On Essential and Inessential Polygons in Embedded Graphs

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In this article, we present a number of results of the following type: A given subgraph of an embedded graph either is embedded in a disc or it has a face chain containing a non-contractible closed path. Our main application is to prove that any two faces of a 4-representative embedding are simultaneously contained in a disc bounded by a polygon. This result is used to prove the existence of \( (r-1)/8 \) pairwise disjoint, pairwise homotopic non-contractible separating polygons in an \( r \)-representative orientable embedding. Our proof of this latter result is simple and mechanical.

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1. INTRODUCTION

In their pioneering work on graph minors, Robertson and Seymour [11] made significant use of the concept of the representativity of a graph embedded in a surface. This has stimulated much discussion, but important results seem to be hard to come by. In particular, it seems difficult to use the representativity to ensure the existence of particular homotopy types of polygons in the embedded graph.

In this work, we investigate some “local” properties of embeddings. In particular, we are interested in statements of the type, “If \( G \) is embedded in

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\( \Sigma \) and \( H \) is a subgraph of \( G \), then either there is a polygon of \( G \) bounding a disc containing \( H \) or there is some simple property in \( H \) which shows such a polygon does not exist.” This property is usually the existence of some non-contractible closed path which is “contained” in \( H \) in some way.

Before getting to precise statements of our results, some specific concepts are required.

Let \( \Sigma \) be a surface (i.e., a compact connected two-dimensional manifold without boundary). A path is a continuous function \( \gamma : [0, 1] \to \Sigma \) (thus, we use path in the topological and not in the graphical sense). We adopt the standard topological notions of ends of a path, closed paths, simple paths, inessential (=contractible) and essential (=non-contractible) paths and path composition [7].

Throughout this article, we refer to \( \Omega \subset \Sigma \) as an O-arc if \( \Omega \) is the image of a simple closed path. We use \( (\Omega) \) to denote a particular choice for such a path. Two O-arcs \( \Omega_1 \) and \( \Omega_2 \) are homotopic if \( (\Omega_1) \) and \( (\Omega_2) \) may be chosen to be homotopic. Note that the property of being essential or inessential is independent of the choice of \( (\Omega) \), so \( \Omega \) is essential if, for some (and hence any) choice, \( (\Omega) \) is essential. Similarly, \( \Phi \) is an I-arc if \( \Phi \) is the image of a path \( (\Phi) \) which is a homeomorphism of the closed unit interval and two I-arcs \( \Phi_1 \) and \( \Phi_2 \) are homotopic if \( (\Phi_1) \) and \( (\Phi_2) \) can be chosen to be homotopic. (When referring to homotopy of two paths, we usually mean with fixed basepoint in the case of closed curves and always mean with fixed endpoints in the case of paths that are not closed. For simple closed paths and O-arcs, we sometimes mean freely homotopic, i.e., allowing the basepoint to move as well. This is in the context of having homotopic, but disjoint, simple closed curves in the surface.)

Note that O-arcs and I-arcs are point sets in the surface. A polygon in a graph \( G \) is a subgraph of \( G \) which is mapped to an O-arc by any embedding of \( G \) in \( \Sigma \). Similarly, an arc is the graph theoretic counterpart of an I-arc.

In an abuse of language we will also use the graph theoretic terms polygon, arc and even \( G \) to refer to their images under an embedding of \( G \). Thus, for a given embedding of \( G \) a polygon is essential if its image under the embedding is an essential O-arc and two arcs are homotopic if their images are homotopic I-arcs.

We have chosen to adopt this slightly cumbersome terminology because we feel it is important to distinguish between point sets in the surface and traversals of the same point sets. (For images of closed curves that are not simple, different traversals can behave very differently with respect to homotopy, e.g., one traversal of a given figure eight in the torus can be null homotopic while another is not.) Thus, we choose polygon rather than the more standard cycle, since a cycle is usually defined by a traversal of the corresponding polygon. And similarly we wish to distinguish between objects in the surface and objects in the graph.
Suppose a graph $G$ is embedded in a surface $\Sigma$ other than the sphere. The representativity of the embedding $G$ is $r(G) = \min\{|\Omega \cap G| : \Omega \text{ is an essential O-arc}\}$. We do not define the representativity of embeddings in the sphere. An embedding $G$ in a surface $\Sigma$ is $k$-representative if $r(G) \geq k$.

As a main result of this work we show: if $G$ is a 4-representative embedding, then, for any two faces $F$ and $F'$, there is a polygon of $G$ that bounds a disc containing $F \cup F'$ (where a disc is homeomorphic to the closed unit disc in $\mathbb{R}^2$).

An O-arc $\Omega$ is separating if $\Sigma \setminus \Omega$ is not connected. It follows from the work of Robertson and Seymour [11] that if $G$ is embedded in $\Sigma$ with sufficiently high representativity (as a function of the genus of $\Sigma$), then $G$ contains an essential separating polygon. One wonders how high the representativity must be. In an earlier version of this article, Richter and Vitray [10] showed that if $r(G) \geq 11$ and the genus of the orientable surface is at least 2, then $G$ contains an essential separating polygon. Immediately following that, Zha and Zhao [17] improved this result by showing that if $r(G) \geq 7$ and the (orientable or nonorientable) surface $\Sigma$ has genus at least 2, then $G$ contains an essential separating polygon. Subsequently, Brunet et al. [3] extended the results to the following:

1. if $r(G) \geq 5$ and $\Sigma$ is a nonorientable surface of genus at least 2, then $G$ contains an essential separating polygon;
2. if $r(G) \geq 6$ and $\Sigma$ is an orientable surface of genus at least 2, then $G$ contains an essential separating polygon;
3. if $\Sigma$ is an orientable surface, then $G$ contains $[(r(G) - 1)/2]$ pairwise disjoint, pairwise homotopic, essential polygons; and
4. if $\Sigma$ is a surface of genus at least two, then $G$ contains $[(r(G) - 1)/8] - 1$ pairwise disjoint, pairwise homotopic, essential separating polygons.

Zha and Zhao [17] proved that every 5-representative embedding in an orientable surface has a pair of disjoint homotopic essential polygons. Cutting out the cylinder bounded by these two polygons and gluing in discs, our result that any two faces of a 4-representative embedding are simultaneously in a closed disc bounded by a polygon gives the orientable results (2)–(4) stated above in a simple way.

In the version of [3] originally submitted for publication, the bound of approximately $r/8$ in (4) was stated as $r/4$. In the meantime, this work was undertaken and we proved the $r/8$ bound. It was natural to then review the proof in that version of [3] and an error was discovered. The referee of the original version of [3], whose report came later, provided a counter-example to the $r/4$ estimate. The final version of [3] contains the $r/8$ estimate given above.
Zha [17] has conjectured that if \( r(G) \geq 3 \) and \( \Sigma \) is an orientable surface of genus at least 2, then \( G \) contains an essential separating polygon. It seems difficult even to prove \( r(G) \geq 5 \) suffices in the orientable case.

2. CLOSED DISCS CONTAINING SPECIFIED SUBGRAPHS

In this section, we prove several results about the existence of closed discs containing specified subgraphs. We believe that they will find applications to other questions about embeddings of graphs beyond those considered here.

The following fact seems intuitively obvious, but is slippery.

**Proposition 1.** Let \( G \) be a 2-connected graph embedded in a surface \( \Sigma \). Either \( G \) contains an essential polygon or there is a disc \( D \subset \Sigma \) such that \( G \cap D \) is bounded by a polygon of \( G \).

The proof of Proposition 1 and its corollary require the following result of Whitney.

**Whitney’s Theorem.** Let \( G \) be a 2-connected graph and let \( H \) be a 2-connected subgraph of \( G \). Either \( G = H \) or there is a 2-connected subgraph \( K \) of \( G \) containing \( H \) and an arc \( A \) that is disjoint from \( K \) except for its ends such that \( G = K \cup A \).

Thus, every 2-connected graph is either a polygon or the union of a 2-connected subgraph and an arc. This yields the ear decomposition of a 2-connected graph.

**Proof of Proposition 1.** We will proceed by induction on \( |E(G)| \).

If \( G \) is a polygon, the result follows from Epstein [4]: every inessential O-arc in \( \Sigma \) bounds a disc in \( \Sigma \). Suppose that \( G \) is not a polygon. By Whitney’s Theorem, there is a 2-connected proper subgraph \( H \) of \( G \) and an arc \( A \) such that \( A \cap H \) consists of the ends of \( A \) and \( G = H \cup A \). If \( H \) has an essential polygon, then so does \( G \) and the result follows. Therefore, we can assume \( H \) has no essential polygon. Also, \( |E(H)| < |E(G)| \) so the inductive hypothesis applied to \( H \) yields a disc \( A_H \) in \( \Sigma \) containing \( H \) and bounded by a polygon \( P \) of \( H \).

Since \( A \) is disjoint from \( H \) except for its ends, either \( A \subset A_H \) or \( A \cap A_H \) is just the ends \( u \) and \( v \) of \( A \) and both \( u \) and \( v \) are vertices of \( P \). In the first case, \( G \subset A_H \) and we are done. In the second case, let \( A_1 \) and \( A_2 \) be the two arcs of \( P \) having ends \( u \) and \( v \). Consider the polygon \( P_1 = A \cup A_1 \). If this is essential, then we are done. Therefore, we can assume \( P_1 \) is inessential and so, by [4], bounds a disc \( A_1 \). Note that \( A_1 \subset A_H \cap A_1 \).
If \( A_H - A_1 \) is disjoint from \( A_1 \), then \( A_H \cap A_1 = A_1 \), which implies \( A_H \cup A_1 \) is a disc containing \( G \) bounded by the polygon \( A_2 \cup A_3 \), as required.

If \( A_H - A_1 \) is not disjoint from \( A_1 \), then there is a point \( y \) in \((A_H - A_1) \cap A_1 \). We claim that this implies \( A_H \subset A_1 \). For if \( x \) is any point of \( A_H - A_1 \), there is an I-arc \( \Phi \) in \( A_H - A_1 \) with ends \( x \) and \( y \). Since \( \Phi \) is disjoint from \( P_i \), it must be wholly contained in the same region of \( \Sigma - P_i \) as \( y \), namely within \( A_1 \). Since \( A \) is also contained in \( A_1 \) we conclude that \( A_1 \) is the disc containing \( G \) and bounded by a polygon of \( G \), namely \( P_i \).

Before proceeding we require a generalization of the notion of I-arc. Suppose that \( \Phi_1, \Phi_2, ..., \Phi_n \) are I-arcs in a surface \( \Sigma \). Also, for \( i = 1, 2, ..., n \), let \( \tilde{\gamma}_i : [0, 1] \to \Phi_i \) be a homeomorphism such that, for \( i = 1, 2, ..., n-1 \), \( \tilde{\gamma}_i(1) = \tilde{\gamma}_{i+1}(0) \). For a particular choice of \( \tilde{\gamma}_i's \), we use \((\Phi_1, \Phi_2, ..., \Phi_n)\) to represent the path \( \tilde{\gamma}_1 \ast \tilde{\gamma}_2 \ast \cdots \ast \tilde{\gamma}_n \), where ‘\( \ast \)’ is the standard topological operation of path composition.

Suppose \( \tilde{\gamma}_1, \tilde{\gamma}_2, \) and \( \tilde{\gamma}_3 \) are paths with common ends. From elementary homotopy theory if \( \tilde{\gamma}_1 \) is homotopic to \( \tilde{\gamma}_2 \), then \( \tilde{\gamma}_1 \ast (\tilde{\gamma}_2)^{-1} \) is homotopic to \( \tilde{\gamma}_2 \ast (\tilde{\gamma}_3)^{-1} \) (where \( (\tilde{\gamma}_3)^{-1} \) is the reverse of the path \( \tilde{\gamma}_3 \)). We can rephrase this result in the language of arcs.

**Observation 2.** If \( P, Q \), and \( R \) are arcs with common ends and \( P \) is homotopic to \( Q \), then \((P, R)\) can be chosen homotopic to \((Q, R)\).

As an application, we obtain the following.

**Corollary 1.1.** Let \( G \) be a 2-connected graph and let \( H \) be a 2-connected subgraph of \( G \). If \( G \) is embedded so that some polygon of \( G \) is essential but no polygon of \( H \) is essential, then there is an arc \( A \) of \( G \), disjoint from \( H \) except for its ends, such that \( H \cup A \) contains an essential polygon.

**Proof.** By Whitney’s Theorem, there is a sequence \( A_1, A_2, ..., A_k \) of arcs of \( G \) such that, for each \( i = 1, 2, ..., k \), \( A_i \) is disjoint from \( H \cup A_1 \cup \cdots \cup A_{i-1} \) except for its ends. \( H \cup A_1 \cup \cdots \cup A_k \) is 2-connected and \( H \cup A_1 \cup \cdots \cup A_k \) contains an essential polygon.

Choose \( k \) so that \( H \cup A_1 \cup \cdots \cup A_{k-1} \) does not contain an essential polygon. By Proposition 1, there is a polygon in \( H \cup A_1 \cup \cdots \cup A_{k-1} \) that bounds a disc \( A \) containing \( H \cup A_1 \cup \cdots \cup A_{k-1} \). Let \( A' \) and \( A'' \) be totally disjoint arcs in \( H \cup A_1 \cup \cdots \cup A_{k-1} \) joining the ends of \( A_k \) to vertices in \( H \), so that \( A'' = A' \cup A_k \cup A'' \) is an arc disjoint from \( H \) except for its ends. (Note that \( A' \) or \( A'' \) may be empty if \( A_k \) is not disjoint from \( H \).) Let \( A'' \) be an arc in \( H \) joining the ends of \( A'' \), so \( A'' \cup A'' \) is a polygon.

There is an essential polygon \( P \) in \( H \cup A_1 \cup \cdots \cup A_k \). Clearly, \( P \) contains \( A_k \) and is, therefore, of the form \( A_k \cup A \), for some arc \( A \) in \( A \). The arc
$A' \cup A^+ \cup A^*$, which also joins the ends of $A_k$, is also contained in $A$; hence it is homotopic to $A$. By Observation 2, $(A', A^+, A^*, A_k)$ may be chosen homotopic to $(A, A_k)$ and so must be essential. Thus $A^+ \cup A^* = A' \cup A^+ \cup A^* \cup A_k$ is an essential polygon contained in $H \cup A^*$ as required.

For the remainder of this article, we shall assume basic results about connectivity and representativity as outlined in [12]. We shall always have both connectivity and representativity at least 2 which is equivalent to the condition that all closed faces are discs bounded by polygons. One important result that we need in this work is the following, first proved in [15], and also proved in [5, 6].

**Nested Polygons Theorem.** Let $G$ be an $r$-representative embedding and let $F$ be any face of $G$. Then there are discs $D_1, \ldots, D_{[r/2]}$ bounded by polygons of $G$ such that $F \subseteq D_1 \subseteq \cdots \subseteq D_{[r/2]}$ and, for each $i = 2, 3, \ldots, [r/2]$, $D_{i-1}$ is contained in the interior of $D_i$.

For a vertex $v$ of a 3-connected 3-representative embedding $G$, the union of the faces of $G$ that are incident with $v$ is a disc bounded by a polygon of $G$. This disc is the wheel neighborhood of $v$ and $v$ is its hub. For a wheel neighborhood $W$, we may sometimes use $W$ or the wheel neighborhood to mean the graph $G \cap W$.

A slice of a wheel neighborhood $W$ in a 3-connected and 3-representative embedding is a 2-connected union of a subset of the faces of $W$. (Thus, necessarily, it is a union of consecutive faces in the cyclic rotation around the hub of $W$.) By Proposition 1, every slice is contained in a disc bounded by a polygon of the slice.

Our next result is one of the two main technical tools in this article.

**Theorem 3.** Let $G$ be a 3-connected 3-representative embedding in $\Sigma$ and let $T$ be a tree whose vertices are slices of wheel neighborhoods of $G$ such that:

1. if $S$ and $S'$ are adjacent vertices of $T$, then $S$ and $S'$ have a face of $G$ in common;
2. if $S$ and $S'$ are at distance at least 3 in $T$, then $S$ and $S'$ are disjoint in $G$; and
3. if $S$ is adjacent to both $S'$ and $S''$ in $T$, then there is a disc in $\Sigma$ containing $S \cup S' \cup S''$.

Then there is a disc in $\Sigma$ containing $\bigcup_{S \in V(T)} S$.

We remark that we do not assume that if $S$ and $S'$ are not disjoint, then they are adjacent in $T$. However, Condition 2 implies they are at distance at most two in $T$. Furthermore, if $H$ and $K$ are 2-connected subgraphs of $a$
graph \( G \) having at least 2 vertices in common, then \( H \cup K \) is also a 2-connected subgraph of \( G \); thus, \( \bigcup_{S \in V(T)} S \) is a 2-connected subgraph of \( G \).

**Proof of Theorem 3.** We proceed by induction on \( n = |V(T)| \), the base case \( n \leq 3 \) being trivial by Condition (3). If \( \bigcup_{S \in V(T)} S \) does not contain an essential polygon, then we are done by Proposition 1. Thus, we assume \( \bigcup_{S \in V(T)} S \) contains an essential polygon and seek a contradiction.

Let \( S^* \) be a vertex that has degree 1 in \( T \). By the inductive assumption, \( \bigcup_{S \in V(T) \setminus \{S^*\}} S \) is contained in a disc and hence does not contain an essential polygon. By Corollary 1.1, there is an essential polygon \( P \) in \( \bigcup_{S \in V(T)} S \) such that \( P \) consists of an arc \( A' \) in \( \bigcup_{S \in V(T) \setminus \{S^*\}} S \) and an arc \( A \) in \( S^* \) that is disjoint from \( \bigcup_{S \in V(T) \setminus \{S^*\}} S \) except for its ends \( u \) and \( v \).

We let \( S_1, S_2, \) and \( S_3 \) be the (not necessarily distinct) slices in \( V(T) \setminus \{S^*\} \) such that \( u \) is in \( S_1 \), \( v \) is in \( S_2 \) and \( S_3 \) is the unique neighbor of \( S^* \) in \( T \). By Condition (2), \( S_2 \) and \( S_3 \) either equal \( S_1 \) or are neighbors of \( S_1 \) in \( T \). Let \( w \) be a vertex in \( S^* \cap S_1 \). Let \( A_1 \) be an arc in \( S^* \) with ends \( u \) and \( w \) and let \( A_2 \) be an arc in \( S^* \) with ends \( w \) and \( v \); because \( S^* \) is 2-connected, we can assume \( A_1 \) and \( A_2 \) are disjoint except for their common end \( w \). Let \( A'_1 \) be an arc in \( S_1 \cup S_2 \) with ends \( u \) and \( w \) and let \( A'_2 \) be an arc in \( S_3 \cup S_4 \) with ends \( w \) and \( v \).

Both \( A_1 \) and \( A'_1 \) are contained in the disc containing \( S^* \cup S_1 \cup S_2 \), so \( A_1 \) is homotopic to \( A'_1 \). Similarly, \( A_2 \) is homotopic to \( A'_2 \); hence, \( (A_1, A_2) \) may be chosen homotopic to \( (A'_1, A'_2) \). Also, \( A \) and \( A_1 \cup A_2 \) are contained in \( S^* \) so \( (A) \) may be chosen homotopic to \( (A'_1, A'_2) \).

By assumption, \( P \) is essential, so any choice of \( (A', A) \) is essential. By two applications of Observation 2 we see that \( (A', A) \) may be chosen homotopic to \( (A'_1, A'_2) \). This is a contradiction since \( A' \cup A'_1 \cup A'_2 \) is contained \( \bigcup_{S \in V(T) \setminus \{S^*\}} S \) which in turn is contained in a disc.  

**Corollary 3.1.** Let \( G \) be a 3-connected 3-representative embedding in a surface \( \Sigma \). Let \( S_1, S_2, \ldots, S_n \) be a chain of slices of wheel neighborhoods with distinct hubs \( v_1, v_2, \ldots, v_n \), respectively. Suppose that

1. for each \( i = 2, 3, \ldots, n-1 \), there is a disc in \( \Sigma \) containing \( S_{i-1} \cup S_i \cup S_{i+1} \) and
2. for \( |i-j| > 2 \), \( S_i \) and \( S_j \) are disjoint.

Then there is a disc in \( \Sigma \) containing \( S_1 \cup S_2 \cup \cdots \cup S_n \).
Corollary 3.2. Let $G$ be a 3-connected 3-representative embedding, let $F$ be a face of $G$ and let $S_1, S_2, \ldots, S_n$ be slices such that, for $i = 1, 2, \ldots, n, F \subseteq S_i$. Either $\bigcup_{1 \leq i \leq n} S_i$ is contained in a disc or there exist $i, j \in \{1, 2, \ldots, n\}$ such that $S_i \cup S_j$ is not contained in a disc.

Proof. Let $T$ a tree with vertices $F, S_1, S_2, \ldots, S_n$ and an edge joining $F$ to each $S_i$. Obviously, $T$ satisfies Conditions (1) and (2) of Theorem 3. If $T$ satisfies (3), then $\bigcup_{1 \leq i \leq n} S_i$ is contained in a disc. If $T$ does not satisfy (3), then, since $F$ is the only vertex in $T$ of degree greater than one, there exist $i, j \in \{1, 2, \ldots, n\}$ such that $S_i \cup F \cup S_j$ is not contained in a disc. By construction, $F \subseteq S_j \cup S_i$; hence, $S_i \cup S_j$ is not contained in a disc.

We remark that, since $\bigcup_{S \in \phi(T)} S$ is 2-connected, we can assume the discs of Theorem 3 and its corollaries are bounded by polygons of $G$.

3. LOCATING ESSENTIAL CENTRAL O-ARCS

A face chain is an alternating sequence $v_0, F_1, v_1, F_2, \ldots, F_n, v_n$ of vertices and faces. Its length is $n$. It is closed if $v_0 = v_n$ and simple if all the $v_i$ (except perhaps $v_0$ and $v_n$) and all the $F_i$ are distinct. For a face chain, $v_0, F_1, v_1, F_2, \ldots, F_n, v_n$, with $n \geq 2$, there is an I-arc $\Phi_i$, contained in $F_i$, joining $v_{i-1}$ and $v_i$ but otherwise disjoint from the boundary of $F_i$. If the face chain is simple and closed, then the union $\Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_n$ is a central O-arc of the face chain. Since any two central O-arcs of a given face chain are homotopic keeping $v_0, v_1, \ldots, v_n$ fixed (assuming every closed face is a disc), we sometimes speak of the central O-arc of the face chain. If the face chain $\mathcal{F}$ has an essential central O-arc, then $\mathcal{F}$ is an essential face chain.

A face chain $v_0, F_1, v_1, \ldots, F_{n-1}, v_n$ is determined by a chain of slices $S_0, S_1, \ldots, S_n$ if, for $i = 0, 1, \ldots, n$, $v_i$ is the hub of $S_i$ and, for each $i = 1, 2, \ldots, n$, $F_i \subseteq S_{i-1} \cap S_i$.

A $\Theta$-arc is the union of an O-arc and an I-arc that have only the ends of the I-arc in common. A $\Theta$-graph is the union of three arcs with common ends but which are otherwise disjoint. (Thus, an embedded $\Theta$-graph is a $\Theta$-arc.) Using Observation 2, it is easy to see that if one of the three distinct O-arcs in a $\Theta$-arc is essential then at least two of them must be essential. We employ this fact in the following proof of our second main technical engine.

Theorem 4. Let $G$ be a 3-connected 3-representative embedding. Let $S_1, S_2, \ldots, S_n$ be a chain of slices of wheel neighborhoods with distinct hubs $v_1, v_2, \ldots, v_n$ such that $S_1 \cup S_2 \cup \cdots \cup S_n$ contains an essential polygon, but neither $S_1 \cup S_2 \cup \cdots \cup S_n$, nor $S_1 \cup S_2 \cup \cdots \cup S_{n-1}$ does. Let $F_i$ be a face in
$S_{i-1} \cap S_i$ for $i = 2, 3, \ldots, n$. If the faces $F_2, F_3, \ldots, F_n$ can be chosen to be distinct, then there exist a face $F_1$ in $S_1 \setminus (S_2 \cup S_3 \cup \cdots \cup S_n)$, a face $F_{n+1}$ in $S_n \setminus (S_1 \cup S_2 \cup \cdots \cup S_{n-1})$ and a vertex $v$ such that $v, F_1, v_1, F_2, \ldots, v_{n-1}, F_n, v_n, F_{n+1}, v$ is an essential, simple, closed face chain.

**Proof.** By Corollary 1.1, there is an essential polygon $P$ in $\bigcup_{i=1}^n S_i$ such that $P$ consists of the arc $A = P \cap \bigcup_{i=2}^n S_i$ and an arc $A' \subseteq S_1$ disjoint from $\bigcup_{i=2}^n S_i$ except for its ends $u$ and $v$.

If $u$ and $v$ are both contained in $\bigcup_{i=2}^{n-1} S_i$, then let $A'$ be an arc contained in $\bigcup_{i=2}^{n-1} S_i$ joining $u$ and $v$. The arcs $A'$ and $A$ are both contained in $\bigcup_{i=2}^n S_i$ and so are homotopic. Therefore, any choices of $(A, A')$ and $(A, A')$ are homotopic, which is impossible since $P = \bigcup_{i=1}^n S_i$ is essential and $A \cup A'$ is a subset of the disc containing $\bigcup_{i=2}^{n-1} S_i$. Thus, without loss of generality, we assume $u$ is a vertex in $S_n \setminus \bigcup_{i=2}^{n-1} S_i$.

Let $e$ be the edge of $A$ incident with $u$. By construction, $e$ is contained in $S_1 \setminus \bigcup_{i=2}^n S_i$; hence, there is a face $F_u$ in $S_1 \setminus \bigcup_{i=2}^n S_i$ which has $e$ and, therefore, $u$ in its boundary. Let $\Phi$ be an I-arc from $u$ to $v_1$ through $F_u$. Similarly, there is a face $F_v$ in $S_1 \setminus \bigcup_{i=2}^n S_i$ which has $v$ in its boundary and an I-arc $\Phi_v$ from $v$ to $v_1$ through $F_v$. As an added condition, we choose $\Phi$, so that $\Phi_v \cap \Phi_u = v_1$ (which can be done even if $F_u = F_v$ since the faces of $G$ are discs bounded by polygons).

Let $F'_v$ be a face in $S_1 \setminus \bigcup_{i=2}^{n-1} S_i$ which has $u$ in its boundary and let $\Phi'_v$ be an I-arc from $v$ to $u$ through $F'_v$. Note that, since $F'_u$ is in $S_u$ and $F_u$ is not, $\Phi'_u \cap \Phi_u = u$.

Let $j = \min \{i \in \{2, 3, \ldots, n\} : v \in S_i\}$. Let $F'_v$ be a face in $S_1 \setminus \bigcup_{i=2}^{j-1} S_i$ which has $v$ in its boundary and let $\Phi'_v$ be an I-arc from $v_j$ to $v$ through $F'_v$ chosen so that $(\Phi'_v \cap \Phi_u)$ \(\setminus v_n = \emptyset\). Once again, since $F'_u$ is in $S_j$ and $F_v$ is not, $\Phi'_u \cap \Phi_v = v$.

Turning our attention to constructing the part of the central $O$-arc disjoint from $u$ or $v$ we let $\Phi_1$ be an I-arc in $F_1$ joining $v_{n-1}$ to $v_1$ for $i \in \{2, 3, \ldots, n\}$. Using these I-arcs we let $\beta = \bigcup_{i=2}^n \Phi_1$ and, if $j < n$, we let $\beta' = \bigcup_{i=1}^{n-j-1} \Phi_1$.

First, suppose that $j < n$. The I-arcs $A$ and $\Phi_u \cup \beta \cup \Phi'_v$ are contained in $\bigcup_{i=1}^n S_i$ which is contained in a disc; hence, $A$ is homotopic to $\Phi_u \cup \beta \cup \Phi'_v$. Similarly, $\tilde{A}$ is homotopic to $\Phi'_v \cup \beta' \cup \Phi_u$. Thus, $(A, \tilde{A})$ may be chosen homotopic to $(\Phi_u, \beta, \Phi'_v, \beta', \Phi_u)$ which is homotopic to $(\Phi_u, \beta, \beta', \Phi_u)$. From the assumption that $P$ is essential we conclude that $(\Phi_u, \beta, \beta', \Phi_u)$ is an essential path, so $\Phi_u \cup \beta \cup \beta' \cup \Phi_u$ provides the required essential central $O$-arc.

For the remaining case, we have $j = n$. This implies that $\Phi'_v$ is in $S_n \setminus \bigcup_{i=2}^{n-1} S_i$. Thus the I-arcs $A$ and $\Phi_u \cup \Phi'_v$ are both in the disc containing $\bigcup_{i=2}^n S_i$ and so are homotopic. Similarly, $\tilde{A}$ is homotopic to $\Phi_u \cup \Phi_u$, which implies $(A, \tilde{A})$ may be chosen homotopic to $(\Phi_u, \Phi_u, \Phi'_v, \Phi_u)$; hence,
\[ \Phi_u \cup \Phi_v \cup \Phi_u' \cup \Phi_v' \] must be an essential O-arc. By construction, \( \beta \cup \Phi_u \cup \Phi_v \cup \Phi_u' \cup \Phi_v' \) is a \( \Theta \)-arc and so either \( \beta \cup \Phi_u \cup \Phi_u' \) or \( \beta \cup \Phi_v \cup \Phi_v' \) must also be essential and provides the required essential central O-arc.

In all cases, \( F \) is either \( F_u \) or \( F_v \) and so is in \( S_i \setminus (S_{i-1} \cup \cdots \cup S_n) \).

Also, by construction we have \( F_n \) in \( S_n \setminus (S_{n-1} \cup \cdots \cup S_1) \). If \( F_n \) is in \( S_i \), then the entire essential central O-arc is contained in \( S_i \setminus (S_{i-1} \cup \cdots \cup S_n) \), contrary to hypothesis. Thus, \( F_n \) in \( S_{n-1} \setminus (S_{n-2} \cup \cdots \cup S_1) \).

Putting Corollary 3.1 and Theorem 4 together, we have:

**Corollary 4.1.** Let \( G \) be a 3-connected 3-representative embedding in \( \Sigma \). Let \( S_1, S_2, \ldots, S_n \) be a chain of slices of wheel neighborhoods, with distinct hubs \( v_1, v_2, \ldots, v_n \), respectively, satisfying Condition (2) of Theorem 3. Then one of the following holds:

1. there is a disc containing \( S_1 \cup S_2 \cup \cdots \cup S_n \);
2. there exists \( i \in \{1, 2, \ldots, n-1\} \) such that \( S_i \cup S_{i+1} \) contains an essential, simple, closed face chain \( v, F, v, F, \ldots, v \);
3. there exists \( i \in \{2, 3, \ldots, n-1\} \) such that \( S_{i-1} \cup S_i \cup S_{i+1} \) contains an essential, simple, closed face chain \( v, F, v, F, \ldots, v \).

**Proof.** If (1) does not hold, then, by Corollary 3.1, there is some \( i \in \{2, 3, \ldots, n-1\} \) such that \( S_{i-1} \cup S_i \cup S_{i+1} \) is not contained in a disc of \( \Sigma \). If either \( S_{i-1} \cup S_i \) or \( S_i \cup S_{i+1} \) is not contained in a disc, then Theorem 4 implies Conclusion (2). If both \( S_{i-1} \cup S_i \) and \( S_i \cup S_{i+1} \) are contained in discs, then Theorem 4 implies Conclusion (3).

Taking the slices in Corollary 3.2 and Theorem 4 to be entire wheel neighborhoods, we get the following corollary.

**Corollary 4.2.** Let \( G \) be a 3-connected 5-representative embedding in \( \Sigma \) and let \( W_1 \) and \( W_2 \) be any wheel neighborhoods of \( G \). Then there is a disc \( D \) bounded by a polygon of \( G \) such that \( W_1 \cup W_2 \subseteq D \).

**Proof.** Let \( W_i = N_0, N_1, \ldots, N_k = W_2 \) be a shortest chain of slices of wheel neighborhoods, where the slices \( N_i \) are complete wheel neighborhoods. If \( N_i \) and \( N_j \) are not disjoint, for some \( i < j-2 \), then they have a vertex \( u \) in common. Let \( N \) be the wheel neighborhood with hub \( u \). The chain \( N_0, \ldots, N, N, N_j, \ldots, N_k \) is a shorter chain, a contradiction. Thus, the original chain satisfies Condition (2) of Theorem 3.

Since \( G \) is 5-representative, neither (2) nor (3) of Corollary 4.1 can hold.

Specializing Corollary 4.2 yields the following fact, which will be improved in Theorem 6.
Corollary 4.3. Let $G$ be a 3-connected 5-representative embedding in $\Sigma$ and let $F$ and $F'$ be any faces of $G$. Then there is a disc $A$ bounded by a polygon of $G$ such that $F \cup F' \subseteq A$.

4. FINDING CLOSED DISCS CONTAINING SPECIFIED FACES

In this section we improve the result of Corollary 4.3 by showing that any two faces of a 4-representative embedding are simultaneously in a closed disc bounded by a polygon of the graph.

To attain the improvement in representativity from 5 to 4 we must be a bit more clever in the way we choose the chain of slices joining the two faces.

Let $F$ and $F'$ be any two faces of a 3-connected 3-representative embedding $G$. Let $S_0, S_1, \ldots, S_k$ be a shortest chain of slices such that $F \subseteq S_0$ and $F' \subseteq S_k$. Note that a face chain determined by a shortest chain of slices necessarily has distinct faces. Furthermore, if $S_i$ and $S_j$ have a vertex $v$ in common with $|i - j| > 2$, then (as in the proof of Corollary 4.2) we can find a shorter chain of slices from $F$ to $F'$ through $v$. Thus, any such shortest chain of slices must satisfy Condition (2) of Corollary 3.1.

Among all such shortest chains, a canonical choice for $F$ and $F'$ makes $(|S_0|, |S_1|, \ldots, |S_k|)$ lexicographically least, where $|S_i|$ denotes the number of faces in $S_i$. We remark that the lexicographic minimality implies there is a unique face chain $v_0, F_1, \ldots, F_k, v_k$ determined by the chain of slices. Moreover, each slice $S_i$ in the chain consists of consecutive faces from $F_i$ to $F_{i+1}$ in the cyclic ordering of faces around $v_i$ (with $F_0 = F$ and $F_{k+1} = F'$).

Proposition 5. Let $G$ be a 3-connected 3-representative embedding and let $F$ and $F'$ be distinct faces of $G$. Suppose that, for any face $F''$ such that $F \cap F'' = \emptyset$ and $F' \cap F'' = \emptyset$, there is a polygon bounding a disc containing the closed face $F''$ in its interior.

(1) Let $S_0, S_1, \ldots, S_k$ be the slices of a canonical choice for the faces $F$ and $F'$. If both $S_0 \cup S_1$ and $S_{k-1} \cup S_k$ are contained in discs, then there is a polygon in $G$ bounding a disc that contains $S_0 \cup S_1 \cup \cdots \cup S_k$. In particular, there is a polygon bounding a disc that contains $F \cup F'$.

(2) Suppose a shortest chain of slices joining $F$ and $F'$ has at least three slices. If there is no polygon in $G$ bounding a disc containing $F \cup F'$, then there is an essential, closed face chain $v_0', F_1', v_1', \ldots, F_n', v_n'$ such that $n \leq 4$ and $\{F, F'\} \cap \{F_1', F_2', \ldots, F_n'\} = \emptyset$.

For 4-representative embeddings, the condition on $F''$ is implied by the Nested Polygons Theorem and holds for all faces, not just those disjoint from $F$ and $F'$. Thus, Proposition 5 is stronger than what is required for
the improvement of Corollary 4.3 to 4-representative embeddings. We will use the full strength of Proposition 5 in the next section.

Proof. (1) For \( i \in \{1, 2, \ldots, k-2\} \), the face \( F_{i+1} \) in \( S_i \cap S_{i+1} \) is disjoint from \( F \) and \( F' \). Therefore, there exists a disc bounded by a polygon \( P \) which contains \( F_{i+1} \) in its interior and, therefore, contains \( S_i \cup S_{i+1} \). By hypothesis, \( S_0 \cup S_1 \) and \( S_{k-1} \cup S_k \) are contained in discs. Thus, Conclusion (2) of Corollary 4.1 does not hold.

Suppose, Conclusion (3) of Corollary 4.1 holds. Let \( v_0, F_1, \ldots, F_k, v_k \) be the face chain determined by \( S_0, S_1, \ldots, S_k \). Then, for some \( i \in \{1, 2, \ldots, k-1\} \), there is a simple closed face chain \( v, F^+, v_{i-1}, F_i, v_i, F_{i+1}, v_{i+1}, F^*, v \) in \( S_{i-1} \cup S_i \cup S_{i+1} \) having essential central O-arc. Note that because the face chain is simple \( F^+ \neq F_i \) and \( F^* \neq F_{i+1} \).

Consider the chain of slices \( S_0, S_1, \ldots, S_{i}, S_{i+1}, S_{i+2}, \ldots, S_k \), where \( S_{i-1} \) is the minimal subslice of \( S_{i-1} \) containing both \( F^+ \) and \( F_{i+1} \), \( S_{i+1} \) is the minimal subslice of \( S_{i+1} \) containing both \( F^* \) and \( F_{i+2} \) and \( S_i \) is any minimal slice with hub \( v \) containing both \( F^+ \) and \( F^* \). From the remark preceding the proposition we see that \( F_i \) is not in \( S_{i-1} \); hence, these choices give a lexicographically smaller chain of slices than a canonical one, which is impossible. Therefore, \( S_0 \cup S_1 \cup \cdots \cup S_k \) is contained in a disc, as claimed.

(2) Among all shortest chains of slices \( S_0, S_1, \ldots, S_k \) so that \( F \subseteq S_0, F' \subseteq S_k \), choose one so that \((|S_1|, |S_{k-1}|)\) is lexicographically least. By Corollary 3.1, there exists an \( i \in 1, 2, \ldots, n-1 \) such that \( S_{i-1} \cup S_i \cup S_{i+1} \) is not contained in a disc.

Suppose that both \( S_{i-1} \cup S_i \) and \( S_i \cup S_{i+1} \) are contained in discs. By Theorem 4, there is a face chain \( v, F^+, v_{i-1}, F_i, v_i, F_{i+1}, v_{i+1}, F^*, v \) with essential central O-arc. If either \( F^+ = F \) or \( F^* = F' \), then (as before) there is a shorter chain of slices joining \( F \) and \( F' \), a contradiction. Similarly, \( F^* \neq F \) and \( F^* \neq F' \). Thus, this face chain satisfies (2).

If it is not the case that both \( S_{i-1} \cup S_i \) and \( S_i \cup S_{i+1} \) are contained in discs, then fix \( j \in \{0, 1, \ldots, n-1\} \) such that \( S_j \cup S_{j+1} \) is not contained in a disc. By Theorem 4 there is a simple face chain \( v, F^+, v_j, F_{j+1}, v_{j+1}, F^*, v \) with essential central O-arc such that \( F^* \subseteq (S_j \setminus S_{j+1}) \) and \( F^* \subseteq (S_{j+1} \setminus S_j) \).

If either \( F^+ = F' \) or \( F^* = F \), then we can find a shorter chain of slices joining \( F \) and \( F' \), a contradiction. Therefore, \( F^* \neq F' \) and \( F^* \neq F \). If \( F^* \neq F' \) and \( F^* \neq F' \), then the face chain satisfies the conclusions of the theorem.

Suppose \( F^+ = F \). If \( j \geq 1 \), then again there is a shorter chain of slices joining \( F \) and \( F' \). If \( j = 0 \), we let \( S'_1 \) be the minimal slice with hub \( v_1 \) contained in \( S_1 \) and containing \( F^* \cup F_2 \), where \( F_2 \) is the face in \( S'_1 \cap S_2 \). Since \( F^* \subseteq S_1 \setminus S_2, F^* \neq F_1 \). The minimality condition on \((|S_1|, |S_{k-1}|)\) implies that \( F^* \) is between \( F_1 \) and \( F_2 \) in the cyclic order of faces of \( S_1 \), which implies \( S'_1 \)
is a proper subslice of $S_i$. Thus the chain of slices $S_i, S_i', S_2, ..., S_k$ contradicts the lexicographic minimality of $\mid S_i \mid, \mid S_{k-1} \mid$.

Finally, suppose $F^* = F'$. Repeating the argument from the previous paragraph interchanging $F^*$ and $F^+$ and replacing each index $j$ with $k-j$, we obtain a chain of slices $S_0, ..., S_{k-1}', S_k$, where $S_{k-1}'$ is a proper subslice of $S_{k-1}$, once again contradicting the lexicographic minimality of $\mid S_i \mid, \mid S_{k-1} \mid$. (Notice that the assumption $k \geq 2$ ensures that $\mid S_i \mid$ remains the same or is reduced by the new choice of slices since only for $i = k$ can $\mid S_i' \mid > \mid S_i \mid$.)

Using this result we obtain the following improvement on Corollary 4.3.

**Corollary 5.1.** Let $F$ and $F'$ be distinct faces of a 3-connected 4-representative embedding $G$. Then there is a polygon of $G$ bounding a disc containing $F \cup F'$.

**Proof.** Let $S_0, S_1, ..., S_k$ be a shortest chain of slices such that $F \subseteq S_0$ and $F' \subseteq S_k$. By the Nested Polygons Theorem, each face in a 4-representative embedding is in the interior of a disc bounded by a polygon. Further, this applies to a face in $S_0 \cap S_i$ (respectively, $S_{k-1} \cap S_i$). In particular, the disc contains $S_0 \cup S_i$ (respectively, $S_{k-1} \cup S_i$).

Thus, the hypotheses of (1) of Proposition 5 hold. Therefore, there is a polygon in $G$ that bounds a disc containing $F \cup F'$, as required.

The connectivity condition can be easily disposed of by considering the essential cleavage unit.

**Theorem 6.** Let $F$ and $F'$ be any faces of a 4-representative embedding $G$. Then there is a polygon in $G$ bounding a disc containing $F \cup F'$.

**Proof.** Let $H$ denote the essential cleavage unit of $G$ as described in [12]; $H$ is a 3-connected graph which has a subdivision contained in $G$ and the representativity of this subdivision is that of $G$. Let $F_H$ and $F'_H$ be the faces of $H$ containing $F$ and $F'$, respectively. If $F_H = F'_H$, then the closed face $F_H$ provides the required disc. Otherwise, Corollary 5.1, applied to $H$ and the faces $F_H$ and $F'_H$, yields the conclusion for $G$.

It is clear that if $G$ has representativity equal to 2, then there is a pair $F, F'$ of faces such that $F \cup F'$ is not contained in any disc, which leaves the following question. Is being 3-representative enough to ensure that any two faces are contained in a disk? An easy argument shows that this question is dual to the following problem considered by Barnette [1, 2].

**Problem.** Let $G$ be a 3-connected 3-representative embedding in a surface and let $u$ and $v$ be vertices of $G$. Is there all arc $A$ in $G$ with ends $u$ and $v$ such that, for every face $F$ of $G$, $A \cap F$ is either empty or an arc?
Barnette prove that such an arc always exists if the surface is either the projective plane or the torus. Pulapaka and Vince [8, 9] have shown the arc always exists if the surface has nonnegative Euler characteristic and have examples for which the arcs do not exist in every surface with Euler characteristic at most $-2$. The duals of these examples show that being 3-representative is not sufficient to ensure every pair of faces is contained in a disc. (Upon hearing of our Theorem 6, they proved the dual; i.e., they proved that there is always an affirmative answer to Barnette’s Problem with the additional assumption that $G$ is 4-representative.)

5. EXISTENCE OF ESSENTIAL SEPARATING POLYGONS

In this section, we describe how our results can be used to prove:
(a) every 6-representative embedding in an orientable surface $S$ of genus at least 2 has an essential separating polygon, i.e., an essential polygon $P$ such that $S \setminus P$ is not connected; (b) that every $r$-representative embedding in an orientable surface of genus at least 2 has at least $\lceil \frac{r-1}{2} \rceil$ pairwise disjoint, pairwise homotopic essential separating polygons $P$; and (c) that an $r$-representative embedding has $\lceil \frac{r-1}{2} \rceil$ pairwise disjoint, pairwise homotopic essential polygons.

These results appear (in the case of (c) in a more refined version) in [3], although we provide a slight improvement (by one) in (b). As our results are not significantly different from those of [3], we do not prove them in detail.

The following observation of Zha and Zhao provides an important improvement of similar statements from an earlier version of this paper and plays a critical role in the arguments of this section.

**Lemma 7** [17]. Let $G$ be a 3-connected 5-representative embedding in an orientable surface. There are disjoint essential polygons $P$ and $Q$ and a cylinder $\mathcal{C}$, bounded by $P$ and $Q$, such that any two vertices of $P$ are joined by a face chain in $\mathcal{C}$ of length at most $(r(G)/2)+1$ and any two vertices in $P \cup Q$ are joined by a face chain in $\mathcal{C}$ of length at most $(r(G)/2)+2$.

We use Lemma 7 to obtain the fundamental construction of this section. Thus, for $G$ satisfying the conditions of the lemma, we construct $F(\mathcal{C})$ by deleting the interior of $\mathcal{C}$ and capping both $P$ and $Q$ with discs. The discs provide two new faces $F$ and $F'$ in $G(\mathcal{C})$ which are bounded by $P$ and $Q$, respectively. Any essential face chain in $G(\mathcal{C})$ which does not use $F$ or $F'$ is also a face chain in $G$ and must have length at least $r = r(G)$. Any other essential face chain must use one or both of the new faces. The limiting case is an essential face chain of length $r(G(\mathcal{C}))$ that uses $F'$ and not $F$. By
Lemma 7, such a face chain corresponds to an essential face chain in $G$ with length at most $r(G(\mathcal{C})) - 1 + \lceil \frac{r}{2} \rceil + 2$, which implies $r(G(\mathcal{C})) \geq \lceil \frac{r}{2} \rceil - 1$.

We use our construction along with Proposition 5 to obtain the first (and most important) of the three results.

**Theorem 8.** Let $G$ be a 6-representative embedding in an orientable surface $\Sigma$ with genus at least 2. Then $G$ contains an essential separating polygon.

**Sketch of Proof.** We shall conduct the argument with the assumption that $r \geq 7$. After the proof, we will indicate how to handle the case $r = 6$.

Consider our fundamental construct obtained from Lemma 7. If $P$ is separating, then we are done, so we may assume $P$ is not separating.

Using $r(G(\mathcal{C})) \geq \lceil \frac{r}{2} \rceil - 1$, we see that $r \geq 7$ implies $r(G(\mathcal{C})) \geq 3$. To obtain the connectivity condition of Proposition 6, we consider the essential cleavage unit $K$ of $G(\mathcal{C})$. A straightforward, though somewhat technical, argument shows that $K$ contains polygons $P$ and $Q$ that are contractions of $P$ and $Q$.

Let $F^*$ be any face disjoint from both $F$ and $F'$. Since $r \geq 6$, the Nested Polygons Theorem applied to $G$ implies that there are at least three pairwise disjoint polygons, including the boundary $\partial F^*$ of $F^*$, each bounding a disc containing $F^*$; hence, at least two of them bound discs containing $F^*$ in their interiors. The disjointness condition on $F^*$ implies that the smaller of these two discs can be taken to be disjoint from the interior of $\mathcal{C}$; thus, the major hypotheses of Proposition 5 are satisfied.

Let $S_0, S_1, \ldots, S_k$ be a shortest chain of slices in $K$ joining $F$ and $F'$. Since $P$ and $Q$ are disjoint, $F \cap F' = \emptyset$, so $k \neq 0$. If $k = 1$, then some $F_1$ in $S_0 \cap S_1$ has a vertex incident with $F$ and a vertex incident with $F'$. By Lemma 7 we can obtain an essential face chain in $G$ of length no more than $1 + (r/2) + 2 < r$ for $r \geq 7$. Thus, $k \geq 2$ and (2) of Proposition 5 implies there is a polygon $Q^*$ of $G(\mathcal{C})$ bounding a disc that contains $F \cup F'$, for the essential face chain of length 4 cannot exist, since it would also be an essential face chain of $G$. It is easy to check that $Q^*$ is an essential separating polygon of $G$.

Brunet et al. [3] have shown that when $r \equiv 2 \mod 4$, Lemma 7 can be improved slightly to have the conclusion: such that any two vertices of $P$ or any two vertices of $Q$ are joined by a face chain in $\mathcal{C}$ of length at most $r/2 + 1$ and any two vertices in $P \cup Q$ are joined by a face chain in $\mathcal{C}$ of length at most $r/2 + 2$. Thus, for the special case $r = 6$, the face chain in $G(\mathcal{C})$ containing $F'$ and not $F$ corresponds to an essential face chain in $G$ with length at most $r(G(\mathcal{C})) - 1 + (6/2) + 1$ which implies $r(G(\mathcal{C})) \geq 3$. They also provide a technical argument to handle the case that some face of $K$ has a vertex incident with $F$ and a vertex incident with $F'$. These two
points dispose of the only parts of the proof Theorem 8 which require \( r > 6 \).

Once we have a single essential separating polygon, we can easily obtain many more (as long as \( r \) is sufficiently large). Note that \( r \) must be at least 9 for the following theorem to have any content.

**Theorem 9.** Let \( G \) be an \( r \)-representative embedding in an orientable surface of genus at least 2. There are at least

\[
\left\lfloor \frac{r-1}{8} \right\rfloor
\]

pairwise disjoint, pairwise homotopic, essential separating polygons.

**Sketch of Proof.** Let \( \hat{P} \) be the essential separating polygon of \( G \) found in the proof of Theorem 8. Create a new graph embedding \( H \) in a surface \( \Sigma^* \) from \( G \) by replacing with an open disc the component of \( \Sigma \setminus \hat{P} \) containing the cylinder \( C \). Thus, \( \hat{P} \) bounds a face \( F \) of \( H \). Note \( \Sigma^* \) has genus \( \geq 1 \). We begin by showing that \( r(H) \geq \left\lceil \frac{r}{2} \right\rceil - 1 \).

Let \( \mathcal{F} \) be an essential face chain in \( H \), with length \( m = r(H) \). We may assume \( F \) is in \( \mathcal{F} \) since otherwise \( m \geq r \). Recall, from the proof of Theorem 8, that the face \( F \) of \( H \) containing both \( P \) and \( Q \) is obtained by taking a shortest chain of slices \( S_0, S_1, \ldots, S_k \) in \( G(C) \) joining \( P \) and \( Q \).

There are only a few possibilities for \( \mathcal{F} \) and the one that gives the smallest lower bound for \( m \) occurs if one end of \( \mathcal{F} \setminus F \) is incident with \( Q \) and the other is incident with a face \( F_{i-1} \) in the wheel neighborhood of \( v_i \). Since \( k \) is the length of a shortest chain of slices in \( G(C) \) joining \( P \) to \( Q \) and another such chain is obtained by first using a chain of slices containing the face chain \( \mathcal{F} \setminus F \) and then \( S_{i+1}, S_{i+2}, \ldots, S_k \), we have that \( m-1 \geq i \).

On the other hand, let \( F_i \) be a face common to \( S_{i-1} \) and \( S_i \). The face chain \( \mathcal{F} \setminus F \), together with \( F_1, v_2, F_2, v_i, F_i \), and a face chain in \( G \) joining the end of \( \mathcal{F} \setminus F \) to a vertex of \( F_i \cap P \), contains an essential face chain in \( G \). Thus, \( m-1 + i + \left\lceil \frac{r}{2} \right\rceil + 2 \geq r \), so that \( 2m \geq \left\lceil \frac{r}{2} \right\rceil - 2 \), or \( m \geq \left\lceil \frac{r}{2} \right\rceil - 1 \).

From the Nested Polygons Theorem, there are \( \left\lfloor \frac{r^2}{4} \right\rfloor \) pairwise disjoint polygons bounding discs in \( \Sigma^* \) which contain \( F \). These polygons are essential, separating and homotopic when viewed as subgraphs of \( G \). We conclude the proof by noting that \( \left\lfloor \frac{(r-1)}{8} \right\rfloor = \left\lfloor \frac{(r-1)/4}{2} \right\rfloor \).

Even without the use of Proposition 5 our fundamental construct can be used to obtain the following result.

**Theorem 10.** Let \( G \) be an \( r \)-representative embedding in an orientable surface. Then \( G \) contains \( \left\lceil \frac{r}{4} \right\rceil \) pairwise disjoint, pairwise homotopic essential polygons.
Sketch of Proof. If $0 \leq r \leq 2$, then $\lfloor \frac{r-1}{2} \rfloor = 0$ and Theorem 10 is vacuously true. Also, any embedding with representativity greater than 0 must contain at least one essential polygon, which implies Theorem 10 for $3 \leq r \leq 4$. Thus, we can assume $r \geq 5$ and, by restricting our attention to the essential cleavage unit of $G$, we obtain our fundamental construct.

By the Nested Polygons Theorem, there exist at least $\left\lfloor \frac{r-1}{2} \right\rfloor$ disjoint polygons bounding discs containing $F'$. Placing a vertex in the center of $F'$ joined to the vertices on $\partial F'$ and using the additional condition on $P$ from Lemma 7, we obtain an embedding with representativity at least $\left\lfloor \frac{r}{2} \right\rfloor$ and hence containing at least $\left\lfloor \frac{1}{2} \left(\frac{r}{2}\right) \right\rfloor$ disjoint polygons bounding discs containing $F$. The first $\left\lfloor \frac{1}{2} \left(\frac{r}{2}\right) \right\rfloor$ of the polygons around $F$ must be disjoint from the first $\left\lfloor \frac{1}{2} \left(\frac{r}{2}\right) - 1 \right\rfloor$ of the ones around $F'$ or else we can find an essential face chain of $G$ with length less than $r$. Thus, there are at least $\left\lfloor \frac{1}{2} \left(\frac{r}{2}\right) \right\rfloor + \left\lfloor \frac{1}{2} \left(\frac{r}{2}\right) - 1 \right\rfloor = \lfloor \frac{r-1}{2} \rfloor$ pairwise disjoint polygons which are all homotopic to $P$ in $G$.

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REFERENCES