Rank reversal in the Index of Multiple Deprivation

Robert Calvert Jump*

Jo Michell†

November 26, 2019

Abstract

This research note demonstrates that the Index of Multiple Deprivation suffers from rank reversal. We present a simple numerical example and a Monte Carlo analysis for the 2019 English index. We conclude by considering a potential solution to the problem.

1 Introduction

The English, Scottish and Welsh Indices of Multiple Deprivation, and the Northern Ireland Multiple Deprivation Measure, are the official measures of geographic deprivation in the United Kingdom. The first English index was introduced by the Office of the Deputy Prime Minister in 2000, with the most recent version being released by the Ministry of Housing, Communities and Local Government in 2019. The various editions of the index have been used to guide resource distribution by central government, including the identification of local authorities eligible for New Labour's Neighbourhood Renewal Fund and National Strategy for Neighbourhood Renewal (Deas et al., 2003).

In this research note we demonstrate that the Index of Multiple Deprivation suffers from rank reversal. Specifically, the rank position of two neighbourhoods according to their index scores is sensitive to the set of neighbourhoods used to calculate the index. We first describe the construction of the Index of Multiple Deprivation (IMD), followed by a brief discussion of rank reversal. We then present a simple example of rank reversal in the IMD using three neighbourhoods from the 2019 index, followed by a Monte Carlo sensitivity analysis. As rank reversal is widely considered to be an undesirable attribute of poverty and deprivation indices, we conclude by considering a potential solution to the problem.

*Institute of Political Economy, Governance, Finance and Accountability, University of Greenwich. Email: r.g.calvertjump@greenwich.ac.uk.

†Department of Accounting, Economics and Finance, University of the West of England. Email: jo.michell@uwe.ac.uk.

2 The Index of Multiple Deprivation

The English IMD uses lower-level super output areas as its units of observation, which are small neighbourhoods with approximately 1500 inhabitants on average. The index is based on seven domains of deprivation: income deprivation; employment deprivation; education, skills, and training deprivation; health deprivation and disability; crime; barriers to housing and services; and living environment deprivation. This reflects the multi-dimensional nature of deprivation (Rawls, 1971; Townsend, 1979, 1987; Sen, 2009).

Denote a score for domain j in neighbourhood i as z_{ij}. These scores are computed using the indicators listed in table 1, and are generally weighted averages. The domain scores are then ranked such that r_{ij} is the position of neighbourhood i in the set of all English neighbourhoods ranked by domain score j. Normalised domain ranks $R_{ij} \in (0, 1]$ are then computed, where $R_{ij} = 1$ for the neighbourhood with the highest domain score. These normalised ranks are transformed as follows,

$$X_{ij} = -23 \ln \left(1 - R_{ij} \left(1 - e^{-\frac{100}{23}}\right)\right), \tag{1}$$

such that the transformed scores $X_{ij} \in (0, 100]$ have an approximately exponential distribution. The scaling parameter in (1) is equal to 23, which ensures that approximately 10% of the lower-level super output areas fall within the top half of the distribution and 90% in the bottom half. Finally, the IMD score is constructed as a weighted average of these domain scores,

$$\text{IMD}_i = \sum_{j=1}^{7} \omega_j X_{ij}, \tag{2}$$

where the weights ω_j are given in the first column of table 1. We can therefore summarise the IMD score for neighbourhood i as,

$$\text{IMD}_i = \sum_{j=1}^{7} \omega_j f(r_{ij}), \tag{3}$$

i.e. as a weighted average of transformed domain ranks. In turn, the IMD score itself is then ranked and reported as an ordinal variable. Detailed descriptions of the IMD, including a rationale for the transform in (1), can be found in Smith et al. (2015a,b).

3 Rank reversal

Rank reversal is defined by the following:

> If A is ranked above B in the set of alternatives $\{A, B\}$, a *rank reversal* occurs if B is ranked above A in the set of alternatives $\{A, B, C\}$.

See e.g. Wang & Luo (2009). Suppose we find that neighbourhood A is more deprived than neighbourhood B when we calculate the IMD using information from neighbourhoods A and B only. A rank reversal occurs if we find that neighbourhood B is more deprived than neighbourhood A when we calculate the IMD using information from neighbourhoods A, B

Table 1: IMD domains and indicators

Sub-index	Indicators entering sub-index
Income (22.5%):	Income support families,
	Income-based Jobseeker's Allowance families,
	Income-based Employment and Support Allowance families,
	Pension Credit (Guarantee) families,
	Child Tax Credit and Working Tax Credit families, below 60% median income and not counted above,
	Asylum seekers in England in receipt of subsistence support, accommodation support, or both.
Employment (22.5%):	Adult claimants of Jobseeker's Allowance,
	Adult claimants of Employment and Support Allowance,
	Adult claimants of Incapacity Benefit,
	Adult claimants of Severe Disablement Allowance,
	Adult claimants of Carer's Allowance.
Education (13.5%):	Key stage 2 attainment: average points score,
	Key stage 4 attainment: average points score,
	Secondary school absence rate,
	Students staying on in education post 16,
	Students entering higher education,
	Adults with no or low qualifications,
	Adult English language proficiency.
Health and disability (13.5%):	Years of potential life lost,
	Comparative illness and disability ratio,
	Acute morbidity,
	Mood and anxiety disorders.
Housing and services (9.3%):	Road distance to post office, primary school, general store or supermarket, GP surgery,
	Household overcrowding,
	Homelessness,
	Housing affordability.
Crime (9.3%):	Crime rates for violence, burglary, theft and criminal damage.
Living environment (9.3%):	Housing in poor condition,
	Houses without central heating,
	Air quality,
	Road traffic accidents.

Notes: See Smith et al. (2015b) for more details. The domain weights ω_j in (2) are given in brackets in the first column.

and C. Rank reversal is closely related to the independence of irrelevant alternatives axiom in social choice theory, see e.g. Morreau (2016).

There appears to be little discussion of rank reversal in the context of the IMD. However, the OECD acknowledge that dependence on irrelevant alternatives is an undesirable characteristic for composite indicators (OECD, 2008), and the Human Development Index has been criticised for being sensitive to rank reversal (Sayed et al., 2018). From a policy perspective, rank reversal implies the possibility that a neighbourhood might be eligible for the receipt of local government funding when compared with neighbourhoods across the whole of England, but cease to be eligible when compared with neighbourhoods in its locality. In general, rank reversal is an irrational attribute of a poverty or deprivation index, as the following example makes clear.

4 A simple example with three neighbourhoods

Consider our summary of the IMD score in (3). With equal weighting, and when f is the identity transform, this is simply a Borda count over the seven domains (Mas-Colell et al., 1995, pp.794). As Borda counts are known to violate independence of irrelevant alternatives, any deprivation index of the general form in (3) might be expected to suffer from rank reversal. In fact, it is not difficult to identify subsets of neighbourhoods in the English IMD that can be used to demonstrate rank reversal. The example below uses two neighbourhoods in Birmingham and a third neighbourhood in Hertfordshire, where the domain scores are taken from the 2019 Index of Multiple Deprivation:

Calculating IMD ranks using two neighbourhoods:

Neighbourhood	Domain scores	IMD score	IMD rank
Birmingham 046A	$\{0.26, 0.184, 57.0, 1.17, 0.74, 27.0, 35.8\}$	57.4	1
Birmingham 046C	$\{0.27, 0.181, 42.5, 1.11, 0.89, 34.5, 37.0\}$	58.2	2

Calculating IMD ranks using three neighbourhoods:

Neighbourhood	Domain scores	IMD score	IMD rank
Birmingham 046A	$\{0.26, 0.184, 57.0, 1.17, 0.74, 27.0, 35.8\}$	59.1	3
Birmingham 046C	$\{0.27, 0.181, 42.5, 1.11, 0.89, 34.5, 37.0\}$	48.7	2
East Hertfordshire 002B	$\{0.06, 0.037, 10.2, \text{-}1.9, \text{-}0.24, 45.4, 51.3\}$	26.1	1

In this example, we first calculate the IMD ranks for Birmingham 046A and 046C just using the domain scores for these two neighbourhoods, from which we conclude that Birmingham 046C is more deprived than Birmingham 046A.[1] We then add a third neighbourhood in East Hertfordshire to the calculation, and re-compute the ranks. From this we conclude that Birmingham 046A is more deprived than Birmingham 046C, hence a rank reversal has occurred.[2]

[1] Lower-level super output areas - which we refer to as neighbourhoods - do not have official names in the same way as local authorities and wards. Instead, they are referred to by an alpha-numeric code appended to their parent local authority.

[2] In this example we calculate the normalised domain ranks R_{ij} by dividing through by the maximum rank, but the result is robust to various normalisation procedures.

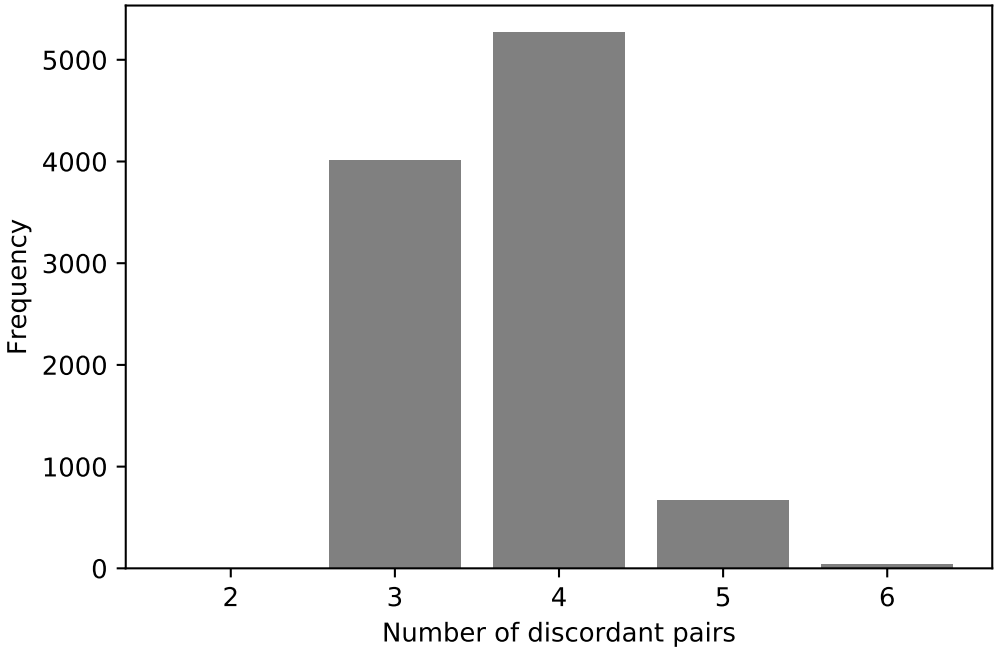

Figure 1: Frequency distribution of discordant pairs of neighbourhoods in the North West of England when comparator neighbourhoods are varied at random.

5 A Monte Carlo sensitivity analysis

Although the simple example above demonstrates that the IMD suffers from rank reversal, it does not illustrate the extent of the problem. We therefore compute a Monte Carlo analysis of the sensitivity of IMD ranks within the North West of England to changes in the set of comparator neighbourhoods.

To do this, we first create two identical sets, each of which contains every neighbourhood in England. From both of these sets we remove 50% of neighbourhoods *outside* the North West at random, and then rank the neighbourhoods in each set according to the IMD in (1) - (2). If we denote the IMD score of neighbourhood i according to the first set by IMD_i^1, and the IMD score of the same neighbourhood according to the second set by IMD_i^2, then we can define a pair of neighbourhoods as discordant if,

$$\text{sign}(\text{IMD}_i^1 - \text{IMD}_j^1) \neq \text{sign}(\text{IMD}_i^2 - \text{IMD}_j^2).$$

We then count the pairs of neighbourhoods in the North West of England that are discordant according to this definition, giving a simple estimate of the extent to which the IMD is sensitive to rank reversal.

Figure 1 plots the frequency distribution of discordant pairs of neighbourhoods when this exercise is repeated 10,000 times. Changing the set of comparator neighbourhoods always results in at least one pair of neighbourhoods in the North West switching rank position, with the modal number of discordant pairs equal to four. While this result suggests that it is not difficult to identify instances of rank reversal in the IMD, it is worth noting that there are 4,497 neighbourhoods in the North West and therefore 10,109,256 different pairs of neighbourhoods.

6 A potential solution

Rank reversal is an undesirable attribute for a poverty or deprivation index. As mentioned above, it is closely related to the independence of irrelevant alternatives axiom in social choice theory, which in the present context can be defined by the following:

> A deprivation index respects *independence of irrelevant alternatives* if the relative ranking of two neighbourhoods depends only on their relative ranking in each domain.

This way of stating the axiom follows Geanakoplos (2005), and it follows that the IMD in (1) - (2) does not satisfy the axiom as the IMD scores are weighted averages of transformed ranks, and thus the domain ranks are treated as cardinal before the IMD scores are themselves ranked[3].

If we are to continue treating deprivation as an aggregation of ordinal domain ranks, then it follows from Arrow's impossibility theorem that dependence on irrelevant alternatives cannot be resolved while simultaneously satisfying the following attributes:

1. *Unrestricted Domain*, i.e. any combination of domain ranks can be used to compute the deprivation index;
2. *Unanimity*, i.e. the deprivation index ranks neighbourhood A above neighbourhood B whenever every domain ranks A above B;
3. *Non-dictatorship*, i.e. the deprivation index is not based entirely on a single domain;
4. *Complete Ordering*, i.e. the index provides an unambiguous ranking between every pair of neighbourhoods.

See e.g. Morreau (2016) or Weymark (2016). The fact that deprivation indices are calculated using empirical data makes it operationally difficult to abandon attribute 1. Attribute 2 is intuitively necessary for any meaningful deprivation index, and it is similarly difficult to abandon attribute 3 given the multi-dimensional nature of deprivation. Thus if we wish to resolve the rank reversal issue - without resorting to a cardinal measure of deprivation - we must abandon attribute 4.

Abandoning completeness does not, however, mean that a transitive quasi-ordering, or partially ordered set, cannot be computed from the domain ranks that comprise the IMD. Consider, for example, the Pareto quasi-ordering in which a neighbourhood i is considered more (less) deprived than a neighbourhood k whenever i is more (less) deprived than k in all seven domains, and is otherwise not comparable. As shown in Weymark (1984), this is the only quasi-ordering which satisfies attributes 1 - 3 above plus independence of irrelevant alternatives, and a further condition that there is no strict subset of the domains that completely determines the quasi-ordering.

Consider, as a simple example, two domains of deprivation and four neighbourhoods indexed by A, B, C, D. The four neighbourhoods are ranked according to the two domains as follows:

[3]This means that the difference in IMD scores between two neighbourhoods depends on the magnitude of the differences between their ranks over the seven domains, not just the sign of these differences, and thus (implicitly) the position of other "irrelevant" neighbourhoods in the domain rankings.

$$\text{Domain 1: } A \succ B \succ C \succ D, \quad \text{Domain 2: } A \succ C \succ B \succ D,$$

where $\succ$ denotes "ranked above", or "more deprived than". According to the Pareto quasi-ordering, our deprivation index has $A \succ B$, $A \succ C$, $A \succ D$, $B \succ D$, and $C \succ D$, which can be summarised as,

$$\text{Deprivation Index: } A \succ \{B, C\} \succ D.$$

This states that neighbourhood A is more deprived than both B and C, and neighbourhoods B and C are both more deprived than neighbourhood D, but we cannot compare B with C. We have therefore divided the four neighbourhoods into three subsets which might be characterised as "most-deprived", "mid-deprived", and "least-deprived". As IMD ranks are often divided into deciles or quartiles for the purposes of interpretation, statistical analysis and policy-making, this type of quasi-ordering might not result in a significant loss of practical information, and is worth investigating further as a solution to the problem of rank reversal in the Index of Multiple Deprivation.

References

Deas, I., Robson, B., Wong, C., & Bradford, M. (2003). Measuring neighbourhood deprivation: a critique of the Index of Multiple Deprivation. *Environment and Planning C: Government and Policy*, *21*, 883–903.

Geanakoplos, J. (2005). Three brief proofs of Arrow's impossibility theorem. *Economic Theory*, *26*(1), 211–215.

Mas-Colell, A., Whinston, M., & Green, J. (1995). *Microeconomic theory*. Oxford university press New York.

Morreau, M. (2016). Arrow's Theorem. In Edward N. Zalta (Ed.) *The Stanford Encyclopedia of Philosophy (Winter 2016 Edition)*.
URL {https://plato.stanford.edu/archives/win2016/entries/arrows-theorem/}

OECD (2008). Handbook on Constructing Composite Indicators: Methodology and User Guide. Tech. rep., Organisation for Economic Cooperation and Development.
URL https://www.oecd.org/sdd/42495745.pdf

Rawls, J. (1971). *A Theory of Justice*. Cambridge, Mass: Belknap Press of Harvard University Press.

Sayed, H., Hamed, R., Hosny, S. H., & Abdelhamid, A. H. (2018). Avoiding ranking contradictions in Human Development Index using goal programming. *Social Indicators Research*, *138*(2), 405–442.

Sen, A. (2009). *The Idea of Justice*. Cambridge, Mass: Belknap Press of Harvard University Press.

Smith, T., Noble, M., Noble, S., Wright, G., McLennan, D., & Plunkett, E. (2015a). The English Indices of Deprivation 2015 Research Report. *Department for Communities and Local Government Report*.

Smith, T., Noble, M., Noble, S., Wright, G., McLennan, D., & Plunkett, E. (2015b). The English Indices of Deprivation 2015 Technical Report. *Department for Communities and Local Government Report*.

Townsend, P. (1979). *Poverty in the United Kingdom*. Harmondsworth: Penguin.

Townsend, P. (1987). Deprivation. *Journal of Social Policy*, *16*, 125–146.

Wang, Y.-M., & Luo, Y. (2009). On rank reversal in decision analysis. *Mathematical and Computer Modelling*, *49*(5-6), 1221–1229.

Weymark, J. A. (1984). Arrow's theorem with social quasi-orderings. *Public Choice*, *42*(3), 235–246.

Weymark, J. A. (2016). Social welfare functions. In *The Oxford Handbook of Well-Being and Public Policy*, (p. 1667). Oxford University Press.