O.R. Applications

An efficient envelope-based Branch and Bound algorithm for non-convex combined heat and power production planning

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Abstract

Combined heat and power (CHP) production is universally accepted as one of the most energy-efficient technologies to produce energy with lower fuel consumption and fewer emissions. In CHP technology, heat and power generation follow a joint characteristic. Traditional CHP production is usually applied in backpressure plants, where the joint characteristic can often be represented by a convex model. Advanced CHP production technologies such as backpressure plants with condensing and auxiliary cooling options, gas turbines, and combined gas and steam cycles may require non-convex models. Cost-efficient operation of a CHP system can be planned using an optimization model based on forecasts for heat load and power price. A long-term planning model decomposes into thousands of single-period models, which can be formulated in the convex case as linear programming (LP) problems, and in the non-convex case as mixed integer programming (MIP) problems.

In this paper, we introduce EBB algorithm, for solving the non-convex single-period CHP models of a long-term planning problem under the deregulated power market. EBB is based on the Branch and Bound (B&B) algorithm where tight lower bounds are computed analytically for pruning the search tree and the LP sub-problems are solved through an efficient envelope-based dual algorithm. We compare the performance of EBB with realistic models against the ILOG CPLEX 9.0 MIP solver and the Power Simplex (PS)-based B&B algorithm (PBB). PS is an efficient specialized primal-based Simplex algorithm developed for convex CHP planning problems. EBB is from 661 to 955 times (with average 785) faster than CPLEX and from 11 to 31 times (with average 24) faster than PBB.

Keywords: Branch and Bound; Mixed integer programming; Envelope; Combined heat and power production; Energy optimization

1. Introduction

The competitive pressures to cut costs and the growing emphasis on environmental protection are compelling the energy industry to develop energy efficient production technologies with lower fuel consumption and fewer emissions. The deregulated power market requires that the marginal production cost of electricity must...
respond to spot market variation. The European energy industry must for example consider the already launched green certificates (European Commissions, 2000) and manage emissions trading starting from year 2005 (Commission of the European Communities, 2000). Combined heat and power (CHP) production is a leading technology to respond to the market demand and environmental concerns because of its high energy efficiency. CHP means the simultaneous production of useful heat and electric power. When steam or hot water is produced for an industrial plant or a residential area, power can be generated as a by-product. Vice versa, surplus heat from a power plant can be used for industrial purposes, or for heating space and water. CHP technology produces a given amount of electric power and heat with 10–30% less fuel than it takes to produce the power and heat separately. The decrease in fuel consumption causes direct savings in money and reduction in emissions. The EU commission encourages the use of more efficient energy technologies, including CHP technology, to respond to the environmental policy. Thus, the EU commission announces to raise the share of electricity produced by CHP technology from 9% to 18% during the years 1997–2010 (CEC Commission of the European Communities, 1997).

Besides CHP plants, a CHP system may also include condensing power plants, hydropower, heat plants and purchase and sales contracts for heat and power. Cost-efficient operation of a CHP system can be planned by using an optimization model. In CHP technology, heat and power generation follow a joint characteristic. Traditional CHP production is usually applied in backpressure plants, where the joint characteristic can typically be represented by convex (linear programming, LP) model (Gardner and Rogers, 1997; Lahdelma and Hakonen, 2003; Rong and Lahdelma, 2007a; Rong et al., 2006). Advanced production technologies and deregulated power market pose new challenges to CHP production planning. Advanced production technologies often require non-convex models, which makes the optimization problem much more difficult than before. Non-convex models can be encoded as mixed integer programming (MIP) problems (Makkonen and Lahdelma, 2006). MIP refers to problems where some of the decision variables are constrained to integral values. Earlier, in the regulated power market, the CHP production was symmetrically driven by heat and power demand. Under the deregulated power market, CHP planning problem is asymmetrical, where heat is produced to meet variable heat demand and power is produced to respond to the volatile power price on the market. Optimization must be faster than before for two reasons. Firstly, rapid re-optimization is required when the situation on the market changes. Secondly, risk analysis through stochastic simulation requires solving a large number of models rapidly (Makkonen and Lahdelma, 1998; Makkonen and Lahdelma, 2001; Rong and Lahdelma, 2005a, 2007b). To facilitate the description, the planning problems under the regulated market and deregulated market are called symmetrical and asymmetrical planning problems, respectively.

Using various decomposition and coordination techniques, a medium- and long-term planning model can be decomposed into a sequence of single-period models. The natural period length is typically one hour, but could be e.g. 30 or 15 minutes in some market areas. In this paper we refer to the single-period models as hourly models, although a different period length could be easily applied. The medium- and long-term planning of a CHP system is based on hourly forecasts for heat demand and power price. The hourly model in this paper can be used as a basic component for the long-term planning problem in more complex settings with dynamic constraints. Dynamic constraints couple the hourly production models together. Dynamic constraints may include e.g. energy storage constraints (Rolfsman, 2004; Rong and Lahdelma, 2005b), power ramp-rate constraints (Rong and Lahdelma, 2007c), and shut-down and start-up constraints. Depending on the decomposition algorithm, and whether the problem includes dynamic constraints, the hourly models must be solved once or multiple times to obtain a solution for the multi-period model. The applicable decomposition techniques include e.g. dynamic programming, Lagrangian decomposition, Dantzig–Wolfe decomposition, Benders’ decomposition, and various heuristic techniques. The interested reader could refer, for example, to Alguacil and Conejo (2000), Baldick (1995), Bloom (1983), Bos et al. (1996), Dantzig (1963), Guan et al. (1992), Guan et al. (1995), Lahdelma and Makkonen (1996), Lahdelma and Ruuth (1989, 1993), Lautala et al. (1992), Perez-Ruiz and Conejo (2000), Rolfsman (2004), Rong and Lahdelma (2005b, 2007c), Shahidehpour and Tong (1992) and Wang et al. (1995).

In this paper, we focus on solving the hourly non-convex model under the deregulated power market. Makkonen and Lahdelma (2006) have presented for the symmetrical CHP problem a customized Power Simplex (PS)-based Branch and Bound (B&B) algorithm (PBB) using LP relaxations. PBB adopts an area-based branching technique, which allows more efficient problem representation than the standard branching tech-
nique. The procedure creates and solves the sub-problems immediately because the PS solver (Lahdelma and Hakonen, 2003) is very efficient when starting from an old basic solution. PS is an efficient specialized primal-based Simplex algorithm for solving the symmetrical hourly convex model. The asymmetrical problems can be solved also using the primal-based algorithms, because the spot market can be modeled as a pair of open purchase and sales contracts. However, the envelope-based dual algorithms ECON/ECOFF (Rong and Lahdelma, 2007a) are more efficient than the primal-based algorithm for the asymmetrical problem.

Here we introduce the envelope-based B&B (EBB) algorithm for solving the asymmetrical non-convex planning problems efficiently. EBB is based on the LP-relaxation B&B paradigm with similar area-based branching as the PBB algorithm. The LP-relaxation sub-problems are solved by using a modified version of on-line envelope construction based algorithm (ECON) (Rong and Lahdelma, 2007a). Tight lower bounds are computed analytically prior to solving the sub-problems. The lower bounds are used during the search for reducing the overhead of recording the unpromising sub-problems, for pruning the search tree and for selecting the most promising sub-problems to be solved.

The approach that EBB algorithm deals with the non-convexity can be used for solving a class of separable non-linear optimization problems in different production, economics and transportation contexts (Sherali, 2001). Because the envelope of the non-convex CHP plant is a generic piecewise linear function, it is structurally identical with the cost functions reflecting variation between specified breakpoints, or economies of scale, or congestion-related delays in the above mentioned different application backgrounds.

The paper is organized as follows. In Section 2, we describe the hourly non-convex CHP production model under the deregulated power market. In Section 3, we describe the envelope of a CHP plant and review the envelope-based algorithms for solving the convex CHP planning problem. In Section 4, we present the EBB algorithm. In Section 5, we report the results on test runs.

2. Problem description

Utilization of CHP technology can result in significant energy savings when both power and heat are needed. The primary concern of the CHP production is to produce heat to satisfy variable demand. Heat lack is in general not allowed and heat surplus may impose a penalty cost. Under the deregulated power market, the objective of CHP production planning is to minimize the overall net acquisition costs for power and heat. The net acquisition costs consist of actual production costs (primarily fuel and emissions costs) and purchase costs of energy subtracted by revenue from selling energy. The planning horizon can be anything from a few days in a medium-term model to several years in a strategic long-term planning model. The medium- and long-term model decomposes into a large number of hourly models. The hourly production level in the CHP plants should be adjusted based on power price, heat demand and production cost. Next we describe the hourly models in detail. Since the generic convex CHP plant model is the building block for the overall CHP system model, we start with the description of the convex CHP plant model.

2.1. Convex CHP plant model

In CHP technology, heat and power generation follows a joint characteristic that defines the dependency between the operating costs and heat and power generation. The plant is convex if the feasible operating region (characteristic area) is convex in terms of heat and power generation and the production cost is a convex function of the generated heat and power. Convexity of the characteristic area means that if the plant can operate at two different points, it can also operate at any point on the line segment connecting them. Convexity of the cost function means that the operating cost on the line segment is not higher than the corresponding linear combination of the operating costs at the endpoints. Fig. 1 shows a typical feasible operating region of a convex CHP plant.

Due to convexity, the hourly power generation $P^u_t$, heat generation $Q^u_t$ and operating costs $C^u_t = C^u(P^u_t, Q^u_t)$ of plant $u$ can be represented as a convex combination of extreme characteristic points $(c^u_t, p^u_t, q^u_t)$ (the corner points of the triangular facets in Fig. 1):
Here the variables $x_j^u$ are used for forming the convex combination and $J_u$ is the index set of the extreme points associated with plant $u$. We observe that the system (1) defines the feasible operating region of a CHP plant as a three-dimensional convex polytope in the $(c,p,q)$ space. However, because each plant should operate as cost-efficiently as possible, only the lower envelope of this polytope (the triangular facets in Fig. 1) is of interest. This formulation allows the shape of the characteristic to change hourly, but assumes that the same number of points $|J_u|$ is used for each hour. If a plant needs fewer points at some hours, extra points can be effectively disabled by fixing those $x_j^u$-variables to zero. This formulation can approximate any convex cost function with arbitrarily good precision if a sufficiently dense set of extreme points is used. In practice, the extreme points can be determined empirically (based on test runs) or calculated based on an analytical model. In either case, the necessary number of extreme points will be reasonably small.

To model relaxed on-off states of CHP plants, we can include the point $(0,0,0)$ into each characteristic. The plant can then be forced to the on-state by fixing the corresponding (slack) variable $x_j^u = 0$, and to the off-state by fixing all other $x_j^u$-variables to zero. Unit commitment considering start-up and shut-down costs and constraints can then be determined by using dynamic programming or other techniques, but this is out of the scope of this paper.

The above modeling technique applies also to separate power and heat components. Such components include condensing power plants, individual hydropower plants, heat plants, demand-side management components, and various bilateral purchase and sales contracts for heat and power. All these can be modeled as special cases of the CHP plant model (1) with either $q_j^u = 0$ (in power components) or $p_j^u = 0$ (in heat components). For example, in power sales contracts, $c_j^u < 0$, $p_j^u < 0$ and $q_j^u = 0$. 

Here are the equations:

\[
\begin{align*}
C_u^u &= \sum_{j \in J_u} c_j^u x_j^u, \\
P_u^u &= \sum_{j \in J_u} p_j^u x_j^u, \\
Q_u^u &= \sum_{j \in J_u} q_j^u x_j^u, \\
\sum_{j \in J_u} x_j^u &= 1, \\
x_j^u &\geq 0, \quad j \in J_u.
\end{align*}
\]
2.2. Non-convex CHP plant model

The characteristic operating area of a simple backpressure CHP plant is often convex. However, for more advanced cogeneration technologies, such as backpressure plants with condensing and auxiliary cooling options, gas turbines, and combined gas and steam cycles, the characteristic may be non-convex. A CHP plant can also have a number of alternative operating modes that shift some or all of the characteristic points. This makes the characteristic non-continuous (and thus non-convex). The shut-down state typically also makes the characteristic non-continuous. However, a non-convex CHP plant model can be divided into multiple convex submodels, which can be encoded as alternative model components. Fig. 2 illustrates the non-convex characteristic of a backpressure plant with different operating modes. The characteristic is projected onto the $p$–$q$ (power-heat) plane and the $c$-coordinate at each extreme point is shown numerically. The characteristic area is divided into three convex sub-areas: A1, A2 and A3. A1 is formed by extreme points 1, 8, 9, 2 and 3. This area includes the normal backpressure operation mode (line between points 1 and 2), gradual shift into condensing mode (area within points 1, 2 and 3) and the reduction mode (area within points 1, 8, 9 and 2). The auxiliary cooling operating mode must be split into two convex sub-areas; A2 is formed by extreme points 1, 3, 6, 5, and 4 and A3 by points 2, 7, 6, and 3. The plant can only operate in one convex sub-area each time, but some extreme points may belong simultaneously to several areas. To enable (or activate) a convex sub-area means enabling the extreme points that define the sub-area and disabling the remaining extreme points. A compact way to represent the convex partition of a non-convex operating area is to tabulate the allocation of the extreme points into different sub-areas. Table 1 shows the coordinate of the extreme points, the net cost at two different power prices (net cost will be explained later) and allocation of extreme points into convex sub-areas.

2.3. Model formulation

Next we describe the hourly non-convex CHP model with multiple plants (and possible other components such as contracts). To make the notation simpler, we omit the time index $t$ in the subsequent formulas. We use the following notation:

Index sets
- $A^N$ characteristic areas of all non-convex plants ($A^N = \bigcup_{u \in U} A_u$)
- $A_j$ characteristic areas that contain extreme point $j$
Table 1
Coordinates of extreme points, net cost of extreme points for two different power prices and allocation of extreme points to convex sub-areas

<table>
<thead>
<tr>
<th>Point no.</th>
<th>Extreme point coordinates</th>
<th>Net cost</th>
<th>Allocation to sub-areas</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>c</td>
<td>p</td>
<td>q</td>
</tr>
<tr>
<td>1</td>
<td>460</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>2190</td>
<td>100</td>
<td>90</td>
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<td>70</td>
<td>40</td>
</tr>
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<td>4</td>
<td>600</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
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<td>50</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2055</td>
<td>70</td>
<td>15</td>
</tr>
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<td>3215</td>
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<td>40</td>
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</tr>
<tr>
<td>9</td>
<td>2030</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>

\( A_u \) characteristic areas of non-convex plant \( u \)
\( J \) set of extreme points of the operating regions of all plants (\( J = \cup_{u \in U} J_u \))
\( J^N \) set of extreme points of the operating regions of all non-convex plants (\( J^N = \cup_{u \in U^N} J_u \))
\( J_a \) set of extreme points in area \( a \)
\( J_u \) set of extreme points of the operating region of plant \( u \)
\( U \) set of all plants
\( U^N \) set of non-convex plants

**Parameters**
\( (c_p, p_j, q_j) \) extreme point \( j \in J_u \) of operating region of plant \( u \) (cost, power and heat)
\( c^p \) power price on market
\( c^q \) penalty for heat surplus \((\geq 0)\)
\( Q \) heat demand

**Decision variables**
\( x_j \) variables encoding the operating level of each plant in terms of extreme points
\( x^p \) net level of power
\( x^q \) heat surplus
\( y_a \) 0–1 variables determining if area \( a \) is in operation, \( a \in A^N \)

The hourly sub-problem for CHP planning to minimize the overall net acquisition costs is formulated as follows:

\[
\text{Min} \quad \sum_{j \in J} c_j x_j - c^p x^p + c^q x^q
\]

s.t.

\[
\sum_{j \in J_u} x_j = 1, \quad u \in U, \quad (3)
\]

\[
\sum_{j \in J} p_j x_j - x^p = 0, \quad (4)
\]

\[
\sum_{j \in J} q_j x_j - x^q = Q, \quad (5)
\]

\[
x_j \geq 0, \quad j \in J, \quad (6)
\]
\[ x^{q+} \geq 0, \]  
\[ x_j \leq \sum_{a \in A_j} y_a, \quad j \in J^N, \]  
\[ \sum_{a \in A_u} y_a = 1, \quad u \in U^N, \]  
\[ y_a \in \{0, 1\}, \quad a \in A^N. \]

In this formulation, constraints (3) and (6) represent the convex combination of the extreme points of the operating region of each plant. The power balance (4) determines the net amount of power that can be traded on the market. The heat balance (5) states that the demand \( Q \) must be satisfied and if the acquisition of heat exceeds the demand, then the surplus \( x^{q+} \) leads to penalty cost \( c^{q+} \) in the objective function (2).

The restriction with the above formulation is that it forces the purchase and sales price on the spot market to be equal. However, this simplification is acceptable in the typical application contexts listed in Rong and Lahdelma (2007a).

The objective function (2) together with constraints (3)–(7) form the hourly convex CHP model under the deregulated power market. This acts as the LP-relaxation sub-problem of the entire MIP problem (2)–(10) when the area-based branching technique is used (discussed later). Constraints (8)–(10) deal with the non-convex plants. Constraints (9) force each non-convex plant to operate exactly within one characteristic area. The variable \( y_a \) equals one when the corresponding area is used and zero otherwise. Constraints (8) disallow operation in other areas or between areas by forcing the \( x_j \)-variables belonging to disallowed areas to zero. Since some extreme points may belong to several areas simultaneously, an \( x_j \)-variable is forced to zero only when all of the corresponding characteristic areas are disabled, i.e., when the sum of the corresponding \( y_a \)-variables (the right-hand side of (8)) is zero.

2.4. Model transformation

The free (unconstrained) variable \( x^p \) corresponding to sales of power to the market allows us to transform the model into a format that can be solved very efficiently using the envelope-based ECON/ECOFF algorithms (Rong and Lahdelma, 2007a). The heat surplus variable can be treated easily as described in Section 3.2. We eliminate constraint (4) by solving \( x^p \) from it and substituting \( x^p \) into the objective function (2) giving

\[ \text{Min } \sum_{j \in J} (c_j - c^p p_j) x_j + c^{q+} x^{q+}. \]

We then introduce

\[ c'_j = c'_j(c^p) = c_j - p_j c^p \]

to denote the net cost of extreme point \((c_j, p_j, q_j)\). The net cost corresponds to the actual production costs of the plant when it is operating at extreme point \( j \) subtracted by the revenue from selling the produced power to the market at market price. Now the objective of the hourly CHP model can be written as

\[ \text{Min } z = \sum_{j \in J} c'_j x_j. \]

The hourly convex CHP model consists now of (12) together with the constraints (3), and (5)–(7).

3. Envelope of a CHP plant

Minimizing the operating cost of a CHP system will bring each plant to operate on the lower envelope of its feasible operating region. The idea in the envelope-based algorithms is to replace the feasible operating region by a representation of its envelope.

To simplify the representation of the envelope and the envelope-based algorithms, we assume that the \( q \)-coordinates of the characteristic points of a CHP plant are distinct. If the \( q \)-coordinates of some extreme points originally coincide, we perturb them by a small value.
3.1. Envelopes of convex and non-convex CHP plants

In the original formulation the lower envelope defines the minimum operating cost of the plant as a function of generated heat and power. For a convex CHP plant, the lower envelope is simply the lower surface of the three-dimensional convex polytope in the \((c, p, q)\) space defined by (1), i.e. the triangular facets illustrated in Fig. 1. After the model transformation, the characteristic operating region is reduced into a two-dimensional convex polygon in the \((c_0, q)\) plane. Accordingly, the lower envelope of a convex plant \(u\) is reduced into a piecewise-linear convex function \(c'_u(q)\). To represent the envelope of convex plant \(u\), we define index array \(g_u\) that sorts the envelope points into ascending order by their \(q\)-coordinate.

The operating region of a non-convex plant is partitioned into convex sub-areas. In each sub-area \(a\), the envelope is a piecewise-linear convex function \(c'_a(q)\). The combined lower envelope of a non-convex plant \(u\) is then a generic piecewise linear function defined as the minimum of the convex envelopes of different areas, i.e., \(c'_u(q) = \min_{a \in A} c'_a(q)\). To represent the non-convex envelope, we define for each convex sub-area an index array \(g_a\) that sorts the \(n_{env}^a \leq |J_a|\) envelope points into ascending order by their \(q\)-coordinate. Observe that these envelope points are a subset of the extreme points of an area. The set of points that define the envelope in each area depends on the power price. For the EBB algorithm, we also define an artificial convex area \(u^*\) that relaxes the non-convex plant into a convex plant. Area \(u^*\) consists of all of points in the plant and array \(g_{u^*}\) sorts the corresponding \(n_{env}^u \leq |J_u| = |J_a|\) envelope points into ascending order by their \(q\)-coordinate.

We use the non-convex plant of Fig. 2 to illustrate the envelope in the \((c_0', q)\) plane. The extreme points are listed in Table 1. The columns \(c'(30)\) and \(c'(35)\) correspond to the net cost (11) for two different values of power price \((c^p = 30, 35)\). Figs. 3a and b show the extreme points and the corresponding envelopes of convex sub-area A1 as piecewise linear convex functions. The feasible operating region is the convex polygon spanned by the extreme points and the envelope consists of the solid line segments (the lowest part of the operating region). Fig. 4 illustrates construction of the envelope of a non-convex plant. Fig. 4a shows the collection of the envelopes of individual areas. Fig. 4b shows the envelope of the plant as a generic piecewise linear function. For minimization of the net costs, the representations in Fig. 4a and b are functionally equivalent. In non-convex CHP planning, the EBB algorithm can construct the envelopes efficiently for the sub-areas, as shown in Fig. 4a. In other applications, such as planning, scheduling and optimization problems in various production, economics and transportation contexts (Sherali, 2001), a piecewise linear cost curve similar to Fig. 4b may be directly available, and the EBB algorithm can then be applied on convex partitions of the curve.

The envelope of a CHP plant is the function of power price \(c^p (c^p \geq 0)\). The envelope of a non-convex plant is the collection of the envelopes for the individual sub-areas. The envelope of the sub-area can be either constructed online each hour based on the power price or looked up from the pre-computed (offline-computed) envelope patterns (Rong and Lahdelma, 2007a).

![Fig. 3. Envelope of one convex sub-area of a non-convex plant.](image-url)
3.2. Properties of the envelopes

The envelopes of the individual convex operating areas can be computed very efficiently using the algorithm presented by Rong and Lahdelma (2007a). The time complexity for the computation is $O(n_{\text{ext}} \log n_{\text{ext}})$, where $n_{\text{ext}}$ is the number of extreme points in the area. Because the $q$-coordinates are distinct (after perturbing any coincident points), we can define the slope between an arbitrary pair of extreme points $(j,k)$ of a CHP plant as

$$\gamma(j,k) = \frac{c'_k - c'_j}{q_k - q_j}.$$  \hspace{1cm} (13)

When $n^a_{\text{env}} \geq 2$ we also introduce

$$\gamma_a(i) = \gamma(g_a(i), g_a(i+1))$$  \hspace{1cm} (14)

to denote the slope of the segment between points $i$ and $i + 1$ on the envelope of any area $a$. We will next identify some properties of the convex envelopes. Based on convexity, the slopes of the line segments along the envelope are in non-decreasing order, i.e.

$$\gamma_a(i-1) \leq \gamma_a(i), \hspace{1cm} i = 2, \ldots, n^a_{\text{env}} - 1.$$  \hspace{1cm} (15)

Because the hourly model contains a heat surplus variable that can dispose an arbitrary amount of surplus heat at price $c^q$, it is not cost-efficient to operate a plant on an envelope segment where $\gamma_a(i) \leq -c^q$. To reduce costs, the plant could then move to point $g_a(i+1)$ and dispose the extra heat. This means that we can eliminate from the convex envelopes any initial segments whose slopes are smaller or equal to $-c^q$. As a result, we get the second property

$$\gamma_a(i) > -c^q, \hspace{1cm} i = 1, \ldots, n^a_{\text{env}} - 1.$$  \hspace{1cm} (16)

3.3. Envelope-based algorithm for solving convex CHP model

The B&B algorithm for solving a MIP problem using LP relaxations needs to solve many LP problems during the search process. In the EBB algorithm, we use the envelope-based algorithm to solve the sub-problems. In the following we summarize the main results that are relevant for the bounding technique later. A more detailed description is found in Rong and Lahdelma (2007a).

In a convex model all plants $u \in U$ are convex and index arrays $g_u$ store their envelope points in increasing order by the $q$-coordinate. The minimum and maximum amount of heat that the system can produce is then

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\[ q_{\text{min}} = \sum_{u=1}^{[U]} q_{g_u(1)}, \tag{17} \]
\[ q_{\text{max}} = \sum_{u=1}^{[U]} q_{g_u(\text{env})}. \tag{18} \]

If the heat demand \( Q > q_{\text{max}} \), then no feasible solution exists.

Otherwise we can apply the theory of the Simplex method for solving LP problems (Dantzig, 1963). When seeking the optimum, it is possible to restrict the search to basic solutions. A basic solution is found by choosing a basic variable corresponding to each linear constraint and setting the remaining variables to zero. The basic variables are then solved from an \( m \times m \) linear equation system, where \( m \) is the number of constraints. The basic variables must be chosen so that the equation system is non-singular, i.e. the corresponding part of the constraint matrix forms a valid basis. When applied to the convex CHP model (12), (3), (5)–(7) with \( m = [U] + 1 \) constraints, \( [J] + 1 \) variables, and a sparse matrix with a special structure, this means that we must choose one basic variable (extreme point) from each plant plus one additional variable. The additional variable can be the heat surplus or a second variable from some plant. Because all plants should operate on their envelopes to minimize the net cost, we only need to consider variables corresponding to envelope points for basis. Also, if two points are selected from some plant, these must be consecutive on the envelope.

If \( Q \leq q_{\text{min}} \), we can identify the optimal solution trivially. Each plant operates on its first envelope point \( g_u(1) \), the heat surplus variable is basic with value \( x^{q_u} = q_{\text{min}} - Q \), and the optimal objective function value is
\[ z_{\text{opt}} = \sum_{u=1}^{[U]} c'_{g_u(1)} + c^{q_u}(q_{\text{min}} - Q). \tag{19} \]

In particular, if \( q_{\text{min}} = q_{\text{max}} \) this implies that the envelope of each plant contains only a single point and either (19) applies or no feasible solution exists.

In the general case, when \( q_{\text{min}} < Q \leq q_{\text{max}} \), the heat surplus will not be basic due to condition (16). The optimal basis must then contain two consecutive envelope points \( g_u(i_x) \) and \( g_u(i_x + 1) \) from some plant \( \tau \) and exactly one envelope point \( g_u(i_a) \) from each of the remaining \([U] - 1\) plants. This means that plant \( \tau \) acts as the regulating plant. Without changing the basis, the system can regulate the heat production within a range from \( q_{\text{LO}} \) to \( q_{\text{UP}} \) determined by
\[ \begin{cases} q_{\text{LO}} = \sum_{u \in U} q_{g_u(i_a)} \\ q_{\text{UP}} = q_{\text{LO}} + q_{g_u(i_x + 1)} - q_{g_u(i_x)} \end{cases}. \tag{20} \]

The feasibility condition requires that
\[ q_{\text{LO}} \leq Q \leq q_{\text{UP}}. \tag{21} \]

Similar to (14), we introduce
\[ \gamma_u(i) = \gamma(g_u(i), g_u(i + 1)) \tag{22} \]

to denote the slope of the \( i \)th line segment on the envelope of plant \( u \). We denote by \( \gamma_\tau = \gamma_u(i_x) \) the basic slope (the slope of the basic envelope segment of the regulating plant). The optimality conditions can now be written in terms of the basic slope and the envelope slopes of the other plants as
\[ \begin{cases} \gamma_u(i) \leq \gamma_\tau \quad \text{if } i < i_u, \ u \in U, \ u \neq \tau \\ \gamma_u(i) \geq \gamma_\tau \quad \text{if } i \geq i_u, \ u \in U, \ u \neq \tau. \end{cases} \tag{23} \]

Then the objective function value for the optimal basis is
\[ z_{\text{opt}} = z_c + c'_{g_u(i_x)} + (Q - q_{\text{LO}})\gamma_\tau = z_c + c'_{g_u(i_x + 1)} + (Q - q_{\text{UP}})\gamma_\tau. \tag{24} \]
where \( z_c \) is the net cost caused by the other plants except plant \( \tau \)
\[
z_c = \sum_{u \in U, u \neq \tau} c'_g(u, i_u).
\] (25)

Let \( q_c \) denote total amount of heat provided by the plants other than \( \tau \) corresponding to net cost \( z_c \). Then
\[
q_c = \sum_{u \in U, u \neq \tau} q_g(u, i_u).
\] (26)

In the following, we summarize the main idea of the algorithm for finding the optimal solution based on envelopes. The detailed description of the algorithm is in Rong and Lahdelma (2007a).

Starting from an arbitrary basic slope \( \gamma_\tau \) for some plant \( \tau \), we can easily find the basic envelope points \( g_\tau(i_u) \) for the remaining plants that satisfy the optimality conditions (23). Then we compute \( q_{LO} \) and \( q_{UP} \) for the current basis based on (20). If \( q_{LO} \leq Q \leq q_{UP} \), we have found a feasible and optimal solution with objective function value given by (24). Otherwise, we advance the basic envelope segment among all envelopes either forward (to increase heat production) or backward (to decrease heat production). To maintain the optimality condition, advancement is done sequentially to the envelope segment with a slope closest to the current one until a feasible solution is found.

To explain the relationship between the envelope-based algorithm and lower bound computation in EBB algorithm, we define four envelope slopes: \( \gamma_{\text{pred}} \), \( \gamma_{\text{succ}} \), \( \gamma_{\text{pred}, \tau} \), and \( \gamma_{\text{succ}, \tau} \) related to the current basic slope \( \gamma_\tau \). \( \gamma_{\text{pred}} \) is the next basic slope in backward search, i.e. the largest slope immediately to the left of the basic envelope points \( g_\tau(i_u) \) for all of plants. Similarly, \( \gamma_{\text{succ}} \) is the next basic slope in forward search, i.e. the smallest slope immediately to the right of the basic envelope points \( g_\tau(i_u + 1) \) for all of plants. Slopes \( \gamma_{\text{pred}, \tau} \) and \( \gamma_{\text{succ}, \tau} \) are corresponding largest and smallest slopes excluding plant \( \tau \). The quantitative relationship between the four slopes is
\[
\gamma_{\text{pred}, \tau} \leq \gamma_{\text{pred}} \leq \gamma_\tau \leq \gamma_{\text{succ}} \leq \gamma_{\text{succ}, \tau}.
\] (27)

Next we describe the basic idea for computing tight lower bounds for the EBB algorithm. Assume that the individual envelope slopes of all plants are sorted into ascending order and combined into a single envelope. Fig. 5 illustrates for an optimal and feasible basic solution the placement of the four named envelope slopes around the basic slope \( \gamma_\tau \). The values marked on the \( q \)-axis are computed based on (20) according to the corresponding slope. The optimal objective function value \( z_{\text{opt}} \) corresponds to demand \( Q \) on the basic slope. We can see that any of the other slopes can be used for computing a lower bound (marked with cross) for \( z_{\text{opt}} \) by extrapolating the slope to \( Q \), which is equivalent to applying formula (24) with the corresponding envelope slope. To get a tight lower bound, the slope should be close to the optimal slope. In EBB we use \( \gamma_{\text{pred}, \tau} \) and \( \gamma_{\text{succ}, \tau} \) to compute lower bounds for sub-problems.

![Fig. 5. Lower bound estimates using the envelope-based algorithm.](image-url)
4. Envelope-based B&B algorithm

The B&B algorithm is one of the most popular general-purpose techniques to solve difficult MIP problems including pure integer programming problems (Little et al., 1963; Clausen, 1999; Land and Doig, 1960; Linderoth and Savelsbergh, 1997). The B&B algorithm searches the solution space implicitly by using bounds for the objective function to be optimized combined with the value of the current best MIP-feasible solution. The search results in partial enumeration of all possible alternatives. Therefore, the efficiency of the algorithm can be greatly improved if tight bounds can be found.

LP-relaxation based B&B (Land and Doig, 1960; Linderoth and Savelsbergh, 1997) is one of the most effective paradigms to solve MIP problems. An LP relaxation of a MIP problem is obtained by dropping the integrality constraints. The solution process maintains a search tree whose root node is the LP relaxation of the original MIP problem. The B&B algorithm continuously solves unexplored LP sub-problems (nodes) that form the active set. When the objective function is to be minimized, the solution to the LP sub-problem provides a lower bound on the optimal value of the corresponding MIP problem. If an optimal solution to the LP relaxation satisfies the integrality constraints, then this solution is also an optimal solution to the corresponding MIP problem. If the LP relaxation is infeasible, then also the corresponding MIP problem is infeasible. Otherwise, some of the integrality constraints are not satisfied. The B&B algorithm then branches on the sub-problem by creating two or more sub-problems where some of the infeasible integer variables are restricted to alternative integral values. The success of the LP-relaxation based B&B lies in that the optimal solution to the LP relaxation generally gives reasonably tight bounds and the methodology for solving LP problems is quite efficient and very reliable.

In EBB we follow the LP-relaxation based B&B paradigm. The branching and bounding techniques are coordinated to improve the efficiency of EBB. In the following, we describe the branching and bounding techniques of EBB.

4.1. Area-based branching technique

In the generic B&B algorithm (Linderoth and Savelsbergh, 1997) for model (2)–(10), the LP-relaxation sub-problem is obtained by dropping the integrality constraints on the variables $y_a$ that indicate which convex sub-areas are in use. The LP relaxation of the MIP problem contains $m = |U| + 2 + |U^N| + |J^N|$ variables including the $|A^N| = \sum_{a \in U^N} |A_a|$ 0/1 variables. When an integer variable $y_a$ has a fractional value $\bar{y}_a$ in node $i$, the branching step forms two sub-problems by adding the constraints $y_a \leq \lfloor \bar{y}_a \rfloor = 0$ and $y_a \geq \lceil \bar{y}_a \rceil = 1$ correspondingly. This technique is called variable-branching. The sub-problems are generic LP problems that can be solved using a general-purpose LP solver. The theoretical number of different 0/1 variable combinations may in the worst case result in $2^{|A^N|} - 1$ nodes in the search tree. To reduce the number of processed nodes, some generic B&B algorithms resort to sophisticated techniques such as adding cuts, using special ordered sets and applying different heuristics for selecting branching nodes and variables (Linderoth and Savelsbergh, 1997). Sophisticated techniques also imply more computational effort at each node.

The area-based branching technique was proposed by Makkonen and Lahdelma (2006) when the PS-based B&B (PBB) algorithm was implemented. Fig. 6 is the slight modification of the original search tree based on the area-based branching technique. Here we explicitly demonstrate the role of $z^*$ (best MIP solution up to now) for pruning the tree. The LP-relaxation sub-problems are formed by dropping constraints (8)–(10). This LP relaxation contains only $m_L = |U| + 2$ linear constraints and $n_L = |J| + 2$ continuous variables. MIP-infeasibility is determined directly based on the (continuous) $x_J$ variables. MIP-infeasibility is caused by plants that are operating between two or more areas. The branch step is implemented based on an entire infeasible plant $u$ by forming multiple sub-problems, one for each convex area $a \in A_u$. The number of different area combinations $C_N = \prod_{u \in U^N} |A_u|$ is much smaller than the theoretical number of 0/1 variable combinations $2^{|A^N|}$ that a brute-force algorithm might enumerate.

The most important advantage of the area-based branching technique is that the LP-relaxation sub-problem has a special structure that allows very efficient solution using specialized algorithms, such as PS and the envelope-based algorithms.
In PBB, all sub-problems branching from an infeasible plant are created and solved immediately partially because PS can solve the sub-problem quite efficiently by reusing the old basic solutions and partially because there is no effective method to estimate which sub-problem is most promising using the PS solver. If a lower bound on the sub-problem can be estimated before solving it, then the efficiency of the B&B algorithm can be improved. The logic of the lower bounds on the sub-problems for pruning the tree is similar to that of the solutions to the sub-problems for pruning the tree shown in Fig. 6. With the bounding technique, we only branch, record and solve promising sub-problems with lower bounds better than the currently best known solution. In addition, the bound can be used to select more promising nodes to be solved first. This will cause good solutions to be found earlier, which again allows more efficient pruning of the unpromising nodes.

4.2. Bounding technique

EBB applies the same area-based branching technique as PBB, but computes efficiently tight lower bounds on the sub-problems before solving them. Let index array $g_u$ denote the set of extreme points on the envelope of the currently active sub-area of plant $u$. The active sub-area is either a convex sub-area $A_u$ of a non-convex plant, or the artificial area $u^*$ corresponding to its convex relaxation as discussed in Section 3. We define the range where envelope segment $i$ of plant $u$ can regulate heat production as

$$D_{qu}(i) = \frac{q_{gu}(i+1) - q_{gu}(i)}{C_0}.$$  (28)

Due to the problem structure, only the regulating plant $s$ has two basic envelope points, and thus only the plant $\tau$ can be MIP-infeasible. When this is the case, the artificial sub-area $\tau^*$ is active and the basic slope $\gamma_\tau$ has endpoints $g_s(i)$ and $g_s(i+1)$ belonging to different sub-areas. When the sub-problem is formed, the relaxed envelope of the artificial area is replaced by that of the sub-area to activate. Excluding plant $\tau$, $\gamma_{\text{pred},\tau}$ and $\gamma_{\text{suc},\tau}$ (27) are the envelope slopes logically closest to the current MIP infeasible solution. These slopes remain unchanged in each sub-problem and they are also the best available slopes logically close to the solution to each sub-problem. For estimating the lower bound for each sub-problem, we choose either $\gamma_{\text{pred},\tau}$ or $\gamma_{\text{suc},\tau}$ into the basis, if they exist. With either choice, the basic points of the remaining plants still satisfy the optimality conditions (23). To form an optimal (but potentially infeasible) basis, we only need to determine the envelope point from the activated sub-area $a$ of plant $\tau$. After the basis is formed, we check the feasibility based on $q_{LO}$ and $q_{UP}$ (20). Then we estimate the lower bound using $\gamma_{\text{pred},\tau}$, $\gamma_{\text{suc},\tau}$ or an envelope slope from area $a$ of plant $\tau$ between them.

In the following, we present the method to compute the lower bounds for individual sub-problems and handle the resulting sub-problems based on the relationship between the lower bound and $z^*$ (best MIP solution).
up to now). The sub-problem is discarded if it is LP-infeasible or its lower bound is greater than or equal to \( z^\tau \). \( z^\tau \) is updated if we can identify a better MIP-feasible solution during the lower bound computation. Otherwise, if the lower bound is less than \( z^\tau \), then the sub-problem and its basis are recorded in the active set. The following notations are defined.

\[
\begin{align*}
\text{z}_\text{min} & \quad \text{sum of net cost corresponding to the minimum heat for the active areas of all plants} \\
q_c, z_c & \quad \text{sum of heat and net cost for plants other than } \tau \text{ in the current MIP-infeasible basis, computed based on (26) and (25)} \\
q^R_{\text{min}}, q^R_{\text{max}} & \quad \text{minimum and maximum heat for active areas of plants other than } \tau \\
q^R_{\text{min}}, q^R_{\text{max}} & \quad \text{minimum and maximum heat for area } a \text{ of the MIP-infeasible plant } \tau \text{ to activate} \\
z^*_L & \quad \text{lower bound for the optimal objective function value in the sub-problem enabling sub-area } a \text{ of plant } \tau \\
\end{align*}
\]

The following cases are considered.

(a) If \( q^a_{\text{max}} + q^R_{\text{max}} < Q \), then the sub-problem is LP-infeasible.
(b) If \( q^a_{\text{min}} + q^R_{\text{min}} \geq Q \), then \( z^*_L = z_{\text{min}} + \epsilon^\tau (q^R_{\text{min}} + q^a_{\text{min}} - Q) \) based on (19). This is a MIP-feasible solution.
(c) If \( q^a_{\text{min}} + q^R_{\text{min}} < Q \leq q^a_{\text{max}} + q^R_{\text{max}} \)

(i) If \( q^a_{\text{min}} = q^R_{\text{max}} (= q_c) \) (isolated points for the active areas in the remaining plants), then neither \( \gamma^\text{pred,}\tau \) nor \( \gamma^\text{succ,}\tau \) exists. From area \( a \) of plant \( \tau \) find the first envelope point \( i_a \) satisfying \( q_c + q_{g_a(i_a)} \geq Q \) and compute \( z^*_L = z_c + c_{g_a(i_a)}^\tau + \gamma^a(i_a - 1)(Q - q_{g_a(i_a)} - q_c) \) based on (24). This is a MIP-feasible solution.

(ii) If \( q^a_{\text{min}} < q^R_{\text{max}} \), this implies that \( \gamma^\text{pred,}\tau \) or \( \gamma^\text{succ,}\tau \) or both exist. The lower bound is computed based on the following sub-cases.

First, consider the situation where \( \gamma^\text{pred,}\tau \) exists.

Bring envelope point \( g_a(i_a) \) for area \( a \) of plant \( \tau \) into basis satisfying optimality conditions (23) with \( \gamma^\text{pred,}\tau \) chosen as the basic slope. Let \( \text{q}_{\text{UP}} = q_c + q_{g_a(i_a)} \).

- If \( Q \leq \text{q}_{\text{UP}} \), then \( z^*_L = z_c + c_{g_a(i_a)}^\tau + \gamma^\text{pred,}\tau(Q - \text{q}_{\text{UP}}) \). Based on \( \gamma^\text{pred,}\tau \), find the corresponding plant \( v \) and envelope points \( g_v(i_v - 1) \) and \( g_v(i_v) \) of its currently active area. In particular, if \( Q \geq \text{q}_{\text{UP}} - \Delta q_d(i_v - 1) \) and plant \( v \) is feasible, then this is a MIP-feasible solution.
- If \( Q > \text{q}_{\text{UP}} \), then \( z^*_L = z_c + c_{g_a(i_a)}^\tau + \gamma^\text{succ,}\tau(Q - \text{q}_{\text{UP}}) \) with \( \gamma^\text{succ,}\tau = \min\{\gamma^\text{succ,}\tau, \gamma^a(i_a)\} \) (at least one of these two envelope slopes exists). Based on \( \gamma^\text{succ,}\tau \), find the corresponding plant \( v \) and envelope points \( g_v(i_v) \) and \( g_v(i_v + 1) \) of its currently active area. In particular, if \( Q < \text{q}_{\text{UP}} + \Delta q_d(i_v) \) and \( v \) is a feasible plant, then this is a MIP-feasible solution.

Second, consider the situation where \( \gamma^\text{pred,}\tau \) does not exist. Then \( \gamma^\text{succ,}\tau \) must exist.

Bring envelope point \( g_a(i_a) \) for area \( a \) of plant \( \tau \) into basis satisfying optimality conditions (23) with \( \gamma^\text{succ,}\tau \) chosen as the basic slope. Let \( \text{q}_{\text{LO}} = q_c + q_{g_a(i_a)} \).

- If \( Q \geq \text{q}_{\text{LO}} \), then \( z^*_L = z_c + c_{g_a(i_a)}^\tau + \gamma^\text{succ,}\tau(Q - \text{q}_{\text{LO}}) \). Based on \( \gamma^\text{succ,}\tau \) find the corresponding plant \( v \) and envelope points \( g_v(i_v) \) and \( g_v(i_v + 1) \) of its currently active area. In particular, if \( Q \leq \text{q}_{\text{LO}} - \Delta q_d(i_v + 1) \) and \( v \) is a feasible plant, then this is a MIP-feasible solution.
- If \( Q < \text{q}_{\text{LO}} \), then \( z^*_L = z_c + c_{g_a(i_a)}^\tau + \gamma^a(i_a - 1)(Q - \text{q}_{\text{LO}}) \). In particular, if \( Q \geq \text{q}_{\text{LO}} - \Delta q_d(i_a - 1) \), then this is a MIP-feasible solution.

After \( z^*_L \) is computed, the decision is made about handling the corresponding sub-problem based on the relationship between \( z^*_L \) and \( z^\tau \) as described before.

To illustrate above bounding technique, we use optimization process of three plants as an example. Heat demand \( Q = 110 \). Table 2 shows the \((c', q)\) coordinates of envelope points for three plants at some power price.
Fig. 7 shows the envelopes of the sub-areas of the three plants. The real number besides the line segment is the slope of the segment.

When the solution begins, the artificial areas of three plants are active and \( z^* = 1 \). The envelopes for three artificial areas are \( g_{U1} = [1, 6, 7] \), \( g_{U2} = [1, 4, 5] \) and \( g_{U3} = [1, 4, 5] \) as shown in Fig. 7. Based on envelope-based algorithm in Section 3.3, in the optimal basis, the basic slope \( c_s = 0.5 \) comes from plant \( U1 \) with end points \( g_{U1}(1) \) and \( g_{U1}(2) \). The points in the basis for plants \( U2 \) and \( U3 \) are points \( g_{U2}(1) \) and \( g_{U3}(2) \), respectively. Plant \( U1 \) is infeasible. The total amount of heat provided by plants \( U2 \) and \( U3 \) is \( q_c = 20 + 60 = 80 \) with the cost \( z_c = 20 + 30 = 50 \). The remaining heat \( Q = q_c = 110 - 80 = 30 \) is provided by plant \( U1 \) with the cost \( z_c = 0.5(30 - q_c) + 10 = 20 \). Therefore, the objective function value is \( 20 + 50 = 70 \) (MIP-infeasible).

Plant \( U1 \) will be branched. \( c_{pred, s} = c_{U3}(1) = 0.5 \) from plant \( U3 \) and \( c_{succ, s} = \min\{c_{U2}(1), c_{U3}(2)\} = \min\{0.8, 3.0\} = 0.8 \) from plant \( U2 \). Using \( c_{pred, s} = 0.5 \) as the basic slope, the first sub-problem is formed by activating area \( a = A11 \) in plant \( U1 \), then point in the basis for plant \( U1 \) is \( g_{A11}(1) \) based on optimality conditions (23). The maximum heat provided by this basis is \( q_{UP} = q_c + q_{g_{A11}(1)} = 80 + 10 = 90 \). Since \( Q > q_{UP} \), we need to find \( z^* = \min\{z_{succ}, z_{A11}(1)\} = \min\{0.8, 1.0\} = 0.8 \) from plant \( U2 \). If the sub-problems are evaluated sequentially from area \( a = A11 \) to \( A13 \). The node

<table>
<thead>
<tr>
<th>Plant no.</th>
<th>Areas</th>
<th>Envelope point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>(10, 10)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(30, 20)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(40, 25)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(35, 35)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(25, 36)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(25, 40)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(45, 50)</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(20, 20)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(80, 40)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(79.5, 55)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(52, 60)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(97, 90)</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>(70, 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(90, 20)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(62.5, 25)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(30, 60)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(105, 85)</td>
</tr>
</tbody>
</table>

for activating \( a = A11 \) is created and recorded in the active set. After the sub-problem for activating \( a = A12 \) is evaluated, \( z^* \) is updated and \( z^* = 77 \). After the sub-problem for activating \( a = A13 \) is evaluated, since \( z_a = 80 > 77 = z^* \), this sub-problem is discarded. That means, only the sub-problem for activating \( a = A11 \) is recorded and it will be removed from the active set in next turn of iteration and the solution process continues.

4.3. Reusing the previous solutions

Often the problems for subsequent hours are very similar. The heat demand and power price vary somewhat from hour to hour and some constraints and cost coefficients change occasionally, but model structure and most coefficients remain the same. In order to speed up solving an hourly model, we can test if the optimal MIP solution from the previous hour is feasible for the current hour, and if it is, we can use that solution to prune unpromising branches of the search tree.

5. Computational results

Here we construct the envelope of each non-convex plant on-line based on the power price each hour when the model is solved each hour. That is, we implement the on-line envelope-based (ECON) B&B algorithm, called EBB, using C++ and Microsoft Visual Studio. Similarly, the envelope pattern of the plant can also be prepared offline as mentioned in Section 3.1, which can increase the speed of the algorithm for solving the underlying convex model. For solving the convex model, the speed ratio of offline envelope-based algorithm (ECOFF) and ECON is about 3 for reusing the old basis (Rong and Lahdelma, 2007a) because looking up envelopes from the pre-computed envelope patterns for ECOFF is more efficient than re-constructing the envelopes each hour for ECON. Both ECON and ECOFF use the same procedures for finding the optimal solution. Consequently, the performance of offline envelope-based B&B algorithm can be more efficient than EBB with almost the same factor as ECOFF against ECON if the same branching and bounding technique is used.

To test the performance of EBB, we use ILOG CPLEX 9.0 MIP solver and Power Simplex (PS) based B&B algorithm (PBB) (Makkonen and Lahdelma, 2006) as benchmarks. CPLEX MIP is a general-purpose commercial software for solving a general MIP problem. PS is an efficient specialized primal-based Simplex algorithm that utilizes the special structure of the symmetrical convex CHP planning problem to implement fast inversion procedures and to reuse intelligently the solutions from previous hours (Lahdelma and Hakonen, 2003). Both PBB and EBB use area-based branching technique. All test runs were performed in a 2.2 GHz Pentium 4 PC under the Windows XP operating system.

5.1. Test problems

Our test problems are generated based on three real CHP plants, one is backpressure (BP) plant (A1) and the other two are combined steam and gas cycle (CSG) (B1 and C1), each of which consists of 14 convex sub-areas. From the real plants we have constructed three slightly modified plants (A2, B2, and C2) by perturbing the extreme points and restricting the plants to operate with auxiliary cooling mode. Each of the modified plants consists of 8 convex sub-areas. Table 3 summarizes the properties of the six plants.

Then six test problems are generated based on different combination of above six plants, where the second problem (D2) consists of three real plants. All of plants in the test problems are non-convex. Table 4 shows the dimensions of the test problems: for each hourly model the number of plants \( |U| = |U^N| \), the number of convex sub-areas for the non-convex plants \( |A^N| = \sum_{u \in U^N} |A_u| \), the number of areas in plant \( u \) in all of plants, the number of linear constraints \( (m_L = |U| + 2) \), the number of continuous variables \( (n_L) \), the number of constraints \( (m = m_L + |U^N| + n_L - 2) \), the number of variables \( (n = n_L + |A^N|) \), and the number of different combinations \( |C_N| = \prod_{u \in U^N} |A_u| \) of convex sub-areas from different plants. The number of integer variables in each hourly model equals \( |A^N| \). \( C_N \) is the total enumeration of all possible alternatives if the area-based branching technique is used. To form a valid test problem, the heat demand is generated based on history data.
of a Finnish energy company and power price is generated based on the spot price history of the Nordic power market (Nord Pool, 2004).

5.2. Computational results

We have solved a sequence of 8760 hourly models by means of EBB, PBB and ILOG CPLEX MIP solver. This is equivalent to a yearly planning problem without dynamic constraints. ILOG OPL (optimization programming language) supports two default algorithms for integer programming and mixed integer programming: the CPLEX MIP algorithm and a B&B algorithm, called SOLVER MIP, which uses cooperation between a Simplex algorithm and constraint-based domain reduction. Based on computational results with our test problems, the CPU time difference between these two algorithms is marginal (the relative difference is within 2%). We have used default settings for the remaining options in CPLEX. In the following we report the results of CPLEX MIP solver, called CPLEX for short. With EBB, PBB and CPLEX, we have solved the subsequent hourly models both starting from scratch, and by using the previous solutions. For CPLEX, reusing the previous solution is implemented by testing whether the solution to the previous model is feasible for the current model. If it is feasible, then the corresponding objective function value is used as a bound to constrain the current model finding the solution not exceeding this bound. To reduce the effect of random (small) variations in CPU time measurements, each test problem was run ten times and the average CPU time was computed. Table 5 gives the CPU time for solving the 8760 hourly models.

Based on Table 5, EBB is the fastest of the three algorithms. Starting from scratch, the speed ratio of EBB against CPLEX is in range [537,828] with average 651; the speed ratio of EBB against PBB, is in range [14,31] with average 25. Reusing the previous solutions speeds up both EBB and PBB significantly, by 23% and 29%, respectively. Reusing previous solutions has only marginal effect on CPLEX and only test problem D2 shows speed-up. On average, CPLEX slows down by 1%. EBB and PBB benefit from reusing previous solutions, because a good MIP-solution from reusing the previous solution helps prune unpromising sub-problems earlier and thus reduce the number of processed nodes in the search tree (Table 6). For CPLEX, the reduction of nodes cannot make up for the overhead due to additional processing time at each node for reusing previous solutions. To be fair in speed comparisons, we apply the slightly faster from-scratch results for CPLEX. Consequently, for reusing the previous solution, the speed ratio of EBB against CPLEX is in range [661,955] with average 785. The speed ratio of EBB against PBB with reusing the previous solution is in range [11,31] with average 24.
There are two reasons why both EBB and PBB are much faster than CPLEX. Firstly, the area-based branching technique makes the size of the LP-relaxation sub-problems much smaller \((m_L \times n_L)\) than in the a general purpose MIP solver such as CPLEX \((m \times n)\) as shown in Table 4. Secondly, the LP-relaxation sub-problem for EBB and PBB has a special structure that allows much more efficient specialized solvers (PS, ECON/ECOFF) (Rong and Lahdelma, 2007a) to be applied.

In addition, the number of generated nodes during the search process also affects the efficiency of the algorithm. The overall computational time of the algorithm is related to the number of processed nodes and the average time used for processing each node. EBB uses a computational efficient bounding technique to prune the unpromising nodes while CPLEX uses sophisticated techniques such as cutting planes, special ordered sets and heuristics for selecting branching nodes and variables to reduce the number of nodes. On the average, the number of processed nodes (about 12 starting from scratch) for CPLEX is a little smaller than that (13.8 as shown in Table 6) for EBB based on test runs. The sophisticated techniques mean that the time used for processing each node is increased. In addition, the underlying LP-relaxation sub-problems (nodes) of EBB and CPLEX are different. Consequently, straightforward comparison of node counts between EBB and CPLEX is not meaningful, because the branching and bounding techniques are different for two algorithms. In the following we compare the number of created nodes for EBB and PBB.

Table 6 shows the average number of nodes per hourly model for PBB and EBB. PBB creates about 8 times more nodes than EBB starting from scratch and about 10 times more when reusing the previous results. The reason why EBB creates much fewer nodes than PBB is that EBB computes lower bounds for the sub-problems before solving them and uses the bounds to determine whether the nodes should be created. The solution of the node is also delayed until it is selected from the active set. Under the B&B framework, PBB and EBB solve continuously a set of underlying LP problems by ECON and PS respectively. Both ECON and PS can reuse old solutions very efficiently. Based on Rong and Lahdelma (2007a), the speed ratio of ECON against PS is 1.3 on average. Solver speed combined with node count would predict for EBB against PBB a speed ratio of 10 from scratch and 14 for when reusing the previous results. However, EBB is about 25 times faster than PBB. The additional improvement in efficiency relies on the fact that EBB can select based on the lower
bounds more promising sub-problems to be solved first. This will yield good MIP solutions earlier, which in turn allows pruning inferior sub-problems more efficiently, before they are solved. That is, EBB does not solve all of created nodes.

6. Conclusions

Advanced combined heat and power (CHP) production technologies result in the non-convex plant characteristic. We have developed an efficient envelope-based Branch and Bound algorithm (EBB) that is specialized for solving hourly non-convex CHP models under the deregulated power market. In the new market situation, the decision support tool must be more efficient than before (Makkonen, 2005). Efficient solution of hourly CHP models is important, because rapid re-optimization is required when the situation on the market changes, because a long-term planning model requires solving thousands of hourly models, and because various advanced stochastic analyses require solving a large number of long-term models.

In test runs with the realistic CHP production models, we used the ILOG CPLEX 9.0 MIP solver and the Power Simplex (PS) based B&B (PBB) algorithm as benchmark. CPLEX is a general purpose solver for large-scale LP/MIP problems. PS is an efficient professional specialized primal-based Simplex algorithm that has been developed for the convex CHP planning problem. For reusing the previous solutions, EBB is from 661 to 955 times (with average 785) faster than CPLEX and from 11 to 31 times (with average 24) faster than PBB. The approach that EBB deals with the non-convexity can be used for solving a class of separable non-linear programming optimization problems in different production, economics and transportation contexts (Sherali, 2001).

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References

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