow[d, "\widetilde{i}"'] & RL\widetilde{R} \arrow[r, "\widetilde{p}L\widetilde{R}"] \arrow[d, "RL\widetilde{i}"'] & \widetilde{R}L\widetilde{R} \arrow[d, "\widetilde{R}L\widetilde{i}"] \\
R \arrow[r, "\eta R"] \arrow[d, "\widetilde{p}"'] & RLR \arrow[r, "\widetilde{p}LR"] \arrow[d, "R\varepsilon"'] & \widetilde{R}LR \arrow[d, "\widetilde{R}\varepsilon"] \\
\widetilde{R} \arrow[r, "\widetilde{i}"] & R \arrow[r, "\widetilde{p}"] & \widetilde{R}
\end{tikzcd}$$

implies $\widetilde{R}\widetilde{\varepsilon} \circ \widetilde{\eta}\widetilde{R} = I_{\widetilde{R}}$.

Again it is straightforward to show that $\widetilde{\beta}$ is symmetric provided β is so. □

So far we have modified the functors to have new adjunction contexts for the same categories. We may also modify the categories to relate an adjunction context with a proper adjunction.

2.16. Related adjoint functors. *Let (L, R, α, β) be a regular and symmetric adjunction context. Denote by $\tilde{\mathbb{A}}$, $\tilde{\mathbb{B}}$ the full subcategories of $\mathbb{A}$ and $\mathbb{B}$, respectively, with*

$$\mathrm{Obj}(\tilde{\mathbb{A}}) = \{A \in \mathrm{Obj}(\mathbb{A}) \mid L(A) \xrightarrow{L\eta_A} LRL(A) \xrightarrow{\varepsilon L_A} L(A) = I_{L(A)}\},$$
$$\mathrm{Obj}(\tilde{\mathbb{B}}) = \{B \in \mathrm{Obj}(\mathbb{B}) \mid R(B) \xrightarrow{\eta R_B} RLR(B) \xrightarrow{R\varepsilon_B} R(B) = I_{R(B)}\}.$$

Then restriction and corestriction of L and R yield functors

$$\tilde{L} : \tilde{\mathbb{A}} \to \tilde{\mathbb{B}}, \quad \tilde{R} : \tilde{\mathbb{B}} \to \tilde{\mathbb{A}},$$

and $(\tilde{L}, \tilde{R})$ is an adjoint pair of functors.

Proof. For every $A \in \tilde{\mathbb{A}}$, we see that

$$RL(A) \xrightarrow{RL\eta_A} RLRL(A) \xrightarrow{R\varepsilon L_A} RL(A).$$

is the identity. By the symmetry of α, this implies $L(A) \in \mathrm{Obj}(\tilde{\mathbb{B}})$.

Similarly, for $B \in \mathrm{Obj}(\tilde{\mathbb{B}})$, we derive that

$$LR(B) \xrightarrow{L\eta R_B} LRLR(B) \xrightarrow{LR\varepsilon_B} LR(B)$$

is the identity map and by symmetry of β, this implies $R(B) \in \mathrm{Obj}(\tilde{\mathbb{A}})$.

From the identities in 2.5 one easily sees that $\alpha \circ \beta(I_{\tilde{R}}) = I_{\tilde{R}}$ for any $B \in \tilde{\mathbb{B}}$ and $\beta \circ \alpha(I_{\tilde{L}}) = I_{\tilde{L}}$. This shows that $(\tilde{L}, \tilde{R})$ is an adjoint pair of functors. □

3. Quasi-monads

Monads F on any category $\mathbb{A}$ are characterised by the fact that they induce a free functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ which is left adjoint to the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$, where $\mathbb{A}_F$ denotes the category of (unital) F-modules. In this section we consider, for endofunctors F, a category of *quasi-modules* which allows for an adjunction context and we study the interplay between properties of this context and the monad properties. Throughout $\mathbb{A}$ and $\mathbb{B}$ denote any categories.

3.1. Quasi-monads. A triple (F, μ, η) is called a *quasi-monad* on $\mathbb{A}$ provided $F : \mathbb{A} \to \mathbb{A}$ is an endofunctor with natural transformations $\mu : FF \to F$ and $\eta : I_{\mathbb{A}} \to F$ where μ is associative. μ is called the *product* and η the *quasi-unit* of this quasi-monad. They (always) define natural transformations

$$\vartheta : F \xrightarrow{F\eta} FF \xrightarrow{\mu} F, \quad \underline{\vartheta} : F \xrightarrow{\eta F} FF \xrightarrow{\mu} F.$$

3.2. Morphisms of quasi-monads. Given two quasi-monads (F, μ, η), (F', μ', η') on $\mathbb{A}$, a natural transformation $h : F \to F'$ is called a *morphism of quasi-monads* if it induces commutativity of the diagrams

$$\begin{tikzcd} FF \arrow[r, "hh"] \arrow[d, "\mu"'] & F'F' \arrow[d, "\mu'"] \\ F \arrow[r, "h"] & F', \end{tikzcd} \qquad \begin{tikzcd} I_{\mathbb{A}} \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & F \arrow[d, "h"] \\ & F'. \end{tikzcd}$$

Similar to the situation for monads, quasi-monads are in close relation to adjunction contexts. For this we define:

3.3. Quasi-modules. Let F be an endofunctor on $\mathbb{A}$ and $\mu : FF \to F$ an associative natural transformation. A *quasi-F-module* is an object $A \in \mathbb{A}$ with a morphism $\varrho : F(A) \to A$ inducing commutativity of the left hand diagram

$$\begin{tikzcd} FF(A) \arrow[r, "F\varrho"] \arrow[d, "\mu_A"'] & F(A) \arrow[d, "\varrho"] \\ F(A) \arrow[r, "\varrho"] & A, \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\varrho"'] & F(A') \arrow[d, "\varrho'"] \\ A \arrow[r, "f"] & A', \end{tikzcd}$$

F-module morphisms between F-quasi-modules (A, ϱ), (A', ϱ') are $\mathbb{A}$-morphisms $f : A \to A'$ for which the right hand diagram is commutative and the set of all these is denoted by $\mathrm{Mor}_F(A, A')$. With these morphisms, quasi-F-modules form a category which we denote by $\underset{\to}{\mathbb{A}}_F$.

By the associativity condition on μ, for every $A \in \mathbb{A}$, $F(A)$ is a quasi-F-module.

The data considered above lead to an adjunction context generalising the Eilenberg-Moore construction.

3.4. Quasi-monads and adjunction contexts. *Let (F, μ, η) be a quasi-monad. Then the free functor*

$$\phi_F : \mathbb{A} \to \underset{\to}{\mathbb{A}}_F, \quad A \mapsto (F(A), \mu_A : FF(A) \to F(A)),$$

and the forgetful functor

$$U_F : \underset{\rightarrow}{\mathbb{A}}_F \to \mathbb{A}, \quad (A, \varrho) \mapsto A,$$

form an adjunction context $(\phi_F, U_F, \alpha_F, \beta_F)$ *with the maps*

$$\begin{aligned} \alpha_F &: \mathrm{Mor}_F(F(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)), \quad f \mapsto f \circ \eta_A, \\ \beta_F &: \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)) \to \mathrm{Mor}_F(F(A), B), \quad g \mapsto \varrho \circ F(g), \end{aligned}$$

where $A \in \mathbb{A}$ *and* $(B, \varrho) \in \underset{\rightarrow}{\mathbb{A}}_F$.

A first example for quasi-monads is given by

3.5. Adjunction contexts and quasi-monads. *Let* $L : \mathbb{A} \to \mathbb{B}$, $R : \mathbb{B} \to \mathbb{A}$ *be functors forming an adjunction context* (L, R, α, β) *with quasi-unit* η *and quasi-counit* ε *(see 2.3).*

(i) $(RL, R\varepsilon L, \eta)$ *is a quasi-monad.*

(ii) *There is a (comparison) functor*

$$K : \mathbb{B} \to \mathbb{A}_{RL}, \quad B \mapsto (R(B), R\varepsilon : RLR(B) \to R(B)),$$

inducing commutativity of the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "L"] \arrow[dr, "\phi_{RL}"'] & \mathbb{B} \arrow[r, "R"] \arrow[d, "K"] & \mathbb{A} \\ & \underset{\rightarrow}{\mathbb{A}}_{RL} \arrow[ur, "U_{RL}"'] & \end{tikzcd}$$

Proof. This follows essentially from 2.3. □

For convenience we record some values of the compositions of α_F and β_F.

3.6. Composing α_F and β_F. Let (F, μ, η) be a quasi-monad. Then the values of α_F and β_F in 3.4 on identity transformations yield, for $A \in \mathbb{A}$, $(B, \varrho) \in \underset{\rightarrow}{\mathbb{A}}_F$:

$$\begin{aligned} \alpha_F(I_{F(A)}) &= A \xrightarrow{\eta_A} F(A), \\ \beta_F \circ \alpha_F(I_{F(A)}) &= F(A) \xrightarrow{F\eta_A} FF(A) \xrightarrow{\mu_A} F(A), \\ \alpha_F \circ \beta_F \circ \alpha_F(I_{F(A)}) &= A \xrightarrow{\eta_A} F(A) \xrightarrow{F\eta_A} FF(A) \xrightarrow{\mu_A} F(A), \\ \beta_F(I_{U_F(B)}) &= F(B) \xrightarrow{\varrho} B, \\ \alpha_F \circ \beta_F(I_{U_F(B)}) &= B \xrightarrow{\eta_B} F(B) \xrightarrow{\varrho} B, \\ \beta_F \circ \alpha_F \circ \beta_F(I_{U_F(B)}) &= F(B) \xrightarrow{F\eta_B} FF(B) \xrightarrow{\mu_B} F(B) \xrightarrow{\varrho} B. \end{aligned}$$

3.7. Monads and adjunctions. *Let* (F, μ, η) *be a quasi-monad with related adjunction context* $(\phi_F, U_F, \alpha_F, \beta_F)$. *The following are equivalent:*

(a) $\beta_F \circ \alpha_F = I$ *and* $\alpha_F \circ \beta_F = I$;

(b) (F, μ, η) *is a monad;*

(c) $\phi_F : \mathbb{A} \to \mathbb{A}_F$, $U_F : \mathbb{A}_F \to \mathbb{A}$ *is an adjunction, where* $\mathbb{A}_F$ *denotes the subcategory of unital* F*-modules of* $\underset{\rightarrow}{\mathbb{A}}_F$.

Proof. These assertions are well-known. □

3.8. Definitions. Let (F, μ, η) be a quasi-monad. Then we call

$$\begin{array}{rll}
\eta \textit{ regular} & \text{if} & I_{\mathbb{A}} \xrightarrow{\eta} F = I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{F\eta} FF \xrightarrow{\mu} F; \\
\eta \textit{ symmetric} & \text{if} & F \xrightarrow{F\eta} FF \xrightarrow{\mu} F = F \xrightarrow{\eta F} FF \xrightarrow{\mu} F; \\
\mu \textit{ regular} & \text{if} & FF \xrightarrow{\mu} F = FF \xrightarrow{F\eta F} FFF \xrightarrow{\mu F} FF \xrightarrow{\mu} F; \\
\mu \textit{ symmetric} & \text{if} & FF \xrightarrow{F\eta F} FFF \xrightarrow{F\mu} FF = FF \xrightarrow{F\eta F} FFF \xrightarrow{\mu F} FF; \\
(F, \mu, \eta) \textit{ regular} & \text{if} & \eta \text{ and } \mu \text{ are regular;} \\
(F, \mu, \eta) \textit{ symmetric} & \text{if} & \eta \text{ and } \mu \text{ are symmetric.}
\end{array}$$

In [10, Definition 2.3], the quasi-unit η is called a *preunit* provided it is regular and symmetric. In [4, Definition 2.1], (F, μ, η) is called a *premonad* provided it is regular and η is symmetric. In both papers, under the assumptions that idempotent morphisms split, adjoint functors are related to the quasi-monads under consideration (similar to the constructions in 2.15).

From the observations in 2.7 we obtain:

3.9. Properties of regular quasi-units. *Let (F, μ, η) be a quasi-monad with related adjunction context $(\phi_F, U_F, \alpha_F, \beta_F)$ (see 3.4).*

1. *η is regular if and only if α_F is regular.*
2. *If η is regular, then*
   - (i) *$\vartheta : F \xrightarrow{F\eta} FF \xrightarrow{\mu} F$ and $\underline{\vartheta} : F \xrightarrow{\eta F} FF \xrightarrow{\mu} F$ are idempotent;*
   - (ii) *$\vartheta \circ \eta = \eta = \underline{\vartheta} \circ \eta$.*
3. *η is symmetric if and only if α_F is symmetric.*
4. *If η is regular and symmetric, then ϑ is an idempotent quasi-monad morphism.*

Notice that in 3.9 no (additional) conditions on the quasi-F-modules are imposed. On the other hand, to get an adjunction for a monad F (see 3.7) we had to refer to a subcategory (of unital modules) of $\underset{\to}{\mathbb{A}}_F$. A similar procedure can be applied under more general conditions.

3.10. Regular quasi-modules. Let (F, μ, η) be a quasi-monad. A quasi-F-module (B, φ) is called

$$\begin{array}{rll}
\textit{regular} & \text{if} & F(B) \xrightarrow{\varphi} B = F(B) \xrightarrow{F\eta_B} FF(B) \xrightarrow{\mu_B} F(B) \xrightarrow{\varphi} B, \\
\textit{symmetric} & \text{if} & F(B) \xrightarrow{F\eta_B} FF(B) \xrightarrow{F\varphi} F(B) = F(B) \xrightarrow{F\eta_B} FF(B) \xrightarrow{\mu_B} F(B).
\end{array}$$

With $\vartheta = \mu \circ F\eta$ (see 3.1), these conditions can be written as

$$\varphi = \varphi \circ \vartheta_B, \quad F\varphi \circ F\eta_B = \vartheta_B.$$

We denote by $\underline{\mathbb{A}}_F$ the full subcategory of $\underset{\to}{\mathbb{A}}_F$ whose objects are regular quasi-F-modules.

- (i) Clearly, $(F(A), \mu_A)$ is a regular (symmetric) quasi-F-module for all $A \in \mathbb{A}$ if and only if the product μ is regular (symmetric).
- (ii) If μ is regular, then with $\underline{\vartheta} = \mu \circ \eta F$ (see 3.1),

$$FF \xrightarrow{\vartheta F} FF \xrightarrow{\mu} F = FF \xrightarrow{\mu} F = FF \xrightarrow{F\vartheta} FF \xrightarrow{\mu} F.$$

(iii) If μ is regular and η is symmetric, then for any $(A,\varphi)\in\underline{\mathbb{A}}_F$,

$$F(A)\xrightarrow{\varphi}A = F(A)\xrightarrow{\varphi}A\xrightarrow{\eta_A}F(A)\xrightarrow{\varphi}A.$$

Assertion (iii) follows from the commutative diagram

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_{F(A)}"] \arrow[d, "\varphi"'] & FF(A) \arrow[r, "\mu_A"] \arrow[d, "F\varphi"'] & F(A) \arrow[d, "\varphi"] \\
A \arrow[r, "\eta_A"] & F(A) \arrow[r, "\varphi"] & A.
\end{tikzcd}$$

As an easy consequence of the definitions we mention that, for any (proper) monad (F,μ,η), all quasi-F-modules are regular and symmetric (but not unital).

3.11. Regular quasi-monads and adjunction contexts. *Let (F,μ,η) be a regular quasi-monad.*

(1) *The (obvious) free and forgetful functors*

$$\phi_F:\mathbb{A}\to\underline{\mathbb{A}}_F,\quad U_F:\underline{\mathbb{A}}_F\to\mathbb{A},$$

form a regular adjunction context $(\phi_F,U_F,\alpha_F,\beta_F)$.

(2) *If η is symmetric, then the quasi-monad morphism $\vartheta:F\to F$ induces the identity functor on $\underline{\mathbb{A}}_F$.*

Proof. (1) is obvious from the observations in 3.9 and 3.10.

(2) The quasi-monad morphism ϑ transfers any quasi-module $\varphi:F(A)\to A$ to $F(A)\xrightarrow{\vartheta}F(A)\xrightarrow{\varphi}A$ which – by regularity – is equal to $\varphi:F(A)\to A$. □

If μ or η is regular, the other one can be modified to be also regular.

3.12. Proposition. *Let (F,μ,η) be a quasi-monad.*

(1) *If η is regular (see 3.8), then, for*

$$\tilde{\mu}:FF\xrightarrow{F\eta F}FFF\xrightarrow{F\mu}FF\xrightarrow{\mu}F,$$

$(F,\tilde{\mu},\eta)$ is a regular quasi-monad.

(2) *If μ is regular, then, for*

$$\tilde{\eta}:I_{\mathbb{A}}\xrightarrow{\eta}F\xrightarrow{F\eta}FF\xrightarrow{\mu}F,$$

$(F,\mu,\tilde{\eta})$ is a regular quasi-monad.

(3) *If (F,μ,η) is a regular quasi-monad, then for*

$$\hat{\mu}:FF\xrightarrow{\eta FF\eta}FFFF\xrightarrow{\mu FF}FFF\xrightarrow{\mu F}FF\xrightarrow{\mu}F,$$

$(F,\hat{\mu},\eta)$ is a regular quasi-monad with η symmetric.

Proof. (1) and (2) follow from Proposition 2.12, assertion (3) can be easily verified. □

As a special case we consider quasi-monads on the category ${}_R\mathbb{M}$ of modules over a commutative ring R with unit.

3.13. Quasi-algebras. A *quasi-algebra* (A,m,u) is an R-module A with associative multiplication $m:A\otimes_R A\to A$ and R-linear map $u:R\to A$. Putting $e:=u(1_R)\in A$ we have:

(1) u is regular if and only if $e = u(1_R)$ is an idempotent in A.

(2) u is regular and symmetric if and only if e is a central idempotent (then Ae is a unital R-subalgebra of A).

(3) μ is regular if and only if $ab = aeb$ for all $a, b \in A$.

(4) μ is symmetric if and only if $A \otimes_R eA = Ae \otimes_R A$.

(5) If u is regular, then $\tilde{m}(a \otimes b) = aeb$, for $a, b \in A$, defines a regular quasi-algebra $(A, \tilde{m}, u)$.

(6) If u is regular, then $\hat{m}(a \otimes b) = eaebe$, for $a, b \in A$, defines a regular quasi-algebra $(A, \tilde{m}, u)$ with u symmetric.

Clearly, the quasi-algebras (A, m, u) over R correspond to the quasi-monads $(A \otimes_R -, m \otimes -, u \otimes -)$ on ${}_R\mathbb{M}$ and thus we get:

3.14. Quasi-modules. Let (A, m, u) be a regular quasi-algebra over R. For the category ${}_A\underline{\mathbb{M}}$ of regular quasi-A-modules, the free functor

$$\phi_A : {}_R\mathbb{M} \to {}_A\underline{\mathbb{M}}, \quad X \mapsto (A \otimes_R X, m_A \otimes I_X),$$

together with the forgetful functor $U_A : {}_A\underline{\mathbb{M}} \to {}_R\mathbb{M}$ yield a regular adjunction context $(\phi_A, U_A, \alpha_A, \beta_A)$ with the maps, for $X \in {}_R\mathbb{M}$, $(M, \rho) \in {}_A\underline{\mathbb{M}}$,

$$\begin{aligned} \alpha_A &: \mathrm{Mor}_{\mathbb{A}}(A \otimes_R X, M) \to \mathrm{Mor}_R(X, M), \quad f \mapsto f \circ (u \otimes A), \\ \beta_A &: \mathrm{Mor}_R(X, M) \to \mathrm{Mor}_{\mathbb{A}}(A \otimes_R X, M), \quad g \mapsto \rho \circ (A \otimes g). \end{aligned}$$

3.15. Quasi-monads acting on functors. Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and (G, μ', η') a quasi-monad on $\mathbb{B}$. We call T a left *quasi-G-module* if there exists a natural transformation $\varrho : GT \to T$ such that

$$GGT \xrightarrow{G\varrho} GT \xrightarrow{\varrho} T = GGT \xrightarrow{\mu' T} GT \xrightarrow{\varrho} T,$$

and we call it a *regular quasi-G-module* if in addition

$$GT \xrightarrow{\varrho} T = GT \xrightarrow{G\eta'} GGT \xrightarrow{\mu' T} GT \xrightarrow{\varrho} T.$$

Note that the quasi-monad G may be seen as quasi-monad on the category of functors $\mathbb{A} \to \mathbb{B}$ and the (regular) quasi-G-module T is a (regular) quasi-module for this quasi-monad.

3.16. Proposition. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and (G, μ', η') a regular quasi-monad on $\mathbb{B}$. Then there is a functor $\overline{T} : \mathbb{A} \to \underline{\mathbb{B}}_G$ with commutative diagram*

$$\begin{tikzcd} & \underline{\mathbb{B}}_G \arrow[d, "U_G"] \\ \mathbb{A} \arrow[ur, "\overline{T}"] \arrow[r, "T"'] & \mathbb{B} \end{tikzcd}$$

if and only if T is a regular quasi-G-module.

Proof. Given T as a regular quasi-G-module with $\varrho : GT \to T$ the natural transformation, the functor

$$\overline{T} : \mathbb{A} \to \underline{\mathbb{B}}_G, \quad A \;\mapsto\; (T(A), \varrho_A : GT(A) \to T(A))$$

has the required property.

Now assume there exists a functor $\overline{T}$ making the diagram commutative. Then for $A \in \mathbb{A}$, the are morphisms $\rho_A : GT(A) \to T(A)$ and they define a natural transformation $\rho : GT \to T$. For this we have to show that, for any morphism $f : A \to \widehat{A}$, the middle rectangle is commutative in the diagram

$$\begin{tikzcd}
GGT(A) \arrow[rr, "\mu'_{T(A)}"] & & GT(A) \arrow[dl, "\rho_A"'] \arrow[ddd, "GT(f)"] \\
GT(A) \arrow[u, "G\eta' T_A"] \arrow[r, "\rho_A"] \arrow[d, "GT(f)"'] & T(A) \arrow[d, "T(f)"] & \\
GT(\widehat{A}) \arrow[r, "\rho_{\widehat{A}}"] \arrow[d, "G\eta' T_{\widehat{A}}"'] & T(\widehat{A}) & \\
GGT(\widehat{A}) \arrow[rr, "\mu'_{T(\widehat{A})}"] & & GT(\widehat{A}). \arrow[ul, "\rho_{\widehat{A}}"]
\end{tikzcd}$$

The top and bottom diagrams are commutative by regularity of the quasi-G-modules, and the right trapezium is commutative since $T(f)$ is an G-morphism. Thus the inner diagram is commutative showing naturality of ρ. $\square$

For an easy example of the notion introduced in Proposition 3.16, observe that for any regular quasi-monad (G, μ', η'), G is a regular quasi-G-module.

4. QUASI-COMONADS

Having seen how to extend the theory of monads to quasi-monads, it is quite obvious how a similar step is to be done for quasi-comonads. Recall that a comonad G on any category $\mathbb{A}$ induces a free functor $\phi^G : \mathbb{A} \to \mathbb{A}^G$ which is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$, where $\mathbb{A}^G$ denotes the category of (counital) G-comodules. Again $\mathbb{A}$ denotes any category.

4.1. Quasi-comonads. A triple (G, δ, ε) is called a *quasi-comonad* on $\mathbb{A}$ provided $G : \mathbb{A} \to \mathbb{A}$ is an endofunctor with natural transformations $\delta : G \to GG$ and $\varepsilon : G \to I_{\mathbb{A}}$ where δ is co-associative. δ is called the *coproduct* and ε the *quasi-counit* of this quasi-comonad. They always define natural transformations

$$\gamma : G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G, \quad \underline{\gamma} : G \xrightarrow{\delta} GG \xrightarrow{\varepsilon G} G.$$

4.2. Morphisms of quasi-comonads. Given two quasi-monads (G, δ, ε) and $(G', \delta', \varepsilon')$ on $\mathbb{A}$, a natural transformation $k : G \to G'$ is called a *morphism of quasi-comonads* if it induces commutativity of the diagrams

$$\begin{tikzcd}
G \arrow[r, "k"] \arrow[d, "\delta"'] & G' \arrow[d, "\delta'"] \\
GG \arrow[r, "kk"] & G'G',
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\varepsilon"] \arrow[d, "k"'] & I_{\mathbb{A}} \\
G' \arrow[ur, "\varepsilon'"'] & .
\end{tikzcd}$$

Similar to the situation for comonads, quasi-comonads are in close relation to adjunction contexts. For this we define:

4.3. Quasi-comodules. Let G be an endofunctor on $\mathbb{A}$ and $\delta : G \to GG$ a co-associative natural transformation. A *quasi-G-comodule* is an object $A \in \mathbb{A}$ with a morphism $\upsilon : A \to G(A)$ such that

$$A \xrightarrow{\upsilon} G(A) \xrightarrow{G\upsilon} GG(A) = A \xrightarrow{\upsilon} G(A) \xrightarrow{\delta} GG(A).$$

G-comodule morphisms between quasi-G-comodules (A, υ), (A', υ') are morphisms $g : A \to A'$ with

$$A \xrightarrow{g} A' \xrightarrow{\upsilon'} G(A') = A \xrightarrow{\upsilon} G(A) \xrightarrow{G(g)} G(A')$$

and the set of all these is denoted by $\mathrm{Mor}^G(A, A')$. With these morphisms, quasi-G-comodules form a category which we denote by $\underrightarrow{\mathbb{A}}^G$.

By the co-associativity condition on δ, for every $A \in \mathbb{A}$, $G(A)$ is a quasi-G-module.

4.4. Quasi-comonads and adjunction contexts. *Let (G, δ, ε) be a quasi-comonad. Then the (cofree) functor*

$$\phi^G : \mathbb{A} \to \underrightarrow{\mathbb{A}}^G, \quad A \;\mapsto\; (G(A), G(A) \xrightarrow{\delta_A} GG(A)),$$

and the forgetful functor

$$U^G : \underrightarrow{\mathbb{A}}^G \to \mathbb{A}, \quad (A, \rho^A) \mapsto A,$$

form an adjunction context $(U^G, \phi^G, \alpha_G, \beta_G)$ where, for $A \in \mathbb{A}$ and $(B, \upsilon) \in \underrightarrow{\mathbb{A}}^G$,

$$\alpha^G : \mathrm{Mor}_{\mathbb{A}}(U^G(B), A) \to \mathrm{Mor}^G(B, G(A)),\; B \xrightarrow{f} A \;\mapsto\; B \xrightarrow{\upsilon} G(B) \xrightarrow{G(f)} G(A),$$

$$\beta^G : \mathrm{Mor}^G(B, G(A)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(B), A),\; B \xrightarrow{g} G(A) \;\mapsto\; G(B) \xrightarrow{G(g)} G(A) \xrightarrow{\varepsilon_A} A.$$

Proof. All assertions are easily derived from the definitions (dual to 3.4). □

As an interesting (motivating) example for comonads we obtain:

4.5. Adjunction contexts and quasi-comonads. *Let (L, R, α, β) be an adjunction context between the categories $\mathbb{A}$ and $\mathbb{B}$ with quasi-unit η and quasi-counit ε (see 2.3). Then:*

(i) *$(LR, L\eta R, \varepsilon)$ is a quasi-comonad.*

(ii) *There is a (comparison) functor*

$$\widetilde{K} : \mathbb{A} \to \underrightarrow{\mathbb{B}}^{LR}, \quad A \mapsto (L(A), L\eta : L(A) \to LRL(A)),$$

inducing commutativity of the diagram

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "R"] \arrow[dr, "\phi^{LR}"'] & \mathbb{A} \arrow[r, "L"] \arrow[d, "\widetilde{K}"] & \mathbb{B} \\
& \underrightarrow{\mathbb{B}}^{LR} \arrow[ur, "U^{LR}"'] &
\end{tikzcd}$$

Proof. This follows essentially from 2.3. □

For convenience we record some values of the compositions of α_G and β_G.

4.6. Composing α^G and β^G. Let (G,δ,ε) be a quasi-comonad. Then the values of α^G and β^G in 4.4 on the identity transformations yield, for $(B,\upsilon)\in \underrightarrow{\mathbb{A}}^G$,

$$\begin{aligned}
\alpha^G(I_{U^G(B)}) &= B \xrightarrow{\upsilon} G(B),\\
\beta^G\circ\alpha^G(I_{U^G(B)}) &= B \xrightarrow{\upsilon} G(B) \xrightarrow{\varepsilon_B} B,\\
\alpha^G\circ\beta^G\circ\alpha^G(I_{U^G(B)}) &= B \xrightarrow{\upsilon} G(B) \xrightarrow{\delta_B} GG(B) \xrightarrow{G\varepsilon_B} G(B),\\
\beta^G(I_G) &= G \xrightarrow{\varepsilon} I_{\mathbb{A}},\\
\alpha^G\circ\beta^G(I_G) &= G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G,\\
\beta^G\circ\alpha^G\circ\beta^G(I_G) &= G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G \xrightarrow{\varepsilon} I_{\mathbb{A}}.
\end{aligned}$$

4.7. Comonads and adjunctions. *Let (G,δ,ε) be a quasi-comonad with related adjunction context $(U^G,\phi^G,\alpha^G,\beta^G)$. The following are equivalent:*

(a) *α^G is invertible with invers β^G;*

(b) *(G,δ,ε) is a comonad;*

(c) *$U^G:\mathbb{A}^G\to\mathbb{A}$, $\phi^G:\mathbb{A}\to\mathbb{A}^G$ is an adjunction, where $\mathbb{A}^G$ denotes the subcategory of counital G-comodules of $\underrightarrow{\mathbb{A}}^G$.*

Proof. These are well-known characterisations of comonads. □

4.8. Definitions. Let (G,δ,ε) be a quasi-comonad with related adjunction context $(U^G,\phi^G,\alpha^G,\beta^G)$. Then we call

$$\begin{array}{rll}
\varepsilon\ \textit{regular} & \text{if} & G\xrightarrow{\varepsilon} I_{\mathbb{A}} = G\xrightarrow{\delta} GG\xrightarrow{G\varepsilon} G\xrightarrow{\varepsilon} I_{\mathbb{A}};\\
\varepsilon\ \textit{symmetric} & \text{if} & G\xrightarrow{\delta} GG\xrightarrow{G\varepsilon} G = G\xrightarrow{\delta} GG\xrightarrow{\varepsilon G} G;\\
\delta\ \textit{regular} & \text{if} & G\xrightarrow{\delta} GG = G\xrightarrow{\delta} GG\xrightarrow{\delta G} GGG\xrightarrow{G\varepsilon G} GG;\\
\delta\ \textit{symmetric} & \text{if} & GG\xrightarrow{G\delta} GGG\xrightarrow{G\varepsilon G} GG = GG\xrightarrow{\delta G} GGG\xrightarrow{G\varepsilon G} GG;\\
(G,\delta,\varepsilon)\ \textit{regular} & \text{if} & \varepsilon \text{ and } \delta \text{ are regular};\\
(G,\delta,\varepsilon)\ \textit{symmetric} & \text{if} & \varepsilon \text{ and } \delta \text{ are symmetric.}
\end{array}$$

In [10, Definition A.3], the quasi-counit ε is called a *pre-counit* provided it is regular and symmetric.

The observations in 2.9 read here as follows.

4.9. Properties of regular quasi-counits. *Let (G,δ,ε) be a quasi-comonad with related adjunction context $(U^G,\phi^G,\alpha^G,\beta^G)$ (see 4.4). Then:*

(1) *ε is regular if and only if β^G is regular.*

(2) *If ε is regular, then*

 (i) *$\gamma: G\xrightarrow{\delta} GG\xrightarrow{G\varepsilon} G$ and $\underline{\gamma}: G\xrightarrow{\delta} GG\xrightarrow{\varepsilon G} G$ are idempotent;*

 (ii) *$\varepsilon\circ\gamma=\varepsilon=\varepsilon\circ\underline{\gamma}$.*

(3) *ε is symmetric if and only if β^G is symmetric.*

(4) *If ε is regular and symmetric, then γ is an idempotent quasi-comonad morphism.*

Similar to the case of quasi-modules (see 3.9), in 4.9 no (additional) conditions on the quasi-G-comodules are imposed. To get an adjunction context with better properties we have to select a subcategory of $\underrightarrow{\mathbb{A}}^G$.

4.10. Regular quasi-comodules. Let (G, δ, ε) be a quasi-comonad. A quasi-G-comodule (B, υ) is called

$$\begin{aligned} &\textit{regular} \quad \text{if} \quad B \xrightarrow{\upsilon} G(B) = B \xrightarrow{\upsilon} G(B) \xrightarrow{\delta_B} GG(B) \xrightarrow{G\varepsilon_B} G(B); \\ &\textit{symmetric} \quad \text{if} \quad G(B) \xrightarrow{G\upsilon} GG(B) \xrightarrow{G\varepsilon_B} G(B) = G(B) \xrightarrow{\delta_B} GG(B) \xrightarrow{G\varepsilon_B} G(B). \end{aligned}$$

With $\gamma = G\varepsilon \circ \delta$ (see 4.1) this conditions are written as

$$\upsilon = \gamma_B \circ \upsilon, \quad G\varepsilon_B \circ G\upsilon = \gamma_B.$$

We denote by $\underline{\mathbb{A}}^G$ the full subcategory of $\underset{\rightarrow}{\mathbb{A}}^G$ whose objects are regular quasi-G-comodules.

(i) Clearly, $(G(A), \delta_A)$ is a regular (symmetric) quasi-G-comodule for each $A \in \mathbb{A}$ if and only if the product δ is regular (symmetric).

(ii) If δ is regular, then with $\underline{\gamma} = \varepsilon G \circ \delta$,

$$G \xrightarrow{\delta} GG \xrightarrow{\underline{\gamma} G} GG = G \xrightarrow{\delta} GG = G \xrightarrow{\delta} GG \xrightarrow{G\underline{\gamma}} GG.$$

(iii) If δ is regular and ε is symmetric, then for any $(B, \upsilon) \in \underline{\mathbb{A}}^G$,

$$B \xrightarrow{\upsilon} G(B) = B \xrightarrow{\upsilon} G(B) \xrightarrow{\varepsilon} B \xrightarrow{\upsilon} G(B).$$

Similar to the situation for quasi-modules, for any (proper) comonad (G, δ, ε), all quasi-comodules are regular and symmetric.

4.11. Regular quasi-comonads and adjunction contexts. *Let (G, δ, ε) be a regular quasi-comonad.*

(1) *The (obvious) cofree and forgetful functors*

$$\phi^G : \mathbb{A} \to \underline{\mathbb{A}}^G, \quad U^G : \underline{\mathbb{A}}^G \to \mathbb{A},$$

form a regular adjunction context $(U^G, \phi^G, \alpha^G, \beta^G)$.

(2) *If ε is symmetric, then the quasi-comonad morphism $\gamma : G \to G$ induces the identity functor on $\underline{\mathbb{A}}^G$.*

Proof. In view of 4.9 and 4.10, the proof is dual to that of 3.11. □

If δ or ε is regular, the other one can be modified to be regular, too.

4.12. Proposition. *Let (G, δ, ε) be a quasi-comonad with related adjunction context $(U^G, \phi^G, \alpha^G, \beta^G)$.*

(1) *If ε is regular (see 4.9), then, for*

$$\tilde{\delta} : G \xrightarrow{\delta} GG \xrightarrow{G\delta} GGG \xrightarrow{G\varepsilon G} GG,$$

$(G, \tilde{\delta}, \varepsilon)$ is a regular quasi-comonad.

(2) *If δ is regular, then, for*

$$\tilde{\varepsilon} : G \xrightarrow{\delta} GG \xrightarrow{G\varepsilon} G \xrightarrow{\varepsilon} I_{\mathbb{A}},$$

$(G, \delta, \tilde{\varepsilon})$ is a regular quasi-comonad.

(3) *If (G, δ, ε) be a regular quasi-comonad, then, for*

$$\hat{\delta} : G \xrightarrow{\delta} GG \xrightarrow{G\delta} GGG \xrightarrow{GG\delta} GGGG \xrightarrow{\varepsilon GG\varepsilon} GG,$$

$(G, \hat{\delta}, \varepsilon)$ is a regular quasi-comonad with ε symmetric.

Proof. (dual to Proposition 3.12) (1) and (2) follow from Proposition 2.12, and assertion (3) can be directly verified. □

As a special case we consider quasi-comonads on the category ${}_R\mathbb{M}$ of modules over a commutative ring R with unit.

4.13. Quasi-coalgebras. A *quasi-coalgebra* (C, Δ, ε) is an R-module C with R-linear maps $\Delta : C \to C \otimes_R C$ and $\varepsilon : C \to R$, where the comultiplication Δ is coassociative. Writing for $c \in C$, $\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}$ we have:

1. ε is regular if and only if for any $c \in C$, $\varepsilon(c) = \sum \varepsilon(c_{\underline{1}})\varepsilon(c_{\underline{2}})$.
2. ε is symmetric if and only if $\sum c_{\underline{1}}\varepsilon(c_{\underline{2}}) = \sum \varepsilon(c_{\underline{1}})c_{\underline{2}}$.
3. Δ is regular if and only if $\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}\varepsilon(c_{\underline{3}})$.
4. Δ is symmetric if and only if $\sum c \otimes \varepsilon(d_{\underline{1}})d_{\underline{2}} = \sum c_{\underline{1}}\varepsilon(c_{\underline{2}}) \otimes d$.
5. If ε is regular, then $\tilde{\Delta}(c) := \sum c_{\underline{1}} \otimes \varepsilon(c_{\underline{2}})c_{\underline{3}}$ defines a regular quasi-coalgebra $(C, \tilde{\Delta}, \varepsilon)$.
6. If (C, Δ, ε) is a regular quasi-comonad, then $\hat{\Delta}(c) := \sum \varepsilon(c_{\underline{1}})c_{\underline{2}} \otimes c_{\underline{3}}\varepsilon(c_{\underline{4}})$ defines a regular quasi-coalgebra $(C, \hat{\Delta}, \varepsilon)$ with ε symmetric.

Clearly, the quasi-coalgebras (C, Δ, ε) over R correspond to the quasi-comonads $(C \otimes_R -, \Delta \otimes -, \varepsilon \otimes -)$ on ${}_R\mathbb{M}$ and thus we get:

4.14. Quasi-comodules. Let (C, Δ, ε) be a regular quasi-coalgebra over R. For the category ${}^C\underline{\mathbb{M}}$ of regular left quasi-C-comodules, the cofree functor

$$\phi^C : {}_R\mathbb{M} \to {}^C\underline{\mathbb{M}}, \quad X \mapsto (C \otimes_R X, \Delta \otimes I_X),$$

together with the forgetful functor $U^C : {}^C\underline{\mathbb{M}} \to {}_R\mathbb{M}$ yield a regular adjunction context $(U^C, \phi^C, \alpha^C, \beta^C)$ with the maps, for $X \in {}_R\mathbb{M}$, $(M, \upsilon) \in {}^C\underline{\mathbb{M}}$,

$$\begin{aligned} \alpha^C &: \mathrm{Mor}_R(M, X) \to \mathrm{Mor}^C(M, C \otimes_R X), \quad f \mapsto (C \otimes f) \circ \upsilon, \\ \beta^C &: \mathrm{Mor}^C(M, C \otimes_R X) \to \mathrm{Mor}_R(M, X), \quad g \mapsto (\varepsilon \otimes I_X) \circ (C \otimes g). \end{aligned}$$

4.15. Weak corings and pre-A-corings. Let A be a ring with unit 1_A and $\mathcal{C}$ a quasi-(A, A)-bimodule which is unital as right A-module. Assume there are given (A, A)-bilinear maps

$$\underline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}, \quad \underline{\varepsilon} : \mathcal{C} \to A,$$

where $\underline{\Delta}$ is coassociative.

$(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ is called a *right unital weak A-coring* in [19], provided for all $c \in \mathcal{C}$,

$$(\underline{\varepsilon} \otimes I_{\mathcal{C}}) \circ \underline{\Delta}(c) = 1_A \cdot c = (I_{\mathcal{C}} \otimes \underline{\varepsilon}) \circ \underline{\Delta}(c),$$

which reads in (obvious) Sweedler notation as

$$\sum \underline{\varepsilon}(c_{\underline{1}})c_{\underline{2}} = 1_A \cdot c = \sum c_{\underline{1}}\underline{\varepsilon}(c_{\underline{2}}).$$

From the equations

$$\begin{aligned} (I_{\mathcal{C}} \otimes \underline{\varepsilon} \otimes I_{\mathcal{C}}) \circ (I_{\mathcal{C}} \otimes \underline{\Delta}) \circ \underline{\Delta}(c) &= \textstyle\sum c_{\underline{1}} \otimes 1_A \cdot c_{\underline{2}} \;=\; \sum c_{\underline{1}} \otimes c_{\underline{2}} \;=\; \underline{\Delta}(c), \\ (I_{\mathcal{C}} \otimes \underline{\varepsilon} \otimes I_{\mathcal{C}}) \circ (\underline{\Delta} \otimes I_{\mathcal{C}}) \circ \underline{\Delta}(c) &= \textstyle\sum 1_A \cdot c_{\underline{1}} \otimes c_{\underline{2}} \;=\; 1_A \cdot \underline{\Delta}(c), \end{aligned}$$

it follows by coassociativity that $1_A \cdot \underline{\Delta}(c) = \underline{\Delta}(c)$. Summarising we see that, in this case, $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ is a regular and symmetric quasi-comonad on the category ${}_A\underrightarrow{\mathbb{M}}$ of left quasi-A-modules ($={}_A\underline{\mathbb{M}}$ since A has a unit).

$(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ is called an *A-pre-coring* in [7, Section 6], if

$$(\underline{\varepsilon} \otimes I_{\mathcal{C}}) \circ \underline{\Delta}(c) = c, \quad (I_{\mathcal{C}} \otimes \underline{\varepsilon}) \circ \underline{\Delta}(c) = 1_A \cdot c,$$

which reads in Sweedler notation as

$$c = \sum \underline{\varepsilon}(c_{\underline{1}})c_{\underline{2}}, \quad 1_A \cdot c = \sum c_{\underline{1}}\underline{\varepsilon}(c_{\underline{2}}).$$

Similar to the computation above we obtain that $1_A \cdot \underline{\Delta}(c) = \underline{\Delta}(c)$. Now $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ is a regular quasi-comonad on ${}_A\underrightarrow{\mathbb{M}}$ but neither $\underline{\varepsilon}$ nor $\underline{\Delta}$ are symmetric.

Notice that in both cases considered above, restriction and corestriction of $\underline{\Delta}$ and $\underline{\varepsilon}$ yield an *A-coring* $(A\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ (e.g. [19, Proposition 1.3], compare also 2.16).

Dual to 3.15, the notion of comodule functors (as considered in [16, 3.3]) can be extended to

4.16. Quasi-comonads acting on functors. Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and (G, δ, ε) a quasi-comonad on $\mathbb{B}$. We call T a left *quasi-G-comodule* if there exists a natural transformation $\upsilon : T \to GT$ such that

$$T \xrightarrow{\upsilon} GT \xrightarrow{\upsilon G} GGT = T \xrightarrow{\upsilon T} GT \xrightarrow{\delta} GGT,$$

and we call it a *regular quasi-G-comodule* if in addition

$$T \xrightarrow{\upsilon} GT = T \xrightarrow{\upsilon} GT \xrightarrow{\delta} GGT \xrightarrow{G\varepsilon} GT.$$

A quasi-comonad G may be seen as quasi-comonad on the category of functors $\mathbb{A} \to \mathbb{B}$ and the (regular) quasi-G-comodule T is a (regular) quasi-comodule for this quasi-monad.

4.17. Proposition. *Let $T : \mathbb{A} \to \mathbb{B}$ be a functor and (G, δ, ε) a regular quasi-comonad on $\mathbb{B}$. Then there is a functor $\overline{T} : \mathbb{A} \to \underline{\mathbb{B}}^G$ with commutative diagram*

$$\begin{tikzcd}
& \underline{\mathbb{B}}^G \arrow[d, "U^G"] \\
\mathbb{A} \arrow[ru, "\overline{T}"] \arrow[r, "T"'] & \mathbb{B}
\end{tikzcd}$$

if and only if T is a regular quasi-G-comodule.

Proof. The proof is dual to that of 3.15. □

5. Entwinings with quasi-monads

5.1. Lifting of functors to quasi-modules. Let (F, μ, η) and (G, μ', η') be quasi-monads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively. Denote by $\underrightarrow{\mathbb{A}}_F$, $\underrightarrow{\mathbb{B}}_G$ the categories of the corresponding quasi-modules and by $\underline{\mathbb{A}}_F$, $\underline{\mathbb{B}}_G$ the categories of the regular quasi-modules provided the quasi-monads are regular (see 3.3). Given functors

$$T : \mathbb{A} \to \mathbb{B}, \quad \overrightarrow{T} : \underrightarrow{\mathbb{A}}_F \to \underrightarrow{\mathbb{B}}_G, \quad \overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{B}}_G$$

we say that $\overrightarrow{T}$ or $\overline{T}$ *is a lifting of* T provided the corresponding diagram

(5.1)
$$\begin{tikzcd}
\underrightarrow{\mathbb{A}}_F \arrow[r, "\overrightarrow{T}"] \arrow[d, "U_F"'] & \underrightarrow{\mathbb{B}}_G \arrow[d, "U_G"] & \text{or} & \underline{\mathbb{A}}_F \arrow[r, "\overline{T}"] \arrow[d, "U_F"'] & \underline{\mathbb{B}}_G \arrow[d, "U_G"] \\
\mathbb{A} \arrow[r, "T"] & \mathbb{B} & & \mathbb{A} \arrow[r, "T"] & \mathbb{B}
\end{tikzcd}$$

is commutative, where the U's denote the forgetful functors (see 3.4).

The natural transformations $\vartheta = \mu \circ F\eta : F \to F$ and $\vartheta' = \mu' \circ G\eta' : G \to G$ are quasi-module morphism (see 3.1) and we put

$$\kappa := T\vartheta : TF \to TF.$$

5.2. Proposition. *With the data given in* 5.1, *consider the pair of functors* TF, $GT : \mathbb{A} \to \mathbb{B}$ *and a natural transformation* $\lambda : GT \to TF$. *The quasi-F-module* (F, μ) *induces a G-action on* TF,

$$\chi : GTF \xrightarrow{\lambda F} TFF \xrightarrow{T\mu} TF.$$

(1) *If* (TF, χ) *is a quasi-G-module, then we get the commutative diagram*

$$\begin{tikzcd}
GGT \arrow[r, "G\lambda"] \arrow[d, "\mu' T"'] & GTF \arrow[r, "G\kappa"] & GTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu"] \\
GT \arrow[rr, "\lambda"] & & TF \arrow[r, "\kappa"] & TF.
\end{tikzcd} \tag{5.2}$$

(2) *If* G *is regular and* (TF, χ) *is a regular quasi-G-module, then we have*

$$GT \xrightarrow{\vartheta' T} GT \xrightarrow{\lambda} TF \xrightarrow{\kappa} TF \;=\; GT \xrightarrow{\lambda} TF \xrightarrow{\kappa} TF. \tag{5.3}$$

(3) *If* F *is regular and* (A, φ) *is a regular F-module, then in the diagram*

$$\begin{tikzcd}
GTF(A) \arrow[r, "\lambda F_A"] & TFF(A) \arrow[r, "TF\varphi"] \arrow[d, shift left, "T\mu_A"] & TF(A) \arrow[d, "T\varphi"] \\
GT(A) \arrow[u, "GT\eta_A"] \arrow[r, "\lambda_A"] & TF(A) \arrow[r, "T\varphi"] \arrow[u, shift left, "TF\eta"] & T(A).
\end{tikzcd} \tag{5.4}$$

the outer paths commute and

$$T\varphi \circ \lambda_A = T\varphi \circ \lambda_A \circ GT\varphi \circ GT\eta_A. \tag{5.5}$$

Proof. (1) To make T a left quasi-G-module, associativity of the G-action is required, that is, commutativity of the inner rectangle in the diagram

$$\begin{tikzcd}
GGT \arrow[rr, "G\lambda"] \arrow[dr, "GGT\eta"] \arrow[ddd, "\mu' T"'] & & GTF \arrow[d, "GTF\eta"'] \arrow[dr, "G\kappa"] & \\
& GGTF \arrow[r, "G\lambda F"'] \arrow[dd, "\mu' TF"] & GTFF \arrow[r, "GT\mu"'] & GTF \arrow[d, "\lambda F"] \\
& & & TFF \arrow[d, "T\mu"] \\
GT \arrow[r, "GT\eta"] \arrow[drr, "\lambda"'] & GTF \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu"] & TF \\
& & TF \arrow[u, "TF\eta"] \arrow[ur, "\kappa"'] & .
\end{tikzcd}$$

The other inner diagrams are commutative by functoriality of composition or definition and hence the outer paths yields commutativity of the diagram (5.2).

(2) The regularity condition for the quasi-G-module structure (see 3.10) is commutativity of the inner rectangle in the diagram

$$\begin{tikzcd}
GGT \arrow[rr, "\mu' T"] \arrow[dr, "GGT\eta"] & & GT \arrow[r, "\lambda"] \arrow[d, "GT\eta"'] & TF \arrow[d, "TF\eta"] \\
& GGTF \arrow[r, "\mu' TF"'] & GTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu"] \\
& GTF \arrow[u, "G\eta' TF"] \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu"] & TF \\
GT \arrow[uuu, "G\eta' T"] \arrow[ur, "GT\eta"] \arrow[rr, "\lambda"] & & TF \arrow[u, "TF\eta"] \arrow[ur, "\kappa"'] & ,
\end{tikzcd}$$

while the other subdiagrams are commutative by naturality or definition. Now, by the definition of ϑ', the outer commutative diagram is just equation (5.3).

(3) Commutativity of the partial diagrams in (5.4) is clear by naturality and the definition of quasi-F-modules. Commutativity of the outer diagram follows from regularity of φ, that is, $\varphi = \varphi \circ \mu_A \circ F\eta_A$. Now the final equation is a consequence of the equality $\lambda_A \circ GT\varphi = TF\varphi \circ \lambda_{F(A)}$. □

5.3. Proposition. *Let* (F, μ, η) *and* (G, μ', η') *be regular quasi-monads on the categories* $\mathbb{A}$ *and* $\mathbb{B}$*, respectively, and* $T : \mathbb{A} \to \mathbb{B}$ *any functor. Then a natural transformation* $\lambda : GT \to TF$ *induces a lifting*

$$\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{B}}_G, \quad (A, \varphi) \mapsto (T(A), T\varphi \circ \lambda_A : GT(A) \to T(A))$$

to the regular modules if and only if the diagram (5.2) *is commutative and equation* (5.3) *holds.*

Proof. The necessity of the conditions follows from Proposition 5.2.

Now assume the diagrams addressed to be commutative. Let $\varphi : F(A) \to A$ be a regular quasi-F-module, that is, $\varphi \circ \vartheta_A = \varphi$ and $T\varphi \circ \kappa_A = T\varphi$.

Attaching F to the commutative diagram (5.2) and applying regularity of μ yields the commutative diagram

$$\begin{tikzcd}
GGTF \arrow[r, "G\lambda F"] \arrow[dd, "\mu' TF"'] & GTFF \arrow[r, "GT\vartheta F"] \arrow[dr, "GT\mu"'] & GTFF \arrow[r, "\lambda FF"] \arrow[d, "GT\mu"] & TFFF \arrow[d, "T\mu F"] \\
& & GTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu"] \\
GTF \arrow[rr, "\lambda F"] & & TFF \arrow[ur, "T\vartheta F"] \arrow[r, "T\mu"'] & TF.
\end{tikzcd}$$

From this we get commutativity of the heptagon in the diagram

$$\begin{tikzcd}
GGT(A) \arrow[rr, "G\lambda"] \arrow[dr, "GGT\eta"] \arrow[ddd, "\mu' T"'] & & GTF(A) \arrow[r, "G\kappa"] \arrow[d, "GTF\eta"'] & GTF(A) \arrow[r, "GT\varphi"] \arrow[d, "\lambda F"] & GT(A) \arrow[d, "\lambda"] \\
& GGTF(A) \arrow[r, "G\lambda F"'] \arrow[d, "\mu' TF(A)"'] & GTFF(A) \arrow[ur, "GT\mu"] & TFF(A) \arrow[r, "TF\varphi"] \arrow[d, "T\mu"] & TF(A) \arrow[d, "T\varphi"] \\
& GTF(A) \arrow[r, "\lambda F"] & TFF(A) \arrow[r, "T\mu"] & TF(A) \arrow[r, "T\varphi"] & T(A) \\
GT(A) \arrow[ur, "GT\eta"] \arrow[rr, "\lambda"'] & & TF(A) \arrow[u, "TF\eta"] \arrow[ur, "\kappa"] \arrow[urr, "T\varphi"'] & &
\end{tikzcd} ,$$

in which all the other subdiagrams are commutative by naturality or definition. This shows that $T\varphi \circ \lambda_A$ defines a quasi-G-module structure on $T(A)$.

Regularity of the quasi-G-module $T(A)$ means commutativity of the outer paths in the diagram

$$\begin{tikzcd}
GGT(A) \arrow[r, "\mu' T"] & GT(A) \arrow[r, "\lambda"] & TF(A) \arrow[d, "\kappa"] \arrow[dr, "T\varphi"] & \\
GT(A) \arrow[u, "G\eta' T"] \arrow[ur, "\vartheta' T"] \arrow[r, "\lambda"'] & TF(A) \arrow[r, "\kappa"'] & TF(A) \arrow[r, "T\varphi"'] & T(A);
\end{tikzcd}$$

this holds since the pentagon is just equation (5.3) (hence commutative by assumption) and (A, φ) is regular. □

These observations allow us to extend Applegate's lifting theorem for monads (e.g. [12, Lemma 1]) to quasi-monads and quasi-modules with regularity conditions.

5.4. Theorem. *Let (F, μ, η) and (G, μ', η') be regular quasi-monads on $\mathbb{A}$ and $\mathbb{B}$, and $\underline{\mathbb{A}}_F$ and $\underline{\mathbb{B}}_G$ the categories of the regular quasi-modules, respectively. For any functor $T : \mathbb{A} \to \mathbb{B}$, there are bijective correspondences between*

(i) *liftings of T to $\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{B}}_G$, such that for any $(A, \varphi) \in \mathbb{A}_F$, the regular quasi-$G$-module structure map $\varrho : GTU_F \to TU_F$ induces commutativity of the diagram*

$$\begin{tikzcd}
GTF(A) \arrow[r, "\varrho_{F(A)}"] & TF(A) \arrow[d, "T\varphi"] \\
GT(A) \arrow[u, "GT\eta_A"] \arrow[r, "\varrho_A"] & T(A);
\end{tikzcd} \tag{5.6}$$

(ii) *regular quasi-G-module structures ϱ on $TU_F : \mathbb{A}_F \to \mathbb{B}$ inducing commutativity of the diagram corresponding to (5.6);*

(iii) *natural transformations $\lambda : GT \to TF$ with*

$$\lambda \circ \vartheta' T = \lambda = \kappa \circ \lambda$$

and commutative diagram

(5.7)
$$\begin{tikzcd}
GGT \arrow[r, "G\lambda"] \arrow[d, "\mu' T"'] & GTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu"] \\
GT \arrow[rr, "\lambda"] & & TF.
\end{tikzcd}$$

Proof. (i)⇔(ii) This follows from the right hand diagram in (5.1) and Proposition 3.16.

(ii)⇒(iii) With ϱ (as in (i)), put

$$\lambda := \varrho F \circ GT\eta : GT \xrightarrow{GT\eta} GTF \xrightarrow{\varrho F} TF.$$

By regularity of η and naturality, we get the commutative diagram

$$\begin{tikzcd}
GT \arrow[r, "GT\eta"] \arrow[ddr, "GT\eta"'] & GTF \arrow[r, "\varrho F"] \arrow[d, "GTF\eta"] & TF \arrow[d, "TF\eta"] \\
 & GTFF \arrow[r, "\varrho FF"] \arrow[d, "GT\mu"] & TFF \arrow[d, "T\mu"] \\
 & GTF \arrow[r, "\varrho F"] & TF
\end{tikzcd}$$

from which we obtain

$$\kappa \circ \varrho F = \varrho F \circ G\kappa \quad \text{and} \quad \kappa \circ \lambda = \lambda.$$

In the diagram

$$\begin{tikzcd}
 & GT \arrow[dr, "GT\eta"] & \\
GGT \arrow[ur, "\mu' T"] \arrow[r, "GGT\eta"'] & GGTF \arrow[r, "\mu' TF"'] & GTF \arrow[d, "\varrho F"] \\
GT \arrow[u, "G\eta' T"] \arrow[r, "GT\eta"'] & GTF \arrow[u, "G\eta' TF"] \arrow[r, "\varrho F"'] & TF,
\end{tikzcd}$$

the right square is commutative by regularity of ϱ while the other partial diagrams are commutative by naturality. This shows that $\lambda \circ \vartheta' T = \lambda$.

Consider the diagram

$$\begin{tikzcd}
GGT \arrow[r, "GGT\eta"] \arrow[d, "\mu' T"'] & GGTF \arrow[r, "G\varrho F"] \arrow[d, "\mu' TF"'] & GTF \arrow[r, "GT\eta F"] \arrow[d, "\varrho F"'] & GTFF \arrow[d, "\varrho FF"] \\
GT \arrow[r, "GT\eta"'] & GTF \arrow[r, "\varrho F"'] & TF & TFF, \arrow[l, "T\mu"]
\end{tikzcd}$$

in which the left two squares are commutative by naturality and associativity, respectively, while the right square is commutative as a special case of the diagram (5.6). Reading the diagram in terms of λ we see that (5.7) is commutative.

(iii)⇒(i) By Proposition 5.3 and 3.16, $\varrho_A := T\varphi \circ \lambda_A$ may be considered as regular quasi-G-module structure on TU_F. Commutativity of (5.6) can be written as

$$\varrho_A = T\varphi \circ \varrho F \circ GT\eta_A = \varrho \circ GT\varphi \circ GT\eta_A.$$

Now the equation (5.5) implies commutativity of (5.6).

To show uniqueness of the correspondence, let $\varrho : GTU_F \to TU_F$ be a quasi-G-module structure morphism with commutative diagram (5.6) (in (ii)). With the λ defined in the proof (ii)⇒(iii), we obtain a quasi-G-module structure on TF (see 5.2),

$$\widetilde{\varrho} : GTF \xrightarrow{GT\eta F} GTFF \xrightarrow{\varrho FF} TFF \xrightarrow{T\mu} TF.$$

This fits into the (obviously) commutative diagram

$$\begin{tikzcd}
GT \arrow[r, "GT\eta"] \arrow[d, "GT\eta"'] & GTF \arrow[r, "\varrho F"] \arrow[d, "GTF\eta"] & TF \arrow[d, "TF\eta"] \\
GTF \arrow[r, "GT\eta F"] & GTFF \arrow[r, "\varrho FF"'] \arrow[d, "GT\mu"'] & TFF \arrow[d, "T\mu"] \\
& GTF \arrow[r, "\varrho F"] & TF
\end{tikzcd}$$

which shows that $\widetilde{\varrho} \circ GT\eta = \kappa \circ \varrho F \circ GT\eta = \lambda$. Now commutativity of (5.6) just means $\varrho_A = T\varphi \circ \lambda_A$. □

Clearly the morphism $\kappa = T\vartheta$ (see 5.1) shows the deviation of the quasi-unit from unitality. We list some properties and relations for this entity.

5.5. Lemma. *Let (F, μ, η), (G, μ', η') be quasi-monads and $T : \mathbb{A} \to \mathbb{B}$ a functor with (any) natural transformation $\lambda : GT \to TF$ and consider*

$$\widehat{\kappa} : \; TF \xrightarrow{\eta' TF} GTF \xrightarrow{\lambda F} TFF \xrightarrow{T\mu} TF.$$

1. $\widehat{\kappa} \circ \kappa = \kappa \circ \widehat{\kappa}$.
2. *If $\lambda \circ \eta' T = T\eta$, then $\widehat{\kappa} = T\underline{\vartheta}$.*
3. *If the diagram (5.7) is commutative, then $\lambda \circ \underline{\vartheta}' T = \widehat{\kappa} \circ \lambda$.*
4. *If (5.7) is commutative and η' is regular, then $\widehat{\kappa}$ is idempotent.*

Proof. (1) follows by commutativity of the diagram

$$\begin{tikzcd}
TF \arrow[r, "\eta' TF"] \arrow[d, "TF\eta"'] & GTF \arrow[r, "\lambda F"] \arrow[d, "GTT\eta"] & TFF \arrow[r, "T\mu"] \arrow[d, "TTF\eta"] & TF \arrow[d, "TF\eta"] \\
TFF \arrow[r, "\eta' TFF"] \arrow[d, "T\mu"'] & GTFF \arrow[r, "\lambda FF"] \arrow[d, "GT\mu"] & TFFF \arrow[r, "T\mu_F F"] \arrow[d, "TF\mu"] & TFF \arrow[d, "T\mu"] \\
TF \arrow[r, "\eta' TF"'] & GTF \arrow[r, "\lambda F"'] & TFF \arrow[r, "T\mu_F"'] & TF,
\end{tikzcd}$$

in which the top and the bottom row both yield $\widehat{\kappa}$ and the left and right vertical morphisms are $\kappa = T\vartheta$.

(2) is obvious, for (3) see lower part of the diagram in the proof of (4).

(4) The diagram

$$\begin{tikzcd}
T \arrow[d, "\eta' T"'] \arrow[rrr, "\eta' T"] & & & GT \\
GT \arrow[r, "\eta' GT"] \arrow[d, "\lambda"'] & GGT \arrow[rr, "\mu' T"'] \arrow[d, "G\lambda"'] & & GT \arrow[d, "\lambda"] \\
TF \arrow[r, "\eta' TF"] & GTF \arrow[r, "\lambda F"] & TFF \arrow[r, "T\mu"] & TF,
\end{tikzcd}$$

is commutative by assumption and naturality. Applying the outer morphisms to F yields the upper part commutative in the diagram

$$\begin{tikzcd}
TF \arrow[rr, "\eta' TF"] \arrow[d, "\eta' TF"'] & & GTF \arrow[d, "\lambda F"] \\
GTF \arrow[r, "\lambda F"] & TFF \arrow[r, "\widehat{\kappa} F"] \arrow[d, "T\mu"'] & TFF \arrow[d, "T\mu"] \\
 & TF \arrow[r, "\widehat{\kappa}"] & TF,
\end{tikzcd}$$

while the lower part is commutative by associativity of μ. This shows that $\widehat{\kappa}$ is idempotent. $\square$

As a special case of Theorem 5.4 we consider regular quasi-algebras.

5.6. Regular quasi-modules of quasi-algebras. Let A be an R-module with multiplication $m : A \otimes_R A \to A$ and idempotents e, f. Then (A, m_e, e) and (A, m_f, f) are regular quasi-algebras with multiplications

$$m_e(a \otimes b) := m(a \otimes m(e \otimes b)) \quad \text{and} \quad m_f(a \otimes b) := m(a \otimes m(f \otimes b)),$$

for $a, b \in A$ (see 3.13).

For any R-module T, the twist map $\mathsf{tw} : A \otimes_R T \to T \otimes_R A$ satisfies the equality $m \circ (\mathsf{tw} \otimes A) \circ (A \otimes \mathsf{tw}) = \mathsf{tw} \circ m$ but this does no longer hold when replacing m by m_e and m_f, respectively.

Composing tw with $- \cdot f \otimes T$ and $T \otimes - \cdot e$ from the left and right hand side, respectively, we define

$$\overline{\lambda} : A \otimes_R T \to T \otimes_R A, \quad a \otimes t \longmapsto t \otimes afe,$$

and the diagram

$$\begin{tikzcd}
A \otimes_R A \otimes_R T \arrow[r, "A \otimes \overline{\lambda}"] \arrow[d, "m_f \otimes T"'] & A \otimes_R T \otimes_R A \arrow[r, "\overline{\lambda} \otimes A"] & T \otimes_R A \otimes_R A \arrow[d, "m_e \otimes T"] \\
A \otimes_R T \arrow[rr, "\overline{\lambda}"] & & T \otimes_R A,
\end{tikzcd}$$

is commutative, provided for $a, b \in A$ and $t \in T$,

$$t \otimes afbfe = t \otimes afebfe.$$

This obviously holds, for example, if $fe = f$ or also if e is a central element. In this case the functor $T \otimes_R - : \mathbb{M}_R \to \mathbb{M}_R$ can be lifted to $\overline{T} : \underline{\mathbb{M}}_{(A,m,e)} \to \underline{\mathbb{M}}_{(A,m,f)}$.

Indeed, for a regular (A, m_e, e)-module $A \otimes_R M \to M$ we have $am = aem$ for any $a \in A$, $m \in M$. On $T \otimes_R M$, $\overline{\lambda}$ induces the left A-module structure

$$A \otimes_R T \otimes_R M \to T \otimes_R M, \quad a \otimes t \otimes m \mapsto t \otimes afem = t \otimes afm,$$

which clearly is (A, m_f, f)-regular.

This example shows that centrality of e (that is, symmetry of η in Theorem 5.4) simplifies the situation but is not necessary for the lifting.

6. Entwinings with comonads

6.1. Lifting of functors to quasi-comodules. Let (F, δ, ε) and $(G, \delta', \varepsilon')$ be quasi-comonads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively. Denote by $\underrightarrow{\mathbb{A}}^F$, $\underrightarrow{\mathbb{B}}^G$ the categories of the corresponding quasi-comodules and by $\underline{\mathbb{A}}^F$, $\underline{\mathbb{B}}^G$ the categories of the regular quasi-comodules provided the quasi-comonads are regular (see 4.3). Given functors

$$T : \mathbb{A} \to \mathbb{B}, \quad \overrightarrow{T} : \underrightarrow{\mathbb{A}}^F \to \underrightarrow{\mathbb{B}}^G, \quad \widehat{T} : \underline{\mathbb{A}}^F \to \underline{\mathbb{B}}^G,$$

we say that $\overrightarrow{T}$ or $\widehat{T}$ *is a lifting of* T if the corresponding diagram

$$\begin{tikzcd} \underrightarrow{\mathbb{A}}^F \arrow[r, "\overrightarrow{T}"] \arrow[d, "U^F"'] & \underrightarrow{\mathbb{B}}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{B} \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \underline{\mathbb{A}}^F \arrow[r, "\widehat{T}"] \arrow[d, "U^F"'] & \underline{\mathbb{B}}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{B} \end{tikzcd}$$

is commutative, where the U's denote the forgetful functors (see 3.4).

The natural transformations $\gamma = F\varepsilon \circ \delta$ and $\gamma' = G\varepsilon' \circ \delta'$ are quasi-comodule morphism (see 4.1) and we put

$$\tau := T\gamma : TF \to TF.$$

6.2. Proposition. *With the data given in* 6.1, *consider the pair of functors* $TF, GT : \mathbb{A} \to \mathbb{B}$ *and a natural transformation* $\psi : TF \to GT$. *The quasi-*F*-comodule* (F, δ) *induces a* G*-coaction on* TF,

$$\zeta : TF \xrightarrow{T\delta} TFF \xrightarrow{\psi F} GTF.$$

(1) *If* (TF, ζ) *is a quasi-*G*-comodule, then we get the commutative diagram*

$$\begin{tikzcd} TF \arrow[r, "\tau"] \arrow[d, "T\delta"'] & TF \arrow[rr, "\psi"] & & GT \arrow[d, "\delta'"] \\ TFF \arrow[r, "\psi F"] & GTF \arrow[r, "G\tau"] & GTF \arrow[r, "G\psi"] & GGT. \end{tikzcd} \tag{6.1}$$

(2) *If* G *is regular and* (TF, ζ) *is a regular quasi-*G*-module, then*

$$TF \xrightarrow{\tau} TF \xrightarrow{\psi} GT \xrightarrow{\gamma' T} GT \;=\; TF \xrightarrow{\tau} TF \xrightarrow{\psi} GT. \tag{6.2}$$

Proof. The proof is dual to that of Proposition 6.2. To illustrate the situation and for convenient reference we write out some of the diagrams involved.

(1) Coassociativity of the coaction means commutativity of the inner rectangle in the diagram

$$\begin{tikzcd}
 & TF \arrow[rr, "\psi"] & & GT \arrow[dddd, "\delta' T"] \\
TF \arrow[ur, "\tau"] \arrow[r, "T\delta"'] \arrow[d, "T\delta"'] & TFF \arrow[u, "TF\varepsilon"'] \arrow[r, "\psi F"'] & GTF \arrow[ur, "GT\varepsilon"'] \arrow[dd, "\delta' TF"] & \\
TFF \arrow[d, "\psi F"'] & & & \\
GTF \arrow[r, "GT\delta"] \arrow[dr, "\tau"'] & GTFF \arrow[r, "G\psi F"] \arrow[d, "GTF\varepsilon"] & GGTF \arrow[dr, "GGT\varepsilon"] & \\
 & GTF \arrow[rr, "G\psi"] & & GGT,
\end{tikzcd}$$

and all the other inner diagrams are commutative by definition or naturality. Thus the outer path is commutative and yields (6.1).

(2) Regularity of (TF, ζ) means commutativity of the inner rectangle in the diagram

$$\begin{tikzcd}
TF \arrow[r, "\psi"] & GT \arrow[rr, "\delta' T"] & & GGT \arrow[ddd, "G\varepsilon' T"] \\
TFF \arrow[u, "TF\varepsilon"] \arrow[r, "\psi F"] & GTF \arrow[u, "GT\varepsilon"] \arrow[r, "\delta' TF"] & GGTF \arrow[ur, "GGT\varepsilon"] \arrow[d, "G\varepsilon' TF"] & \\
TF \arrow[u, "T\delta"] \arrow[r, "T\delta"] \arrow[dr, "\tau"'] & TFF \arrow[r, "\psi F"] \arrow[d, "TF\varepsilon"] & GTF \arrow[dr, "GT\varepsilon"] & \\
 & TF \arrow[rr, "\psi"] & & GT,
\end{tikzcd}$$

where all the other inner diagrams are commutative by definition or naturality. The outer path now gives commutativity of (6.2). □

6.3. Proposition. *Let (F, δ, ε) and $(G, \delta', \varepsilon')$ be regular quasi-comonads on the categories $\mathbb{A}$ and $\mathbb{B}$, respectively, and $T : \mathbb{A} \to \mathbb{B}$ any functor. Then a natural transformation $\psi : TF \to GT$ induces a lifting*

$$\widehat{T} : \underline{\mathbb{A}}^F \to \underline{\mathbb{B}}^G, \quad (F, \upsilon) \mapsto (T(A), \psi \circ T\upsilon : T(A) \to GT(A))$$

to the regular quasi-comodules if and only if the diagrams (6.1) *and* (6.2) *are commutative.*

Proof. The proof is dual to that of Proposition 5.3. □

Dualising Theorem 5.4, Applegate's lifting theorem for comonads extends to quasi-monads and quasi-modules.

6.4. Theorem. *Let (F, δ, ε) and $(G, \delta', \varepsilon')$ be regular quasi-comonads on $\mathbb{A}$ and $\mathbb{B}$, and $\underline{\mathbb{A}}^F$ and $\underline{\mathbb{B}}^G$ the categories of the regular quasi-comodules, respectively. For any functor $T : \mathbb{A} \to \mathbb{B}$, there are bijective correspondences between*

(i) *liftings of T to $\widehat{T} : \underline{\mathbb{A}}^F \to \underline{\mathbb{B}}^G$, such that for any $(A, \upsilon) \in \underline{\mathbb{A}}^F$, the regular quasi-$G$-comodule structure map $\upsilon : TU^F \to GTU^F$ induces commutativity of the diagram*

$$\begin{tikzcd}
TF(A) \arrow[r, "\upsilon F_A"] & GTF(A) \arrow[d, "GT\varepsilon"] \\
T(A) \arrow[u, "T\upsilon"] \arrow[r, "\upsilon_A"] & GT(A);
\end{tikzcd}$$

(ii) *regular quasi-G-comodule structures $\upsilon : TU^F \to GTU^F$ inducing commutativity of the diagram corresponding to that in* (i)*;*

(iii) *natural transformations $\psi : TF \to GT$ with*

$$\psi \circ \tau = \psi = \gamma' T \circ \psi$$

and commutative diagram

$$\begin{tikzcd}
TF \arrow[rr, "\psi"] \arrow[d, "T\delta"'] & & GT \arrow[d, "\delta' T"] \\
TFF \arrow[r, "\psi F"] & GTF \arrow[r, "G\psi"] & GGT.
\end{tikzcd}$$

Proof. In view of 6.2 and 6.3 the proof is dual to that of Theorem 5.4. Here we take ψ as the composition $\psi \circ \tau$ (with ψ from 6.2). □

7. Lifting of endofunctors to quasi-modules

In this section we consider the

7.1. Liftings of endofunctors to quasi-modules. Let (F, μ, η) be a regular quasi-monad and T any endofunctor on the category $\mathbb{A}$. A functor $\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{A}}_F$ and $\widehat{T} : \underline{\mathbb{A}}^G \to \underline{\mathbb{A}}^G$ is a lifting of T provided it induces commutativity of the diagram

$$\begin{tikzcd}
\underline{\mathbb{A}}_F \arrow[r, "\overline{T}"] \arrow[d, "U_F"'] & \underline{\mathbb{A}}_F \arrow[d, "U_F"] \\
\mathbb{A} \arrow[r, "T"] & \mathbb{A}.
\end{tikzcd}$$

As an application of Theorem 5.4 we get

7.2. Proposition. *Let (F, μ, η) be a regular quasi-monad, $\underline{\mathbb{A}}_F$ the category of regular quasi-modules, and $T : \mathbb{A} \to \mathbb{A}$ any endofunctor on $\mathbb{A}$. There are bijective correspondences between*

(i) *liftings of T to $\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{A}}_F$, such that for any $(A, \varphi) \in \mathbb{A}_F$, the regular quasi-$F$-module $\varrho : FTU_F \to TU_F$ satisfies*

$$\varrho_A = T\varphi \circ \varrho F \circ FT\eta_A = \varrho \circ FT\varphi \circ FT\eta_A.$$

(ii) *regular quasi-F-module structures $\varrho : FTU_F \to TU_F$ satisfying the equalities in* (i)*;*

(iii) *natural transformations* $\lambda : FT \to TF$ *with*

$$\lambda \circ \vartheta T = \lambda = T\vartheta \circ \lambda$$

and commutative diagram

$$\begin{tikzcd}
FFT \arrow[r, "F\lambda"] \arrow[d, "\mu T"'] & FTF \arrow[r, "\lambda F"] & TFF \arrow[d, "T\mu"] \\
FT \arrow[rr, "\lambda"] & & TF.
\end{tikzcd} \tag{7.1}$$

From Lemma 5.5 we get the

7.3. Lemma. *Let (F, μ, η) be a quasi-monad and $T : \mathbb{A} \to \mathbb{B}$ a functor with (any) natural transformation $\lambda : FT \to TF$ and consider*

$$\widehat{\kappa} : \; TF \xrightarrow{\eta TF} FTF \xrightarrow{\lambda F} TFF \xrightarrow{T\mu} TF.$$

1. $\widehat{\kappa} \circ \kappa = \kappa \circ \widehat{\kappa}$.
2. *If* $\lambda \circ \eta T = T\eta$, *then* $\widehat{\kappa} = T\underline{\vartheta}$.
3. *If the diagram* (7.1) *is commutative, then* $\lambda \circ \underline{\vartheta} T = \widehat{\kappa} \circ \lambda$.
4. *If* (7.1) *is commutative and* η *is regular, then* $\widehat{\kappa}$ *is idempotent.*

Besides the questions considered in the general case (e.g. 5.4), we may now ask when the liftings are quasi-monads.

7.4. Proposition. *Let (F, μ, η) and $(T, \check{\mu}, \check{\eta})$ be regular quasi-monads and assume T can be lifted to $\overline{T} : \underline{\mathbb{A}}_F \to \underline{\mathbb{A}}_F$ by $\lambda : FT \to TF$ (see 7.2). Then, on TF, product and quasi-unit are defined by*

$$\overline{\mu} : TFTF \xrightarrow{T\lambda F} TTFF \xrightarrow{TT\mu} TTF \xrightarrow{\check{\mu}F} TF, \quad \overline{\eta} : I_{\mathbb{A}} \xrightarrow{\eta} F \xrightarrow{F\check{\eta}} FT \xrightarrow{\lambda} TF.$$

1. *If $\check{\mu}F : TTF \to TF$ is a quasi-F-module, then we get the commutative diagram*

$$\begin{tikzcd}
FTT \arrow[r, "\lambda T"] \arrow[d, "F\check{\mu}"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[d, "\check{\mu}F"] \\
FT \arrow[rr, "\lambda"] & & TF.
\end{tikzcd} \tag{7.2}$$

2. *If (7.2) is commutative, then $(TF, \overline{\mu}, \overline{\eta})$ is a quasi-monad with $\overline{\eta}$ regular.*
3. *In* (2), *$\overline{\mu}$ is regular if and only if, in addition,*

$$FT \xrightarrow{F\underline{\check{\vartheta}}} FT \xrightarrow{\lambda} TF \xrightarrow{\underline{\check{\vartheta}}F} TF \; = \; FT \xrightarrow{\lambda} TF \xrightarrow{\underline{\check{\vartheta}}F} TF, \tag{7.3}$$

where $\underline{\check{\vartheta}} = \check{\mu} \circ \check{\eta} T$. In this case $(TF, \overline{\mu}, \overline{\eta})$ is a regular quasi-monad.

Proof. (1) The condition on $\breve{\mu}F : TTF \to TF$ means commutativity of the large inner rectangle in the diagram

$$\begin{tikzcd}
FTT \arrow[rr, "\lambda T"] \arrow[dr, "FTT\eta"] \arrow[ddd, "F\breve{\mu}"'] & & TFT \arrow[r, "T\lambda"] \arrow[d, "TFT\eta"'] & TTF \arrow[d, "TTF\eta"'] \arrow[dr, "T\kappa"] & \\
& FTTF \arrow[r, "\lambda TF"'] \arrow[d, "F\breve{\mu}F"'] & TFTF \arrow[r, "T\lambda F"'] & TTFF \arrow[r, "TT\mu"'] & TTF \arrow[d, "\breve{\mu}F"] \\
& FTF \arrow[rr, "\lambda F"] & & TFF \arrow[r, "T\mu"] & TF \\
FT \arrow[ur, "FT\eta"] \arrow[rr, "\lambda"] & & TF \arrow[ur, "TF\eta"] \arrow[urr, "\kappa"'] & & .
\end{tikzcd}$$

Since all the other subdiagrams are commutative by naturality or definition, and $\kappa \circ \lambda = \lambda$ (see 7.2), the outer path yields commutativity of (7.2).

(2) Associativity of the product $\overline{\mu}$ is obtained by standard diagram manipulations. It is a special case of the corresponding part of the proof of 7.7.

The condition for regularity of the quasi-unit $\overline{\eta}$ is commutativity of the outer path of the diagram

$$\begin{tikzcd}
I_{\mathbb{A}} \arrow[r, "\eta\breve{\eta}"] \arrow[d, "\eta\breve{\eta}"'] & FT \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & TF \arrow[dl, "\kappa"'] \arrow[d, "TF\eta"] \\
FT \arrow[d, "\lambda"'] & TF \arrow[dl, "\widetilde{\kappa}"'] \arrow[d, "TF\breve{\eta}"] & TFF \arrow[l, "\mu"'] \arrow[d, "TF\breve{\eta}"] \\
TF & TFT \arrow[dl, "T\lambda"] & TFFT \arrow[l, "T\mu T"] \arrow[d, "TF\lambda"] \\
TTF \arrow[u, "\breve{\mu}F"] & TTFF \arrow[l, "TT\mu"] & TFTF, \arrow[l, "T\lambda F"]
\end{tikzcd}$$

where the inner quadrangle is commutative by naturality, the pentagon on the bottom is so by commutativity of (7.1) where

$$\widetilde{\kappa} :\ TF \xrightarrow{TF\breve{\eta}} TFT \xrightarrow{T\lambda} TTF \xrightarrow{\breve{\mu}F} TF.$$

Recalling that $\breve{\vartheta} = \breve{\mu} \circ T\breve{\eta}$ (see 3.1), commutativity of the diagram

$$\begin{tikzcd}
FT \arrow[r, "\lambda"] \arrow[d, "FT\breve{\eta}"'] & TF \arrow[d, "TF\breve{\eta}"] & \\
FTT \arrow[r, "\lambda T"] \arrow[d, "F\breve{\mu}"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[d, "\breve{\mu}F"'] \\
FT \arrow[rr, "\lambda"] & & TF
\end{tikzcd}$$

implies

$$FT \xrightarrow{F\breve{\vartheta}} FT \xrightarrow{\lambda} TF = FT \xrightarrow{\lambda} TF \xrightarrow{\widetilde{\kappa}} TF, \tag{7.4}$$

and thus

$$\begin{aligned} I_{\mathbb{A}} \xrightarrow{\eta\check{\eta}} FT \xrightarrow{\lambda} TF \xrightarrow{\widetilde{\kappa}} TF &= I_{\mathbb{A}} \xrightarrow{\eta\check{\eta}} FT \xrightarrow{F\check{\vartheta}} FT \xrightarrow{\lambda} TF \\ &= I_{\mathbb{A}} \xrightarrow{\eta\check{\eta}} FT \xrightarrow{\lambda} TF, \end{aligned}$$

where the last equality follows by regularity of $\check{\eta}$ (see 3.9). This means that the left hand pentagon - and hence the whole diagram - is commutative.

(3) To show that the product $\overline{\mu}$ is regular, consider the commutative diagram

$$\begin{tikzcd}
TF \arrow[r, "TF\check{\eta}"] & TFT \arrow[r, "TF\eta T"] \arrow[dr, "T\vartheta T"'] & TFFT \arrow[r, "TF\lambda"] \arrow[d, "T\mu T"] & TFTF \arrow[r, "T\lambda F"] & TTFF \arrow[d, "TT\mu"] & \\
 & & TFT \arrow[rr, "T\lambda"] & & TTF \arrow[r, "\check{\mu}F"] & TF.
\end{tikzcd}$$

Since $\lambda\circ\vartheta T = \lambda$ (see 7.2) we see that $\overline{\mu}\circ TF\overline{\eta} = \widetilde{\kappa}$. Thus the condition for regularity of $\overline{\mu}$ means commutativity of the diagram

$$\begin{tikzcd}
TFT \arrow[r, "T\lambda"] & TTF \arrow[dr, "\check{\mu}F"] & \\
TFT \arrow[u, "\widetilde{\kappa}T"] \arrow[r, "T\lambda"] & TTF \arrow[r, "\check{\mu}F"] & TF.
\end{tikzcd}$$

The upper path in this fits in the commutative diagram

$$\begin{tikzcd}
TFT \arrow[r, "TF\check{\eta}T"] \arrow[dr, "TF\underline{\check{\vartheta}}"'] & TFTT \arrow[r, "T\lambda T"] \arrow[d, "TF\check{\mu}"] & TTFT \arrow[r, "\check{\mu}FT"] \arrow[d, "TT\lambda"] & TFT \arrow[d, "T\lambda"] \\
 & TFT \arrow[dr, "T\lambda"'] & TTTF \arrow[r, "\check{\mu}TF"] \arrow[d, "T\check{\mu}F"] & TTF \arrow[d, "\check{\mu}F"] \\
 & & TTF \arrow[r, "\check{\mu}F"'] & TF
\end{tikzcd}$$

and hence the regularity condition reads as commutativity of the bottom rectangle in the diagram

$$\begin{tikzcd}
 & & FT \arrow[r, "\lambda"] \arrow[d, "\check{\eta}FT"] & TF \arrow[d, "\check{\eta}TF"] \\
FT \arrow[urr, "F\underline{\check{\vartheta}}"] \arrow[r, "\check{\eta}FT"'] \arrow[d, "\lambda"'] & TFT \arrow[r, "TF\underline{\check{\vartheta}}"] \arrow[d, "T\lambda"] & TFT \arrow[r, "T\lambda"'] & TTF \arrow[d, "\check{\mu}F"] \\
TF \arrow[r, "\check{\eta}TF"] & TTF \arrow[rr, "\check{\mu}F"] & & TF,
\end{tikzcd}$$

while the other subdiagrams are commutative by naturality. This yields (7.3).

On the other hand, equality (7.3) implies

$$\begin{aligned} TFT \xrightarrow{TF\underline{\check{\vartheta}}} TFT \xrightarrow{T\lambda} TTF \xrightarrow{\check{\mu}F} TF &= TFT \xrightarrow{T\lambda} TTF \xrightarrow{T\underline{\check{\vartheta}}F} TTF \xrightarrow{\check{\mu}F} TF \\ &= TFT \xrightarrow{T\lambda} TTF \xrightarrow{\check{\mu}F} TF \end{aligned}$$

where the last equality follows by regularity of $\check{\mu}$ (see 3.10). This shows that $\overline{\mu}$ is regular. □

7.5. Weak distributive laws. Note that Proposition 7.4 generalises the *weak distributive laws* as considered by Street in [18], where a lifting of a (proper) monad $(T, \check{\mu}, \check{\eta})$ to (regular) quasi-modules over a monad (F, μ, η) is considered. The condition (3) in [18, Definition 2.1] means $\widetilde{\kappa} = \widehat{\kappa}$. For regular quasi-monads F, T, this implies (see Lemma 7.3, (7.4))

$$\lambda \circ \underline{\vartheta} T = \widehat{\kappa} \circ \lambda = \widetilde{\kappa} \circ \lambda = \lambda \circ F\check{\vartheta}.$$

Since $\lambda \circ \vartheta T = \lambda$ (see 7.2), imposing the symmetry condition $\vartheta = \underline{\vartheta}$ implies that all these expressions are equal to λ.

7.6. Quasi-monad entwinings. *For regular monads F, T, and a natural transformation $\lambda : FT \to TF$, the following are equivalent:*

(a) $(TF, \overline{\mu}, \lambda \circ \eta\check{\eta})$ *is a regular quasi-monad on $\mathbb{A}$;*

(b) λ *satisfies*

$$\lambda = \lambda \circ \vartheta T = T\vartheta \circ \lambda = \lambda \circ F\check{\vartheta} = \underline{\check{\vartheta}} F \circ \lambda \tag{7.5}$$

and induces commutativity of the diagram (7.1) and the diagram

$$\begin{tikzcd} FTT \arrow[rr, "F\check{\mu}"] \arrow[d, "\lambda T"'] & & FT \arrow[d, "\lambda"] \\ TFT \arrow[r, "T\lambda"] & TTF \arrow[r, "\check{\mu} F"] & TF; \end{tikzcd} \tag{7.6}$$

(c) λ *satisfies the equations (7.5), induces commutativity of the diagram (7.1), and we have natural transformations*

$$\check{\mu} F : TTF \to TF \quad \text{and} \quad \lambda \circ F\check{\eta} : F \to TF$$

where $\check{\mu} F$ is (F, F)-bilinear and $\lambda \circ F\check{\eta}$ is left F-linear.

If these conditions hold, we call (T, F, λ) a *regular quasi-monad entwining,* and

$$\xi := \lambda \circ F\check{\eta} : F \to TF \quad \text{and} \quad \lambda \circ \eta T : T \to TF$$

are quasi-monad morphisms.

Proof. (b)⇒(a) follows from Proposition 7.4 by taking for λ the composition $\underline{\check{\vartheta}} F \circ \lambda$ (with λ from 7.4).

(c)⇒(a) is a special case of 7.7 (see below).

To show that ξ is a monad morphism observe that the diagram

$$\begin{tikzcd} FF \arrow[r, "FF\check{\eta}"] \arrow[dd, "\mu"'] & FFT \arrow[r, "F\lambda"] \arrow[dd, "\mu T"'] & FTF \arrow[r, "F\check{\eta}TF"] \arrow[d, "\lambda F"'] \arrow[dr, "F\underline{\check{\vartheta}}F"'] & FTTF \arrow[r, "\lambda TF"] \arrow[d, "F\check{\mu}F"] & TFTF \arrow[dd, "T\lambda F"] \\ & & TFF \arrow[d, "T\mu"'] & FTF \arrow[d, "\lambda F"'] & \\ F \arrow[r, "F\check{\eta}"] & FT \arrow[r, "\lambda"] & TF & TFF \arrow[l, "T\mu"] & TTFF \arrow[l, "\check{\mu}FF"] \end{tikzcd}$$

is commutative: the rectangles are commutative by naturality and commutativity of (7.6) and (7.1), and the pentagon is commutative since $F\underline{\check{\vartheta}} \circ \lambda = \lambda$ (see (7.5)). This shows that ξ respects the product of the quasi-monads. The condition $\overline{\eta} = \xi \circ \eta$ is clear by the definition of $\overline{\eta}$ and hence ξ is a quasi-monad morphism.

Similar arguments show that $\lambda \circ \eta T$ is also a quasi-monad morphism. □

Given (F,μ,η) and $T:\mathbb{A}\to\mathbb{A}$, the composition TF may have a (regular) quasi-module structure without requiring such a structure on T. For this some other morphisms and conditions are needed.

7.7. Liftings as quasi-monads. *Let (F,μ,η) be a regular quasi-monad, T any endofunctor on $\mathbb{A}$ that can be lifted to $\overline{T}:\underline{\mathbb{A}}_F\to\underline{\mathbb{A}}_F$ by the entwining $\lambda: FT\to TF$ (see 7.2). Assume there are natural transformations*

$$\nu: TTF\to TF, \quad \xi: F\to TF$$

such that ν is (F,F)-bilinear and ξ is left F-linear. The lifting $\overline{T}$ induces a multiplication and a quasi-unit on TF,

$$\widetilde{\mu}: TFTF\xrightarrow{T\lambda F} TTFF\xrightarrow{TT\mu} TTF\xrightarrow{\nu} TF, \quad \widetilde{\eta}: I_{\mathbb{A}}\xrightarrow{\eta} F\xrightarrow{\xi} TF.$$

(1) *$(TF,\widetilde{\mu},\widetilde{\eta})$ is a quasi-monad if and only if the data induce commutativity of the diagrams*

(7.7)
$$\begin{tikzcd}
TTFT \arrow[r, "\nu T"] \arrow[d, "TT\lambda"'] & TFT \arrow[r, "T\lambda"] & TTF \arrow[d, "\nu"] \\
TTTF \arrow[r, "T\nu"] & TTF \arrow[r, "\nu"] & TF.
\end{tikzcd}$$

(2) *$\widetilde{\eta}$ is regular provided $\kappa\circ\xi=\xi$ and ξ induces commutativity of the diagram*

(7.8)
$$\begin{tikzcd}
I_{\mathbb{A}} \arrow[r, "\eta"] \arrow[d, "\eta"'] & F \arrow[r, "\xi"] & TF \arrow[d, "T\xi"] \\
F \arrow[r, "\xi"] & TF & TTF. \arrow[l, "\nu"']
\end{tikzcd}$$

(3) *$\widetilde{\mu}$ is regular if we have commutativity of the diagram*

(7.9)
$$\begin{tikzcd}
TFT \arrow[r, "T\xi T"] \arrow[d, "T\lambda"'] & TTFT \arrow[r, "\nu T"] & TFT \arrow[d, "T\lambda"] \\
TTF \arrow[r, "\nu"] & TF & TTF \arrow[l, "\nu"']
\end{tikzcd}$$

Proof. (1) Left F-linearity of ν is equivalent to commutativity of the diagram

(7.10)
$$\begin{tikzcd}
FTTF \arrow[r, "\lambda TF"] \arrow[d, "F\nu"'] & TFTF \arrow[r, "T\lambda F"] & TTFF \arrow[r, "TT\mu"] & TTF \arrow[d, "\nu"] \\
FTF \arrow[rr, "\lambda F"] & & TFF \arrow[r, "T\mu"] & TF,
\end{tikzcd}$$

whereas right F-linearity of ν corresponds to commutativity of the diagram

(7.11)
$$\begin{tikzcd}
TTFF \arrow[r, "\nu F"] \arrow[d, "TT\mu"'] & TFF \arrow[d, "T\mu"] \\
TTF \arrow[r, "\nu"] & TF.
\end{tikzcd}$$

To prove associativity of the product $\widetilde{\mu}$ on TF, consider the diagram

$$\begin{tikzcd}
TFTFTF \arrow[r, "T\lambda FTF"] \arrow[d, "TFT\lambda F"'] & TTFFTF \arrow[rr, "TT\mu TF"] \arrow[d, "TTF\lambda F"'] & \arrow[d, phantom, "(1)"] & TTFTF \arrow[r, "\nu TF"] \arrow[d, "TT\lambda F"'] \arrow[rd, phantom, "(2)"] & TFTF \arrow[d, "T\lambda F"] \\
TFTTFF \arrow[r, "T\lambda TFF"'] \arrow[d, "TF\nu F"'] & TTFTFF \arrow[r, "TT\lambda FF"] \arrow[rd, phantom, "(\star\star)"] & TTTFFF \arrow[r, "TTT\mu F"] & TTTFF \arrow[d, "T\nu F"] & TTFF \arrow[d, "\nu F"] \\
TFTFF \arrow[rr, "T\lambda FF"] \arrow[d, "TFT\mu"'] & & TTFFF \arrow[r, "TT\mu F"] \arrow[d, "TTF\mu"'] & TTFF \arrow[r, "\nu F"] \arrow[d, "TT\mu"'] \arrow[rd, phantom, "(3)"] & TFF \arrow[d, "T\mu"] \\
TFTF \arrow[rr, "T\lambda F"] & & TTFF \arrow[r, "TT\mu"] & TTF \arrow[r, "\nu"] & TF.
\end{tikzcd}$$

Diagram (1) is commutative by (7.1), diagram $(\star\star)$ is commutative by (7.10) (added T from the left and F from the right), diagram (2) is commutative by assumption (7.7) (applied to F), and commutativity of diagram (3) follows from (7.11). The remaining inner diagrams are commutative by naturality or associativity of multiplication of F. Thus the outer diagram is commutative and this shows associativity of the multiplication $\widetilde{\mu}$.

(2) Regularity of $\widetilde{\eta}$ means commutativity of the outer paths in the diagram

$$\begin{tikzcd}
I_{\mathbb{A}} \arrow[r, "\xi\circ\eta"] \arrow[dd, "\xi\circ\eta"'] & TF \arrow[r, "TF\eta"] \arrow[rd, "\kappa"] \arrow[rdd, "T\xi"'] & TFF \arrow[r, "TF\xi"] \arrow[d, "T\mu"] & TFTF \arrow[dd, "T\lambda F"] \\
 & & TF \arrow[d, "T\xi"] & \\
TF & & TTF \arrow[ll, "\nu"] & TTFF. \arrow[l, "TT\mu"]
\end{tikzcd}$$

Herein the trapezium is just the diagram (7.8) and hence commutative by assumption, the rectangle is commutative since ξ is left F-linear, the upper triangle is commutative by definition, and commutativity of the lower triangle is a consequence of the condition $\kappa\circ\xi=\xi$ in (2).

(3) Referring to (7.7) we can follow the proof of Proposition 7.4(3). Notice that $\widetilde{\kappa}$ corresponds to $\nu\circ T\xi$ from there. □

Note that under the conditions of section 7.6, the maps $\nu := \check{\mu}F$ and $\xi := \lambda\circ F\check{\eta}$ satisfy the conditions required in 7.7. Hence the proof of (c)⇒(a) in 7.6 follows from 7.7.

7.8. Weak crossed products. Given the morphisms $\nu : TTF \to TF$ and $\xi : F \to TF$ in 7.7, we may form

$$\bar{\nu}:\ TT \xrightarrow{TT\eta} TTF \xrightarrow{\nu} TF, \quad \overline{\eta}:\ I_{\mathbb{F}} \xrightarrow{\eta} F \xrightarrow{\xi} TF.$$

From the commutative diagrams

$$\begin{tikzcd}
TTF \arrow[r, "TT\eta F"] \arrow[rd, "TT\underline{\vartheta}"'] & TTFF \arrow[r, "\nu F"] \arrow[d, "TT\mu"] & TFF \arrow[d, "T\mu"] \\
 & TTF \arrow[r, "\nu"'] & TF,
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "\eta F"] \arrow[rd, "\underline{\vartheta}"'] & FF \arrow[r, "\xi F"] \arrow[d, "T\mu"] & TFF \arrow[d, "\mu"] \\
 & F \arrow[r, "\xi"] & TF,
\end{tikzcd}$$

we obtain

$$\nu \circ TT\underline{\vartheta} = T\mu \circ \bar{\nu}F \quad \text{and} \quad \xi \circ \underline{\vartheta} = T\mu \circ \overline{\eta}F.$$

If η is regular (see 3.9), we obtain $\xi \circ \underline{\vartheta} \circ \eta = \xi \circ \eta$ and $\nu \circ TT\underline{\vartheta}$ defines the same product on TF as ν.

Thus $\bar{\nu}$ and $\overline{\eta}$ may be used to define a (regular) quasi-monad structure on TF. This gives another version for the wreath product of a regular quasi-monad with an endofunctor. In this context the conditions for a weak monad structure on TF come out as *cocycle* and *twisted conditions*. For more details we refer, e.g., to [1], [10, Section 3].

8. Lifting of endofunctors to quasi-comodules

Dual to the material in the preceding section we sketch the lifting of endofunctors to the category to quasi-comodules.

8.1. Lifting of endofunctors to quasi-comodules. Let (G, δ, ε) be a regular quasi-comonad and T any endofunctor on the category $\mathbb{A}$. We now consider liftings $\widehat{T} : \underline{\mathbb{A}}^G \to \underline{\mathbb{A}}^G$ to the category of regular quasi-G-comodules, that is, functors which induce commutativity of the diagram

$$\begin{tikzcd}
\underline{\mathbb{A}}^G \arrow[r, "\widehat{T}"] \arrow[d, "U^G"'] & \underline{\mathbb{A}}^G \arrow[d, "U^G"] \\
\mathbb{A} \arrow[r, "T"] & \mathbb{A}.
\end{tikzcd}$$

As a special case of Theorem 6.4 we have the

8.2. Proposition. *Let (G, δ, ε) be a regular quasi-comonad on $\mathbb{A}$ and $\underline{\mathbb{A}}^G$ the category of regular quasi-G-comodules. For any endofunctor $T : \mathbb{A} \to \mathbb{A}$, there are bijective correspondences between*

(i) *liftings of T to $\widehat{T} : \underline{\mathbb{A}}^G \to \underline{\mathbb{A}}^G$, such that for any $(A, \upsilon) \in \underline{\mathbb{A}}^G$, the regular quasi-G-comodule structure map $\upsilon : TU^G \to GTU^G$ induces commutativity of the diagram*

(8.1)
$$\begin{tikzcd}
TG(A) \arrow[r, "\upsilon_{G(A)}"] & GTG(A) \arrow[d, "GT\varepsilon_A"] \\
T(A) \arrow[u, "T\upsilon_A"] \arrow[r, "\upsilon_A"] & GT(A);
\end{tikzcd}$$

(ii) *regular quasi-G-comodule structures $\upsilon : TU^G \to GTU^G$ inducing commutativity of the diagram corresponding to that in* (i)*;*

(iii) *natural transformations $\psi : TG \to GT$ with*

$$\psi \circ T\gamma = \psi = \gamma T \circ \psi$$

and commutative diagram

(8.2)
$$\begin{tikzcd}
TG \arrow[rr, "\psi"] \arrow[d, "T\delta"'] & & GT \arrow[d, "\delta T"] \\
TGG \arrow[r, "\psi G"] & GTG \arrow[r, "G\psi"] & GGT.
\end{tikzcd}$$

Now one may ask under which conditions the lifting is again a comonad.

8.3. Proposition. *Let* (G,δ,ε) *and* $(T,\check{\delta},\check{\varepsilon})$ *be regular quasi-comonads and assume that* T *can be lifted to* $\widehat{T}:\underline{\mathbb{A}}^G \to \underline{\mathbb{A}}^G$ *by* $\psi : TG \to GT$ *(see* 8.2*). Then, on* TG*, coproduct and quasi-counit are defined by*

$$\widehat{\delta} : TG \xrightarrow{\check{\delta}G} TTG \xrightarrow{TT\delta} TTGG \xrightarrow{T\psi G} TGTG, \quad \widehat{\varepsilon} : TG \xrightarrow{\psi} GT \xrightarrow{G\check{\varepsilon}} G \xrightarrow{\varepsilon} I_{\mathbb{A}}.$$

(1) *If* $\check{\delta}G : TG \to TTG$ *is* G*-colinear, then we get the commutative diagram*

$$\begin{tikzcd} TG \arrow[rr, "\psi"] \arrow[d, "\check{\delta}G"'] & & GT \arrow[d, "G\check{\delta}"] \\ TTG \arrow[r, "T\psi"] & TGT \arrow[r, "\psi T"] & GTT; \end{tikzcd} \tag{8.3}$$

(2) *If* (8.1) *is commutative, then* $(TG,\widehat{\delta},\widehat{\varepsilon})$ *is a quasi-comonad with* $\widehat{\varepsilon}$ *regular.*

(3) *In* (2), $\widehat{\delta}$ *is regular if and only if, in addition,*

$$TG \xrightarrow{\underline{\check{\gamma}}G} TG \xrightarrow{\psi} GT \xrightarrow{G\underline{\check{\gamma}}} GT \;=\; TG \xrightarrow{\underline{\check{\gamma}}G} TG \xrightarrow{\psi} GT, \tag{8.4}$$

where $\underline{\check{\gamma}} = \check{\varepsilon}T \circ \check{\delta}$*. In this case* $(TG,\widehat{\delta},\widehat{\varepsilon})$ *is a regular quasi-comonad.*

Proof. The situation is dual to that of Proposition 7.4. □

8.4. Quasi-comonad entwinings. *For regular comonads* (F,δ,ε)*,* $(T,\check{\delta},\check{\varepsilon})$*, and a natural transformation* $\psi : TG \to GT$*, the following are equivalent:*

(a) $(TG,\widehat{\delta},\varepsilon\check{\varepsilon}\circ\psi)$ *is a regular quasi-comonad on* $\mathbb{A}$*;*

(b) ψ *satisfies*

$$\psi = \psi \circ \tau = \gamma' T \circ \psi = G\underline{\check{\gamma}} \circ \psi = \psi \circ \underline{\check{\gamma}}G \tag{8.5}$$

and induces commutativity of the diagrams (8.2) *and* (8.3)*;*

(c) ψ *satisfies the equations* (8.5)*, induces commutativity of the diagram* (8.2)*, and we have natural transformations*

$$\check{\delta}G : TG \to TTG, \quad G\check{\varepsilon}\circ\psi : TG \to G,$$

where $\check{\delta}G$ *is* (G,G)*-bicolinear and* $G\check{\varepsilon}\circ\psi$ *is left* G*-colinear.*

If these conditions hold, we call (T,G,ψ) a *regular quasi-comonad entwining* and

$$G\check{\varepsilon}\circ\psi : TG \to G \quad \textit{and} \quad \varepsilon T\circ\psi : TG \to T$$

are quasi-comonad morphisms.

Proof. The proof is dual to 7.6. □

8.5. Weak crossed coproduct. Similar to the situation for monads, in 8.4 the coproduct on TG can also be expressed by replacing the natural transformations $\check{\delta}G$ and $G\check{\varepsilon}\circ\psi$ by any natural transformations

$$\nu : TG \to TTG \quad \text{and} \quad \zeta : TG \to G.$$

These have to be subject to certain conditions to make the coprocuct on TG coassociative and regular and $\varepsilon \circ \zeta : TG \to I_{\mathbb{A}}$ a regular quasi-counit on TG (dual to the case considered in 7.7).

Given ν and ζ as above, we may form

$$\widehat{\nu}:\ TG \xrightarrow{\ \nu\ } TTG \xrightarrow{TT\varepsilon} TT\ , \quad \widehat{\zeta}:\ TG \xrightarrow{\ \zeta\ } G \xrightarrow{\ \varepsilon\ } I_{\mathbb{A}}\ ,$$

and (dual to 7.8) one can see that these may also be used to define the coproduct and quasi-counit on TG. This leads to the *weak crossed coproduct* as considered (for coalgebras) in [10] and [11], for example.

9. Mixed entwinings and liftings

Throughout this section let (F, μ, η) denote a regular quasi-monad and (G, δ, ε) a regular quasi-comonad on any category $\mathbb{A}$.

9.1. Liftings of monads and comonads. In the diagrams in 7.1 and 8.1, we may consider $T = G$ or $T = F$ yielding the diagrams

$$\begin{tikzcd}
\underline{\mathbb{A}}_F \arrow[r, "\overline{G}"] \arrow[d, "U_F"'] & \underline{\mathbb{A}}_F \arrow[d, "U_F"] \\
\mathbb{A} \arrow[r, "G"] & \mathbb{A},
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathbb{A}}^G \arrow[r, "\widehat{F}"] \arrow[d, "U^G"'] & \underline{\mathbb{A}}^G \arrow[d, "U^G"] \\
\mathbb{A} \arrow[r, "F"] & \mathbb{A}.
\end{tikzcd}$$

In both cases the lifting properties are related to a natural transformation

$$\omega : FG \to GF.$$

The lifting in the left hand case requires commutativity of the diagrams (see Proposition 5.3)

(9.1)
$$\begin{tikzcd}
FFG \arrow[r, "F\omega"] \arrow[d, "\mu G"'] & FGF \arrow[r, "\omega F"] & GFF \arrow[d, "G\mu"] \\
FG \arrow[rr, "\omega"] & & GF,
\end{tikzcd}
\qquad
\begin{tikzcd}
FG \arrow[r, "\omega"] \arrow[d, "\vartheta G"'] \arrow[dr, "\omega"] & GF \arrow[d, "G\vartheta"] \\
FG \arrow[r, "\omega"] & GF,
\end{tikzcd}$$

whereas the lifting to $\underline{\mathbb{A}}^G$ needs commutativity of the diagrams (see Proposition 6.3)

(9.2)
$$\begin{tikzcd}
FG \arrow[rr, "\omega"] \arrow[d, "F\delta"'] & & GF \arrow[d, "\delta F"] \\
FGG \arrow[r, "\omega G"] & GFG \arrow[r, "G\omega"] & GGF,
\end{tikzcd}
\qquad
\begin{tikzcd}
FG \arrow[r, "\omega"] \arrow[d, "F\gamma"'] \arrow[dr, "\omega"] & GF \arrow[d, "\gamma F"] \\
FG \arrow[r, "\omega"] & GF.
\end{tikzcd}$$

To make $\overline{G}$ a quasi-comonad with coproduct δ, the latter has to be a quasi-F-module morphism, in particular, $\delta F : GF \to GGF$ has to be F-linear and this follows by commutativity of the rectangle in (9.2) provided the square in (9.1) is commutative.

To make the lifting $\widehat{F}$ a quasi-monad with multiplication μ, the latter has to be a quasi-G-comodule morphism, in particular, $\mu G : FFG \to FG$ has to be G-colinear and this follows by commutativity of the rectangle in (9.1) provided the square in (9.2) is commutative.

9.2. Natural transformations. The data given in 9.1 allow for natural transformations

$$\begin{aligned} \xi : &\quad G \xrightarrow{\eta G} FG \xrightarrow{\omega} GF \xrightarrow{\varepsilon F} F\ , \\ \widehat{\kappa} : &\quad GF \xrightarrow{\eta GF} FGF \xrightarrow{\omega F} GFF \xrightarrow{G\mu} GF\ , \\ \widehat{\tau} : &\quad FG \xrightarrow{F\delta} FGG \xrightarrow{\omega G} GFG \xrightarrow{\varepsilon FG} FG, \end{aligned}$$

with the properties

$$\begin{gathered} G\mu \circ \widehat{\kappa}F = \widehat{\kappa} \circ G\mu, \quad \widehat{\tau}G \circ F\delta = F\delta \circ \widehat{\tau}, \\ \mu \circ \xi F = \varepsilon F \circ \widehat{\kappa}, \quad \xi G \circ \delta = \widehat{\tau} \circ \eta G. \end{gathered}$$

(i) If the rectangle in (9.1) is commutative, then $\widehat{\kappa}$ is idempotent.

(ii) If the rectangle in (9.2) is commutative, then $\widehat{\tau}$ is idempotent.

Note that (i) is a special case of Lemma 7.3(4) and the proof of (ii) is dual to that for (i).

To make the liftings (regular) quasi-comonads or quasi-monads, respectively, we have to find (regular) quasi-units and quasi-counits. In what follows we consider these questions.

9.3. Quasi-counits for $\overline{G}$. *Assume the diagrams in (9.1) to be commutative. Then the following are equivalent:*

(a) *For any $(A, \varphi) \in \underline{\mathbb{A}}_F$, $\varepsilon_A : G(A) \to A$ is a quasi-F-module morphism;*

(b) *$\varepsilon F : GF \to F$ is F-linear;*

(c) *$\vartheta = \mu \circ F\eta$ induces commutativity of the diagram*

(9.3)
$$\begin{tikzcd} FG \arrow[r, "F\varepsilon"] \arrow[d, "\omega"'] & F \arrow[d, "\vartheta"] \\ GF \arrow[r, "\varepsilon F"] & F. \end{tikzcd}$$

If these conditions are satisfied, then (with ϑ and $\underline{\gamma}$ from 3.1, 4.1)

$$\mu G \circ F\widehat{\tau} = \widehat{\tau} \circ \mu G \quad \textit{and} \quad \widehat{\tau} = \vartheta \underline{\gamma}.$$

Proof. (a)⇒(b) is obvious.

(b)⇒(c) Condition (b) requires commutativity of the right rectangle in the diagram

$$\begin{tikzcd} FG \arrow[r, "FG\eta"] \arrow[d, "\omega"'] & FGF \arrow[r, "F\varepsilon F"] \arrow[d, "\omega F"] & FF \arrow[dd, "\mu"] \\ GF \arrow[r, "GF\eta"] \arrow[dr, "G\vartheta"'] & GFF \arrow[d, "G\mu"] & \\ & GF \arrow[r, "\varepsilon F"] & F, \end{tikzcd}$$

in which the square and the triangle are obviously. By the properties of ω the outer paths show commutativity of the diagram (9.3).

(c)⇒(a) Since φ is regular, F-linearity of ε means commutativity of the outer paths in the diagram

$$\begin{tikzcd}
FG \arrow[r, "\omega"] \arrow[d, "F\varepsilon"'] & GF(A) \arrow[r, "G\varphi"] \arrow[d, "\varepsilon F"'] & G(A) \arrow[d, "\varepsilon"] \\
F(A) \arrow[r, "\vartheta"] & F(A) \arrow[r, "\varphi"] & A;
\end{tikzcd}$$

herein the right hand square is commutative by naturality and the left hand square is commutative by assumption. □

9.4. Lifting to regular quasi-comonads. *Let ε be symmetric (see 4.8) and assume the diagrams in (9.1), (9.2) and (9.3) to be commutative. Then $(\overline{G}, \delta, \varepsilon)$ is a regular quasi-comonad on $\underline{\mathbb{A}}_F$.*

Proof. As mentioned in 9.1, $\overline{G}$ exists and is a quasi-monad. Now consider the diagrams

$$\begin{tikzcd}
GF \arrow[r, "\delta F"] & GGF \arrow[r, "G\delta F"] \arrow[dr, "G\underline{\gamma}F"'] & GGGF \arrow[d, "G\varepsilon GF"] \\
 & & GGF,
\end{tikzcd}
\qquad
\begin{tikzcd}
GF \arrow[r, "\delta F"] \arrow[dr, "\underline{\gamma}F"'] & GGF \arrow[r, "G\varepsilon F"] \arrow[d, "\varepsilon GF"] & GF \arrow[d, "\varepsilon F"] \\
 & GF \arrow[r, "\varepsilon F"] & F.
\end{tikzcd}$$

Since $G\underline{\gamma} \circ \delta = \delta$ and $\varepsilon \circ \underline{\gamma} = \varepsilon$ (see 4.10, (4.9)), these diagrams show that δ and ε are regular. □

Similar to the quasi-counits for $\overline{G}$ we can ask for quasi-units for $\widehat{F}$.

9.5. Quasi-units for $\widehat{F}$. *Assume the diagrams in (9.2) to be commutative. Then the following are equivalent:*

(a) *for any $(A, \upsilon) \in \underline{\mathbb{A}}^G$, $\eta_A : A \to F(A)$ is a quasi-G-comodule morphism;*

(b) *$\eta G : G \to FG$ is G-colinear;*

(c) *$\gamma = G\varepsilon \circ \delta$ induces commutativity of the diagram*

(9.4)
$$\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[d, "\gamma"'] & FG \arrow[d, "\omega"] \\
G \arrow[r, "G\eta"] & GF.
\end{tikzcd}$$

If these conditions are satisfied, then

$$G\widehat{\kappa} \circ \delta F = \delta F \circ \widehat{\kappa} \quad \textit{and} \quad \widehat{\kappa} = \gamma\underline{\vartheta}.$$

Proof. The proof is dual to that of 9.3. Let us just mention that the crucial diagram here is of the form

$$\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[dd, "\delta"'] & FG \arrow[d, "F\delta"'] \arrow[dr, "\gamma"] & \\
 & FGG \arrow[r, "FG\varepsilon"'] \arrow[d, "\omega G"'] & FG \arrow[d, "\omega"] \\
GG \arrow[r, "G\eta G"] & GFG \arrow[r, "GF\varepsilon"] & GF.
\end{tikzcd}$$

□

9.6. Lifting to regular quasi-monads. *Let η be symmetric (see 3.8) and assume the diagrams in (9.1), (9.2) and (9.4) to be commutative. Then $(\widehat{F}, \mu, \eta)$ is a regular quasi-monad on $\underline{\mathbb{A}}^G$.*

Proof. This is dual to 9.4. □

One may consider other choices for a counit for $\overline{G}$ or a unit for $\widehat{F}$.

9.7. Alternative quasi-counits for $\overline{G}$. *Assume η to be symmetric (see 3.8) and the diagrams in (9.1) to be commutative. With the notations from 9.2, the following are equivalent:*

(a) *for any $(A, \varphi) \in \underline{\mathbb{A}}_F$,*

$$\overline{\varepsilon}_A : \; G(A) \xrightarrow{\xi_A} F(A) \xrightarrow{\varphi} A$$

is a quasi-F-module morphism;

(b) $\overline{\varepsilon}F : \; GF \xrightarrow{\xi F} FF \xrightarrow{\mu} F \;\; (= \; GF \xrightarrow{\widehat{\kappa}} GF \xrightarrow{\varepsilon F} F\;)$ *is F-linear;*

(c) *commutativity of the diagram*

$$\begin{tikzcd} FFG \arrow[r, "F\omega"] & FGF \arrow[r, "F\varepsilon F"] & FF \arrow[d, "\mu"] \\ FG \arrow[u, "F\eta G"] \arrow[r, "\omega"] & GF \arrow[r, "\varepsilon F"] & F. \end{tikzcd} \tag{9.5}$$

If these conditions are satisfied, then

$$\widehat{\tau} = \mu G \circ F\widehat{\tau} \circ F\eta G.$$

Proof. (a)⇒(b) is obvious.

(b)⇒(c) Condition (b) on $\overline{\varepsilon}F$ means commutativity of the big rectangle in the diagram

$$\begin{tikzcd} & & FFG \arrow[r, "F\omega"] \arrow[d, "FFG\eta"] & FGF \arrow[r, "F\varepsilon F"] & FF \arrow[d, "FF\eta"] \arrow[dddr, "\mu"] & \\ FG \arrow[urr, "F\eta G"] \arrow[r, "FG\eta"'] \arrow[d, "\omega"] & FGF \arrow[r, "F\eta GF"] \arrow[d, "\omega F"] & FFGF \arrow[r, "F\omega F"'] & FGFF \arrow[r, "F\varepsilon FF"'] & FFF \arrow[dd, "F\mu"] & \\ GF \arrow[r, "GF\eta"] \arrow[dr, "G\vartheta"'] & GFF \arrow[d, "G\mu"] & & & & \\ & GF \arrow[r, "\eta GF"] & FGF \arrow[r, "\omega F"] & GFF \arrow[r, "\varepsilon FF"] & FF \arrow[r, "\mu"] & F \end{tikzcd}$$

where the bottom line can be written as $GF \xrightarrow{\widehat{\kappa}} GF \xrightarrow{\varepsilon F} F$. From Lemma 7.3(3) we know that $\widehat{\kappa} \circ \omega = \omega \circ \underline{\vartheta}G$. By symmetry of η, that is, $\vartheta = \underline{\vartheta}$, the left outer path reads

$$\widehat{\kappa} \circ \kappa \circ \omega = \widehat{\kappa} \circ \omega = \omega \circ \kappa = \omega.$$

Moreover, the right hand triangle is commutative since here $\mu \circ F\vartheta = \mu$ (see 3.10). This means commutativity of the diagram (9.5).

(c)⇒(a) The assertion requires commutativity of the outer paths in the diagram

$$\begin{tikzcd}
FG(A) \arrow[r, "F\eta G_A"] \arrow[d, "\omega_A"'] & FFG(A) \arrow[r, "F\omega_A"] & FGF(A) \arrow[r, "F\varepsilon F_A"] & FF(A) \arrow[d, "\mu_A"] \\
GF(A) \arrow[d, "G\varphi"'] & & GF(A) \arrow[r, "\varepsilon F_A"] \arrow[d, "G\varphi"] & F(A) \arrow[d, "\varphi"] \\
G(A) \arrow[r, "\eta G_A"] & FG(A) \arrow[ur, "\omega_A"] & G(A) \arrow[r, "\varepsilon_A"] & A.
\end{tikzcd}$$

By regularity (see 3.10) the lower path reads as

$$\varepsilon_A \circ G\varphi \circ \omega_A \circ \eta G_A \circ G\varphi \circ \omega_A = \varepsilon_A \circ G\varphi \circ \omega_A = \varphi \circ \varepsilon_A \circ \omega_A$$

and - by commutativity of (9.5) - this is equal to the upper path. This shows commutativity of the diagram as claimed. □

Notice that commutativity of (9.3) implies commutativity of (9.5).

9.8. Alternative quasi-units for $\widehat{F}$. *Assume ε to be symmetric (see 4.8) and the diagrams in (9.2) to be commutative. Then the following are equivalent:*

(a) *For any $(A, \upsilon) \in \underline{\mathbb{A}}^G$,*

$$\widehat{\eta}: \ A \xrightarrow{\upsilon} G(A) \xrightarrow{\xi_A} F(A)$$

is a quasi-G-comodule morphism;

(b) $\widehat{\eta}G : \ G \xrightarrow{\eta G} FG \xrightarrow{\widehat{\tau}} FG \ \ (= \ G \xrightarrow{\delta} GG \xrightarrow{\xi G} FG\ )$ *is G-colinear;*

(c) *commutativity of the diagram*

(9.6)
$$\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[d, "\delta"'] & FG \arrow[r, "\omega"] & GF \\
GG \arrow[r, "G\eta G"] & GFG \arrow[r, "G\omega"] & GGF. \arrow[u, "G\varepsilon F"']
\end{tikzcd}$$

If these conditions are satisfied, then

$$\widehat{\kappa} = G\varepsilon F \circ G\widehat{\kappa} \circ \delta F.$$

Proof. The situation is dual to 9.7. □

Notice that commutativity of (9.4) implies commutativity of (9.6).

9.9. Theorem. *With the data given in 9.1, assume ε to be symmetric and the diagrams in (9.1), (9.2) and (9.5) to be commutative.*

(1) *If (9.6) is commutative, then $\overline{\varepsilon}$ in 9.7 is a regular quasi-counit for δ, and for $\overline{\delta} : G \to GG$ with*

$$\overline{\delta}F : \ GF \xrightarrow{\delta F} GGF \xrightarrow{G\widehat{\kappa}} GGF,$$

$(\overline{G}, \overline{\delta}, \overline{\varepsilon})$ is a regular quasi-comonad.

(2) *If (9.4) is commutative, then*

  (i) *$\overline{\delta} = G\widehat{\kappa} \circ \delta F = \delta F \circ \widehat{\kappa}$ and $\overline{\varepsilon}$ is symmetric;*

  (ii) *if γ and $\underline{\vartheta}$ are the identities, then $(\overline{G}, \overline{\delta}, \overline{\varepsilon})$ is a comonad on $\underline{\mathbb{A}}_F$.*

Proof. (1) Recall that $\overline{\varepsilon}F = \varepsilon F \circ \widehat{\kappa}$ and consider the diagram

$$\begin{tikzcd}
GGF \arrow[r, "G\widehat{\kappa}"] & GGF \arrow[d, "G\varepsilon F"] & & \\
GF \arrow[u, "\delta F"] \arrow[r, "\widehat{\kappa}"] & GF \arrow[r, "\widehat{\kappa}"] & GF \arrow[r, "\varepsilon F"] & F
\end{tikzcd}$$

in which the square is commutative (see 9.8). Now regularity of $\overline{\varepsilon}$ follows by the fact that $\widehat{\kappa}$ is idempotent (see 9.2).

Since $\overline{\varepsilon}$ is a regular quasi-counit, by Proposition 4.12, a regular quasi-coproduct can be defined by $\overline{\delta} = G\overline{\varepsilon}G \circ G\delta \circ \delta$. Writing this out we obtain the commutative diagram

$$\begin{tikzcd}
 & GGG \arrow[r, "G\eta G"] & GFGG \arrow[rr, "G\omega G"] & & GGFG \arrow[r, "G\varepsilon FG"] \arrow[d, "GG\omega"] & GFG \arrow[d, "G\omega"] & \\
G \arrow[r, "\delta"] & GG \arrow[u, "G\delta"] \arrow[r, "G\eta G"] & GFG \arrow[u, "GF\delta"] \arrow[r, "G\omega"] & GGF \arrow[r, "G\delta F"] & GGGF \arrow[r, "G\varepsilon GF"] & GGF \arrow[r, "GG\varphi"] & GG,
\end{tikzcd}$$

where $\varphi : F(-) \to -$ stands for any F-module structure map. By the symmetry of ε, $\omega = \varepsilon GF \circ \delta F \circ \omega$ and we obtain

$$\overline{\delta} : \; G \xrightarrow{\delta} GG \xrightarrow{G\eta G} GFG \xrightarrow{G\omega} GGF \xrightarrow{GG\varphi} GG.$$

This yields $\overline{\delta}F$ as given in (1).

(2)(i) In the diagram

$$\begin{tikzcd}
GF \arrow[r, "\eta GF"] \arrow[dd, "\delta F"'] & FGF \arrow[r, "\omega F"] \arrow[d, "F\delta F"] & GFF \arrow[r, "G\mu"] \arrow[dd, "\delta F"] & GF \arrow[dd, "\delta F"] \\
 & FGGF \arrow[d, "\omega GF"] & & \\
GGF \arrow[r, "G\eta GF"] & GFGF \arrow[r, "G\omega F"] & GGFF \arrow[r, "GG\mu"] & GGF,
\end{tikzcd}$$

the first rectangle is commutative by commutativity of (9.4) (see 9.5), the second one by commutativity of (9.2) and the third one by naturality. This shows the first equality in (i). The second one is shown in 9.5.

Symmetry of $\overline{\varepsilon}$ requires $G\overline{\varepsilon} \circ \overline{\delta} = \overline{\varepsilon}G \circ \overline{\delta}$. The left side means (see diagram in the proof of (1))

$$G\varepsilon F \circ G\widehat{\kappa} \circ G\widehat{\kappa} \circ \delta F = G\varepsilon F \circ G\widehat{\kappa} \circ \delta F = \widehat{\kappa}.$$

The right hand side is the upper path in the diagram

$$\begin{tikzcd}
GF \arrow[r, "\delta F"] \arrow[dr, "\eta GF"'] & GGF \arrow[r, "\eta GGF"] & FGGF \arrow[r, "\omega GF"] & GFGF \arrow[r, "\varepsilon FGF"] \arrow[d, "G\omega F"] & FGF \arrow[d, "\omega F"] & \\
 & FGF \arrow[ur, "F\delta F"] \arrow[r, "\omega F"'] & GFF \arrow[r, "\delta FF"'] & GGFF \arrow[r, "\varepsilon GFF"'] & GFF \arrow[r, "G\mu"'] & GF,
\end{tikzcd}$$

where the left triangle and the right square are commutative by naturality and the pentagon is commutative since so is (9.2). Since $\gamma F \circ \omega = \omega$, the lower path reads as $G\mu \circ \omega F \circ \eta GF = \widehat{\kappa}$. This proves the symmetry of $\overline{\varepsilon}$.

(ii) Since $\widehat{\kappa} = \gamma\underline{\vartheta} = I_G$ (see 9.5), the computations in the proof of (ii) show that $G\overline{\varepsilon} \circ \overline{\delta} = \overline{\varepsilon}G \circ \overline{\delta} = I_G$, that is, $\overline{\varepsilon}$ is indeed a counit for $\overline{\delta}$. □

9.10. Theorem. *With the data given in 9.1, assume η to be symmetric and the diagrams in (9.1), (9.2), and (9.6) to be commutative.*

(1) *If (9.5) is commutative, then $\widehat{\eta}$ in 9.8 is a regular quasi-unit for μ, and for $\widehat{\mu} : FF \to F$ with*

$$\widehat{\mu}G :\ FFG \xrightarrow{F\widehat{\tau}} FFG \xrightarrow{\mu G} FG,$$

$(\widehat{F}, \widehat{\mu}, \widehat{\eta})$ is a regular quasi-monad.

(2) *If (9.3) is commutative, then*
  (i) *$\widehat{\mu} = \mu G \circ F\widehat{\tau} = \widehat{\tau} \circ \mu G$;*
  (ii) *$\widehat{\eta}$ is symmetric;*
  (iii) *if ϑ and $\underline{\gamma}$ are the identities, then $(\widehat{F}, \widehat{\mu}, \widehat{\eta})$ is a monad on $\underline{\mathbb{A}}^G$.*

Proof. This is dual to Theorem 9.9. □

As mentioned after Definition 3.8, a regular quasi-monad (F, μ, η) with η symmetric is called a *premonad* by Böhm in [4] and the preceding theorems may be compared with results there. Here we have shown that regularity of η and ε together with commutativity of (9.1), (9.2), (9.3), and (9.4) imply that $(\overline{G}, \overline{\delta}, \overline{\varepsilon})$ is a regular quasi-comonad on $\underline{\mathbb{A}}_F$ whereas $(\widehat{F}, \widehat{\mu}, \widehat{\eta})$ is a regular quasi-monad on $\underline{\mathbb{A}}^G$. For this, the given conditions are sufficient but not necessary. Equivalent conditions for these assertions are considered in the Corollaries 5.1 and 5.6 in [4] for the case that (G, δ, ε) is a comonad and (F, μ, η) is a monad and the liftings are to the counital G-comodules and unital F-modules $\mathbb{A}_F$, respectively. The latter conditions are also assumed in a recent paper on the subject by Böhm, Lack and Street [5].

Specialising the situation considered in 9.1 to the case $F = G$ suggests the definition of *weak bimonads* and eventually of *weak Hopf monads* on arbitrary categories generalising the notions studied in [16]. Details should be worked out in a subsequent article.

Acknowledgements. The author wants to thank Gabriella Böhm, Tomasz Brzeziński and Bachuki Mesablishvili for their interest in his work and for helpful comments on the subject.

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