MAXIMUM DETERMINANTS OF COMPLEMENTARY ACYCLIC MATRICES OF ZEROS AND ONES

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Received 22 January 1985
Revised 12 August 1985

We show that for \( n \geq 5 \) the maximum determinant of an \( n \times n \) matrix of zeros and ones whose zeros form an acyclic pattern is \( \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \) and characterize the case of equality.

1. Introduction

Let \( A = [a_{ij}] \) be an \( n \times n \) \((0, 1)\)-matrix, that is, a matrix all of whose entries are zeros and ones. Associated with \( A \) there are two bipartite graphs \( G_0(A) \) and \( G_1(A) \) whose edges correspond respectively to the 0's and the 1's of \( A \). The graph \( G_1(A) \) has \( 2n \) vertices \( x_1, \ldots, x_n \) (row vertices) and \( y_1, \ldots, y_n \) (column vertices), where there is an edge joining \( x_i \) and \( y_j \) if and only if \( a_{ij} = 1 \) \((1 \leq i, j \leq n)\). Let \( J_n \) be the \( n \times n \) matrix of all 1's. Then the graph \( G_0(A) \) is defined by \( G_0(A) = G_1(J_n - A) \).

We say that the matrix \( A \) is acyclic if the graph \( G_1(A) \) has no cycles, that is, is a tree or a forest. The matrix \( A \) is called complementary acyclic if the graph \( G_0(A) \) has no cycles. Thus \( A \) is complementary acyclic if and only if \( J_n - A \) is acyclic.

Recall that a perfect matching of a graph is a set of vertex-disjoint edges which meet all vertices. A perfect matching of \( G_1(A) \) corresponds to a set of \( n \) 1's of \( A \) no two of which lie in the same row or same column. Since a forest has at most one perfect matching, it follows that the determinant of an acyclic \((0, 1)\)-matrix is 0, 1, or \(-1\).

In contrast the determinant of a complementary acyclic \((0, 1)\)-matrix can be large. In this paper we determine the largest absolute value \( f_n \) of the determinant of an \( n \times n \) complementary acyclic \((0, 1)\)-matrix and characterize the case of equality. Let \( A \) be an \( n \times n \) complementary acyclic \((0, 1)\)-matrix and let \( B \) be obtained from \( A \) by interchanging two rows. Then \( B \) is complementary acyclic and \( \det B = -\det A \). Hence \( f_n \) is the maximum determinant of a complementary acyclic \((0, 1)\)-matrix.

Before continuing we discuss briefly some previous work\(^1\). Let \( \beta_n \) be the maximum determinant of an \( n \times n \) \((0, 1)\)-matrix, and let \( \alpha_n \) be the maximum

\(^1\) We are indebted to H.J. Ryser for some of the references in this paragraph.
determinant of an \( n \times n \) matrix all of whose entries are 1's and -1's. The problem of determining \( \beta_n \) goes back to Hadamard \([4]\), and it was shown by Williamson \([9]\) that \( \alpha_n = 2^{n-1} \beta_{n-1} \). When an \( n \times n \) Hadamard matrix \([1, 4, 8]\) exists, \( \alpha_n = n^{\frac{n}{2}} \). Bounds in general for \( \alpha_n \) have been obtained by Ehlich \([2, 3]\). In addition, Ryser \([7]\), \([8, \text{p. 125}]\) has obtained an upper bound for the determinant of a (0, 1)-matrix with \( k \) 1's in each row and column and has shown that equality is attained if and only if a combinatorial design with specified parameters exists. Piehler \([6]\) has derived asymptotic results for determinants of (0, 1)-matrices. Additional references are Brenner and Cummings \([1]\) and Payne \([5]\). Because of the severe restrictions placed on a complementary acyclic (0, 1)-matrix, \( f_n \) is much smaller than \( \beta_n \). Hence the known upper bounds for \( \beta_n \) are of little use in the situation treated in this paper.

We say that the matrix \( X \) is permutation equivalent to the matrix \( Y \) if there are permutation matrices \( P \) and \( Q \) such that \( X = PYQ \). An \( n \times n \) (0, 1)-matrix \( A \) is called complementary triangular if \( A \) has only 1's above its main diagonal, that is, \( J_n - A \) is a triangular matrix.

**Lemma 1.1.** Let \( A \) be an \( n \times n \) complementary acyclic (0, 1)-matrix. Then \( A \) is permutation equivalent to a complementary triangular matrix.

**Proof.** Since \( J_n - A \) is an acyclic matrix, \( J_n - A \) is permutation equivalent to a triangular matrix. Hence \( A \) is permutation equivalent to a complementary triangular matrix. \( \square \)

We conclude this section with two simple necessary conditions for an \( n \times n \) (0, 1)-matrix \( A \) to be nonsingular. Recall that the term rank \([8, \text{p. 55}]\) of \( A \) is the maximum number \( \rho(A) \) of 1's of \( A \) with no two from the same row or column. We define the complementary term rank \( \rho_0(A) \) by \( \rho_0(A) = \rho(J_n - A) \). Thus \( \rho_0(A) \) is the maximum number of 0's of \( A \) with no two from the same row or column. By König's theorem \([8, \text{p. 55}]\), \( \rho_0(A) \) is the minimum number of rows and columns which contain all the 0's of \( A \).

**Lemma 1.2.** Let \( A \) be an \( n \times n \) nonsingular (0, 1)-matrix. Then \( \rho_0(A) \geq n - 1 \).

**Proof.** Consider a set of \( e \) rows and \( f \) columns of \( A \) which contain all the 0's. Without loss of generality we may assume that

\[
A = \begin{bmatrix}
J \\
f & J_n - f
\end{bmatrix}^e_{n-e}
\]

where \( J \) is a matrix of all 1's. Since \( A \) is nonsingular, the last \( n - f \) columns of \( A \) are linearly independent. But evidently the submatrix of \( A \) determined by the last \( n - f \) columns contains at most \( e + 1 \) linearly independent rows. Hence \( e + 1 \geq
A pendant vertex of a graph is a vertex which meets exactly one edge.

**Lemma 1.3.** Let $A$ be an $n \times n$ nonsingular $(0, 1)$-matrix. Then each vertex of $G_0(A)$ is adjacent to at most one pendant vertex.

**Proof.** If there were two pendant vertices adjacent to the same vertex, then $A$ would have two identical rows or two identical columns. 

Using Lemma 1.2 we see that the problem of determining $f_n$ can be divided into two cases: $\rho_0(A) = n - 1$, treated in the next section, and $\rho_0(A) = n$, treated in the final section.

2. The case $\rho_0(A) = n - 1$

We begin by considering the following two special cases.

**Lemma 2.1.** Let $A$ be an $n \times n$ complementary acyclic $(0, 1)$-matrix having both a row and column of all ones. If $\rho_0(A) = n - 1$, then $\det A = \pm 1$.

**Proof.** Suppose $\rho_0(A) = n - 1$. Without loss of generality we may assume that the first row and first column of $A$ consist entirely of $1$'s. Let $A(1; 1)$ be the submatrix obtained from $A$ by striking out row 1 and column 1. Since $A(1; 1)$ is complementary acyclic, by Lemma 1.1 $A(1; 1)$ is permutation equivalent to a complementary triangular matrix $T$. Since $\rho_0(A(1; 1)) = n - 1$, $T$ has only $0$'s on its main diagonal. Hence $A$ is permutation equivalent to a matrix of the form

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 0 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 0 \\
\end{bmatrix}
\]

which has only $1$'s above its main diagonal. Subtracting column 1 from each of columns $2, \ldots, n$ in (2.1) we obtain a triangular matrix with determinant $(-1)^{n-1}$. Hence $\det A = \pm 1$. 

In Lemma 2.1 we have evaluated $\det A$ in the case that $G_0(A)$ has both an isolated row vertex and an isolated column vertex. We now evaluate $\det A$ when there is exactly one isolated row vertex but no isolated column vertex (if there were two isolated vertices, $\det A = 0$). We may assume that $x_i$ is the isolated row vertex so that row 1 of $A$ contains only $1$'s. Suppose $x_i$ was a pendant row vertex of $G_0(A)$. Then subtracting row 1 of $A$ from row $i$ we see that $\det A = \pm \det A'$, where $A'$ is an $(n - 1) \times (n - 1)$ complementary acyclic $(0, 1)$-matrix such that
$G_0(A')$ has an isolated row vertex but no isolated column vertex. We therefore assume that $G_0(A)$ has no pendant row vertices. Deleting row vertex $x_1$ from $G_0(A)$ we obtain an acyclic graph $H$ with $2n-1$ vertices and at least $2(n-1)$ edges (at least two edges meeting each of the row vertices $x_2, \ldots, x_n$). In an acyclic graph the number $v$ of vertices and number $e$ of edges satisfy $v \geq e + 1$ with equality if and only if the graph is a tree. It follows that each of the row vertices $x_2, \ldots, x_n$ meets exactly two edges and that $H$ is a tree. Hence apart from the isolated row vertex $x_1$, $G_0(A)$ is a tree.

**Lemma 2.2.** Let $A$ be an $n \times n$ complementary acyclic $(0, 1)$-matrix such that $A$ has a row of all ones and $\rho_0(A) = n - 1$. Take $A$ in the form

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & * & & & \\
0 & & & & \\
& & & * & \\
& & & & 0
\end{bmatrix},
$$

(2.2)

and assume $G_0(A)$ has no pendant row vertices. Let the distance in $G_0(A)$ between column vertex $y_1$ and column vertex $y_i$ be $2d_i$ ($i = 1, \ldots, n$). Then

$$
det A = (-1)^{n-1} \sum_{i=1}^{n} (-1)^{d_i}.
$$

**Proof.** It follows from the previous discussion that apart from the isolated row vertex $x_1$, $G_0(A)$ is a tree. Hence for each $i = 1, \ldots, n$ there is a unique path $y_i$ from vertex $y_1$ to vertex $y_i$, where this path has even length $2d_i$. Note that $d_1 = 0$. Let $B$ be the matrix obtained from $A$ by subtracting row 1 from each of rows 2, $\ldots$, $n$ and then factoring out a $-1$ from each of rows 2, $\ldots$, $n$. Then $B$ is an $n \times n (0, 1)$-matrix with $det A = (-1)^{n-1} det B$, and the graph $G_1(B)$ is obtained from $G_0(A)$ by adding edges joining $x_1$ to each of $y_1, \ldots, y_n$. Note that $B$ has all 1's on its main diagonal and in its first row, and exactly one off-diagonal 1 in each of rows 2, $\ldots$, $n$. Let $B(1; i)$ be the matrix obtained from $B$ by deleting row 1 and column $i$. The graph $H_i = G_1(B(1; i))$ is obtained from $G_0(A)$ by deleting vertices $x_1$ and $y_i$. Since $G_0(A)$ is acyclic, $H_i$ is also acyclic and hence has at most one perfect matching. It follows that each 1 in row 1 of $B$ appears in at most one

![Fig. 1. The cycle $y_i$.](image-url)
nonzero term in the determinant expansion of \( B \). We now show that the 1 in position \((1, i)\) of \( B \) appears in exactly one such nonzero term (equivalently the edge \( x_1 y_i \) is in exactly one perfect matching of \( G_i(B) \)) and it has value \((-1)^{d_i}\). This is clear for \( i = 1 \), since \( F = \{x_1 y_1, \ldots, x_n y_n\} \) is a perfect matching of \( G_1(B) \). Now let \( i > 1 \). By adjoining the vertex \( x_1 \) to the path \( \gamma_i \) from \( y_1 \) to \( y_i \), we obtain a cycle \( \gamma'_i \) of length \( 2d_i + 2 \) in \( G_i(B) \) (see Fig. 1). Since each of the row vertices \( x_2, \ldots, x_n \) has degree 2, every other edge of \( \gamma'_i \) beginning with \( x_1 y_1 \) belongs to \( F \). The edges of \( \gamma'_i \) not belonging to \( F \) along with the edges of \( F \) not belonging to \( \gamma'_i \) form a perfect matching of \( G_i(B) \) containing the edge \( x_1 y_i \). This perfect matching corresponds to a permutation of \( 1, 2, \ldots, n \) whose cycle lengths are \( d_i + 1, 1, \ldots, 1 \). The corresponding term in the determinant expansion of \( B \) is then \((-1)^{d_i}\). Hence

\[
\det B = \sum_{i=1}^{n} (-1)^{d_i},
\]

and the lemma follows. \( \square \)

**Corollary 2.3.** Under the assumptions in Lemma 2.2,

\[
\det A = (-1)^{n-1}(p - q),
\]

where \( p = |\{i: d_i \text{ is even}\}| \) and \( q = |\{i: d_i \text{ is odd}\}| \).

**Theorem 2.4.** Let \( A \) be an \( n \times n \) complementary acyclic \((0,1)\)-matrix such that \( A \) has a row or column of all ones. Then for \( n \geq 3 \),

\[
|\det A| \leq n - 2. \tag{2.3}
\]

For \( n \geq 4 \), equality holds if and only if \( A \) or \( A^t \) is permutation equivalent to

\[
L_n = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
0 & \ddots & \\
\vdots & & J_{n-1} - I_{n-1} \\
0 & & & 0
\end{bmatrix}. \tag{2.4}
\]

**Proof.** We may assume \( A \) has a row of all 1's, for otherwise we may replace \( A \) by \( A^t \). If \( \rho_0(A) < n - 1 \), then we conclude from Lemma 1.2 that \( \det A = 0 \), and the inequality (2.3) is strict. Hence we may assume that \( \rho_0(A) = n - 1 \). First suppose that \( G_0(A) \) has no pendant row vertices. Then \( A \) is permutation equivalent to a matrix having the form (2.2). Let \( d_i (1 \leq i \leq n) \) and \( p, q \) be defined as in Lemma 2.2 and Corollary 2.3. Since \( d_1 = 0 \), \( p \geq 1 \). Since \( d_i \geq 1 \) \((i = 2, \ldots, n)\), it follows that there exists a \( j \) with \( d_j = 1 \) and hence that \( q \geq 1 \). Since \( p + q = n \), (2.3) follows from Corollary 2.3 in this case. Now suppose that \( G_0(A) \) has a pendant row vertex. When \( n = 3 \), one readily checks that \( |\det A| \leq 1 \). Let \( n > 3 \). Using the reduction preceding Lemma 2.2, we see that \( \det A = \pm \det A' \), where \( A' \) is an
\((n - 1) \times (n - 1)\) matrix satisfying the hypotheses of the theorem. It now follows by induction that

\[
\left| \det A \right| = \left| \det A' \right| \leq n - 3.
\]

Hence (2.3) holds, and for \(n \geq 4\) equality implies that \(p_0(A) = n - 1\) and \(G_0(A)\) has no pendant row vertex.

Suppose \(\left| \det A \right| = n - 2\). Then the basic assumptions in Lemma 2.2 hold. After row and column permutations, we may take \(A\) in the form (2.2). Since we have shown above that \(p \geq 1\) and \(q \geq 1\), it follows from Corollary 2.3 that \(p = 1\) and \(q = n - 1\), or \(p = n - 1\) and \(q = 1\). Suppose there were a \(k\) with \(d_k \geq 3\). Then there would exist \(r, s,\) and \(t\) with \(d_r = 3\), \(d_s = 2\), and \(d_t = 1\). Since \(d_1 = 0\), this would imply that \(p \geq 2\) and \(q \geq 2\). It follows that \(d_i \leq 2\) (\(i = 1, \ldots, n\)). First suppose that \(d_i = 2\) for some \(i\). Then \(p \geq 2\) and hence \(p = n - 1\) and \(q = 1\). Without loss of generality, let \(d_2 = 1\) so that \(d_1, d_2, d_3, \ldots, d_n\) is \(0, 1, 2, \ldots, 2\). Then in \(G_0(A)\) there can be no edges from \(y_i\) to any of \(x_3, \ldots, x_n\), and it follows that \(G_0(A)\) is the graph shown in Fig. 2. After interchanging columns 1 and 2 of \(A\), we arrive at the matrix (2.4). Now suppose \(d_i \leq 1\) (\(i = 1, \ldots, n\)) so that \(d_1, d_2, \ldots, d_n\) is \(0, 1, \ldots, 1\). In this case \(G_0(A)\) is the graph in Fig. 2 with labels \(y_1\) and \(y_2\) interchanged, and \(A\) is the matrix (2.4). Since the matrix \(L_n\) in (2.4) has determinant \((-1)^{n-2}(n - 2)\), the theorem follows. \(\square\)

Referring to Theorem 2.4, we note that when \(n = 2\), \(\det A\) can be 0 or \(\pm 1\). When \(n = 3\), \(\left| \det A \right| = 1\) if and only if \(A\) or \(A'\) is permutation equivalent to one of the matrices

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}.
\]

We now obtain the main result of this section which determines the maximum determinant in general when \(p_0(A) = n - 1\).

**Theorem 2.5.** Let \(A\) be an \(n \times n\) complementary acyclic \((0, 1)\)-matrix with

![Fig. 2. The graph \(G_0(A)\) when \(p = n - 1\) and \(q = 1\).](image-url)
\( \rho_0(A) = n - 1. \) Then

\[
|\det A| \leq \begin{cases} 
  n - 2 & \text{if } 3 \leq n \leq 8 \\
  \left\lceil \frac{n - 3}{2} \rightceil \left\lceil \frac{n - 3}{2} \right\rceil & \text{if } n \geq 8.
\end{cases}
\]  

(2.5)

For \( n \geq 4 \) equality holds in (2.5) if and only if \( A \) or \( A^t \) is permutation equivalent to \( L_n \) as defined in (2.4) \((4 \leq n \leq 8)\) or

\[
\begin{bmatrix}
J_{\lfloor (n-1)/2 \rfloor} & 1 & \cdots & J \\
0 & \cdots & 0 & 1 & \cdots & 1 \\
Z & 0 & & & & \vdots \\
& & & & & 0 \\
& & & & & \cdots \\
& & & & & 0 \\
\end{bmatrix} \quad (n \geq 8)
\]  

(2.6)

where \( Z \) has at most one 0.

**Proof.** We may assume \( A \) is nonsingular. Pick a vertex \( r \) as a root for some connected component of the graph \( G_0(A) \) and choose a pendant vertex \( z \) furthest from \( r \). We may assume \( z \) is a row vertex; if not, we can replace \( A \) by \( A^t \). By permuting rows and columns, we may assume that \( z = x_1 \) and that \( x_1 \) is joined by an edge to \( y_1 \). Since \( A \) is nonsingular and \( x_1 \) is a pendant vertex furthest from \( r \), it follows from Lemma 1.3 that \( y_1 \) is joined to at most two vertices. Hence row 1 of \( A \) has exactly one 0, namely in the (1, 1)-position, while column 1 has at most two 0's. The matrix \( A(1; 1) \) obtained from \( A \) by striking out row 1 and column 1 is a complementary acyclic matrix and hence by Lemma 1.1 is permutation equivalent to a complementary triangular matrix. Without loss of generality we may assume \( A(1; 1) \) is complementary triangular. Since \( \rho_0(A) = n - 1 \), there is an integer \( k \) with \( 2 \leq k \leq n \) such that the \( k \)th entry on the main diagonal of \( A \) is 1. Subtracting row 1 from each of rows 2, \ldots, \( n \) and then factoring out a \(-1\) from each of columns 2, \ldots, \( n \) and row 1, we see that \( \det A = \pm \det B \), where \( B \) has the form

\[
\begin{bmatrix}
0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & & & & & & \\
u & B_1 & & 0 \\
0 & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]  

(2.7)

with at most two 0's in column 1 and with \( B_1 \) and \( B_2 \) square triangular matrices. In
particular, $B$ has a $(k - 1) \times (n - k + 1)$ zero submatrix. Let

$$B'_1 = \begin{bmatrix} u & B_1 \\ a & v \end{bmatrix},$$

so that $B'_1$ is a $(k - 1) \times (k - 1)$ matrix with at most one 0 in its first column. Let

$$B'_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ w & B_2 \end{bmatrix},$$

a $(n - k + 1) \times (n - k + 1)$ matrix. It follows that

$$\det A = \pm \det B = \pm (\det B'_1)(\det B'_2). \quad (2.8)$$

Let $B''_2$ be the matrix obtained from $B'_2$ by subtracting row 1 from each of its other rows and then factoring out a $-1$ from these rows. Then $B''_2$ is a complementary acyclic $(0, 1)$-matrix with $\det B''_2 = \pm \det B'_2$. Using Theorem 2.4 we see that

$$\{1 \text{ if } k = n - 1 \text{ or } n \}
\leq
\begin{cases} 
1 & \text{if } 2 \leq k \leq n - 2. 
\end{cases} \quad (2.9)$$

We now estimate $|\det B'_1|$. If $k = 2$ or 3, $|\det B'_1| \leq 1$. Let $4 \leq k \leq n$. First suppose that the first column of $B'_1$ contains no 0's. Let $B''_1$ be the matrix obtained from $B'_1$ by subtracting column 1 from each of the other columns and factoring out a $-1$ from those columns. Then $B''_1$ is a $(k - 1) \times (k - 1)$ complementary acyclic $(0, 1)$-matrix and it follows from Theorem 2.4 that $|\det B''_1| \leq k - 3$. Now suppose that the first column of $B'_1$ contains a 0. Let $C$ be the matrix obtained from $B'_1$ by replacing this 0 with a 1. By defining $B''_1$ from $C$ as we defined $B''_1$ from $B'_1$ above, we conclude that $|\det C| \leq k - 3$. The submatrix of $B'_1$ obtained by striking out column 1 and the row in which its 0 lies is an acyclic $(0, 1)$-matrix whose determinant therefore has absolute value at most 1. It follows that

$$|\det B'_1| \leq \begin{cases} 
(k - 3) + 1 = k - 2 & \text{if } 4 \leq k \leq n \\
1 & \text{if } k = 2 \text{ or } 3, 
\end{cases} \quad (2.10)$$

where the bound $k - 2$ is achieved only when column 1 of $B'_1$ contains exactly one 0. Using (2.8), (2.9), and (2.10), we obtain the following inequalities.

For $4 \leq k \leq n - 2$,

$$|\det A| \leq (k - 2)(n - k - 1) \leq \begin{cases} 
\left\lfloor \frac{n - 3}{2} \right\rfloor \left\lfloor \frac{n - 3}{2} \right\rfloor & \text{if } n \geq 8 \\
\frac{n - 3}{2} & \text{if } n = 6 \text{ or } 7. 
\end{cases} \quad (2.11)$$

For $k = 2$ or $n - 1$ and $n \geq 4$,

$$|\det A| \leq n - 3 < \begin{cases} 
\left\lfloor \frac{n - 3}{2} \right\rfloor \left\lfloor \frac{n - 3}{2} \right\rfloor & \text{if } n \geq 8 \\
\frac{n - 2}{2} & \text{if } n \leq 8. 
\end{cases} \quad (2.12)$$
For $k = 3$ and $n \geq 5$,
\[
|\det A| \leq n - 4 < \begin{cases}
\left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor & \text{if } n \geq 8 \\
(n-2)^2 & \text{if } n \leq 8.
\end{cases}
\] (2.13)

For $k = n$ and $n \geq 3$,
\[
|\det A| \leq n - 2,
\] (2.14)
so that for $n \geq 8$,
\[
|\det A| \leq \left\lfloor \frac{n-3}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor,
\]
with equality only if $n = 8$.

For $k = 2$ and $n = 3$,
\[
|\det A| \leq 1 = n - 2.
\] (2.15)

The inequality (2.5) now follows from (2.11)–(2.15).

Now suppose equality holds in (2.5). First assume $n > 8$. Then equality holds in (2.9), (2.10), and (2.11) and
\[
\{k-2, n-k-1\} = \left\lfloor \frac{n-3}{2} \right\rfloor , \left\lfloor \frac{n-3}{2} \right\rfloor .
\] (2.16)

The first column of $B'_1$ contains exactly one 0,
\]
(by Theorem 2.4) $B'^*_2$ is permutation equivalent to $L_{n-k+1}$,
\] (2.18)
(by Theorem 2.4) $B'^*_2$ is permutation equivalent to $L'_{k-1}$ and hence $B'_1$ is permutation equivalent to a matrix $X$ of the form
\[
X = \begin{bmatrix}
1 & \cdots & 1 \\
I_{k-2} & 1
\end{bmatrix},
\] (2.19)
whose first column has exactly one 0. The cofactor $c$ of this 0 satisfies $\text{sign } c = -\text{sign } \det Y$, where $Y$ is obtained from $X$ by replacing the 0 in column 1 with a 1, otherwise equality would not hold in (2.10).

Evaluating the determinant of $Y$ by the cofactors of the elements of its first column, we obtain
\[
\det Y = (k-2)(-1)^{k-1} + (-1)^k.
\]
Since $n > 8$, it follows from (2.16) that $k \geq 4$ and hence $\text{sign } \det Y = (-1)^{k-1}$. Since the only cofactor whose sign is opposite the sign of $\det Y$ is the cofactor of the last entry in column 1, we conclude from (2.19) that the 0 in column 1 of $X$ is in its last row. We now conclude $A$ or $A^t$ is permutation equivalent to a matrix of
the form

\[
\begin{bmatrix}
0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
1 & & & J & & & \\
1 & & & & & & & \\
1 & & & & & & & \\
& & & & & & & \\
& & & & & & & \\
0 & & & & & & & \\
\end{bmatrix}
\]

(2.20)

where the columns of \( A_1 \) can be permuted to give

\[
\begin{bmatrix}
1 & & & J_{\lfloor(n-1)/2\rfloor} - I_{\lfloor(n-1)/2\rfloor} & & \\
\vdots & & & & & \\
1 & & & & & \\
0 & & & & & \\
\end{bmatrix}
\]

(2.21)

The first column of (2.21) must correspond to the first column of \( A_1 \), for otherwise \( A \) has two identical rows, contradicting the nonsingularity of \( A \). Since \( G_0(A) \) has no cycles, \( Z \) contains at most one 0. Hence \( A \) is permutation equivalent to a matrix of the form (2.6).

Now assume \( 4 \leq n \leq 8 \) and equality holds in (2.5). Then it follows from (2.11) to (2.14) that \( k = n \), or \( n = 8 \) and \( k = 4 \) or 5. First suppose \( k = n \). Then \( A \) has a column of all 1's and it follows from Theorem 2.4 that \( A \) or \( A^t \) is permutation equivalent to \( L_n \). When \( n = 8 \) and \( k = 4 \) or 5, the argument used for \( n > 8 \) shows that \( A \) or \( A^t \) is permutation equivalent to a matrix of the form (2.6). Since \( |\det L_n| = n - 2 \) and the absolute value of the determinant of a matrix of the form (2.6) is \( [(n - 3)/2] [(n - 3)/2] \), the proof of the theorem is complete.

3. The case \( \rho_0(A) = n \)

We consider in this section \( n \times n \) complementary acyclic nonsingular \((0, 1)\)-matrices \( A \) with \( \rho_0(A) = n \). Following the argument used in the beginning of the proof of Theorem 2.5 we may assume that \( A \) has the form

\[
\begin{bmatrix}
0 & 1 & \cdots & 1 \\
0 & \cdots & & \\
v & & & \\
& & & 0 \\
\end{bmatrix}
\]

(3.1)

where \( v \) has at most one 0. Subtracting row 1 from each of rows 2, \ldots, \( n \) and factoring out a \(-1\) from each of columns 2, \ldots, \( n \) and row 1, we obtain a matrix
Maximum determinants of complementary acyclic matrices

B with \( \det A = (-1)^n \det B \), where

\[
B = \begin{bmatrix}
0 & 1 & \cdots & 1 \\
v & C
\end{bmatrix}
\]  

Here \( C \) is an \((n-1) \times (n-1)\) acyclic triangular \((0,1)\)-matrix having all 1's on its main diagonal. We may write \( C = I_{n-1} + X \), where \( X \) is an \((n-1) \times (n-1)\) \((0,1)\)-matrix having 0's on and above its main diagonal.

The usual digraph \( D(X) \) corresponding to \( X \) can be obtained from \( G_1(C) \) as follows: Direct each edge joining an \( x_i \) and a \( y_j \) from \( x_i \) to \( y_j \) (\( 2 \leq i \neq j \leq n \)) and identify \( x_i \) and \( y_i \) to a new vertex \( z_i \) (\( 2 \leq i \leq n \)). Thus the vertices of \( D(X) \) are \( z_2, \ldots, z_n \) with an arc from \( z_i \) to \( z_j \) if and only if \( i \neq j \) and there is an edge of \( G_1(C) \) joining \( x_i \) and \( y_j \). It follows from the construction of \( D(X) \) and the fact that \( G_1(C) \) is acyclic that the graph obtained from \( D(X) \) by removing the directions of its arcs is acyclic. Thus \( D(X) \) is obtained by directing the edges of an acyclic graph.

We begin by proving some results for digraphs of this sort.

For a digraph \( D \) with \( m \) vertices, we let \( p_i = p_i(D) \) be the number of directed paths of length \( i \) in \( D \) (\( i = 0, 1, \ldots, m - 1 \)) and \( p(D) = \sum_{i=0}^{m-1} (-1)^i p_i \). For a vertex \( x \) we let \( p^0(D;x) \) (respectively, \( p^1(D;x) \)) be the contribution to \( p(D) \) of the paths of \( D \) originating (respectively, terminating) at \( x \). We say that a vertex is an inpendant vertex if its indegree is 1 and its outdegree is 0, and an outpendant vertex if its outdegree is 1 and its indegree is 0. If \( x \) is either an inpendant vertex or outpendant vertex, the unique vertex \( y \) for which there is an arc between \( x \) and \( y \) is called the neighbor of \( x \).

**Lemma 3.1.** Let \( D \) be a digraph which is obtained by directing the edges of an acyclic graph with \( m \geq 2 \) vertices none of which is isolated. Then

\[
p(D) \leq \left[ \frac{m-1}{2} \right] \left[ \frac{m-1}{2} \right] + 1
\]

with equality if and only if \( D \) has the form shown in Fig. 3.

**Proof.** We first observe that if \( x \) and \( y \) are vertices of \( D \), then there is at most one directed path from \( x \) to \( y \). Suppose there are two inpendant vertices \( u \) and \( v \) having distinct neighbors \( r \) and \( s \), We may assume \( p^1(D;u) \geq p^1(D;v) \). Let

\[
\{a,b\} = \left\{ \left\lfloor \frac{m-1}{2} \right\rfloor, \left\lceil \frac{m-1}{2} \right\rceil \right\}
\]

Fig. 3.
Let $v_1 = v, v_2, \ldots, v_k$ be all the inpendant vertices whose neighbor is $s$. Let $D'$ be the digraph obtained from $D$ by removing the arcs $(s, v_i)$ and adding the arcs $(r, v_i)$ for $i = 1, \ldots, k$. For all vertices $x$ different from $v_1, \ldots, v_k$, $p'(D'; x) = p'(D'; x)$. Since all directed paths of $D$ of positive length which terminate at $u$, respectively some $v_i$, pass through $r$, respectively $s$, $p'(D'; v_i) \geq p'(D'; v_i)$ for $i = 1, \ldots, k$. Hence $p(D') \geq p(D)$. Similarly, suppose $c$ and $d$ are two outpendant vertices having distinct neighbors $w$ and $z$, respectively. Assume $p(D; c) > p(D; d)$. Let $d_1 = d, d_2, \ldots, d_l$ be all the outpendant vertices whose neighbor is $z$. Let $D''$ be the digraph obtained from $D$ by removing the arcs $(d_i, z)$ and adding the arcs $(d_i, w)$ for $i = 1, \ldots, k$. Then $p(D'') \geq p(D)$.

A finite number of applications of the above two operations leads to a digraph $D^*$ with $p(D^*) \geq p(D)$ where all the inpendant vertices have the same neighbor and all the outpendant vertices have the same neighbor. Since $D$ has no isolated vertices, it follows that $D^*$ is of one of the types shown in Fig. 4.

We first consider the cases when $D^*$ is one of $D_2, D_3, D_4$. For $D_2$ we have $m \geq 4$ and hence

$$p(D) \leq p(D_2) = 2 < \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 1.$$

![Fig. 4. (Arcs of $D_1$ which have no direction indicated may have either direction.)](image-url)
For $D_3$ and $D_4$ we have

$$p(D) \leq p(D_i) = 1 + \left[ \frac{m-1}{2} \right] \left[ \frac{m-1}{2} \right] + 1 \quad (i = 3, 4)$$

with equality if and only if $m = 2$. Hence when the digraph $D^*$ resulting from $D$ is one of $D_2$, $D_3$, or $D_4$, (3.3) holds with strict inequality unless $m = 2$. When $m = 2$, $D$ is the digraph of Fig. 3 with $\{a, b\} = \{0, 1\}$.

For the remainder of the proof we suppose $D^*$ has the form $D_1$. Then $m = a + b + k$, where $k \geq 1$. We first consider the case where the arcs on the undirected path between $e$ and $f$ are all directed from $e$ towards $f$. An easy calculation shows that

\[
(k \text{ odd}) \quad p(D_1) = ab + \frac{k+1}{2} = (a - \frac{1}{2})(b - \frac{1}{2}) + \frac{2m+1}{4}, \quad \text{where } a + b = m - k. \quad (3.4)
\]

Hence the maximum value of $p(D_1)$ occurs when $k = 1$ ($a + b = m - 1$), and $a$ and $b$ are as nearly equal as possible. Therefore

$$p(D_1) \leq \left[ \frac{m-1}{2} \right] \left[ \frac{m-1}{2} \right] + 1,$$

with equality if and only if $\{a, b\} = \{(m-1)/2, (m-1)/2\}$, that is, $D_1$ is the digraph of Fig. 3. In a similar way we obtain

\[
(k \text{ even}) \quad p(D_1) = -ab + a + b + \frac{k}{2} = -(a - \frac{1}{2})(b - \frac{1}{2}) + \frac{2m+1}{4} < \left[ \frac{m-1}{2} \right] \left[ \frac{m-1}{2} \right] + 1. \quad (3.5)
\]

Hence in this case, (3.3) holds with equality if and only if the digraph $D^*$ resulting from $D$ is given by Fig. 3. To complete the proof in this case, we now

Fig. 5.
show that when $D^*$ is given by Fig. 3 and $D$ is different from $D^*$, then equality does not hold in (3.3). In the sequence of digraphs obtained in transforming $D$ to $D^*$, let $D'$ be the digraph immediately preceding $D^*$. Then

$$p(D) \leq p(D') \leq p(D^*)$$

and $D'$ is obtained from the digraph in Fig. 5 by adjoining a nonempty set of inpendant vertices or of outpendant vertices whose neighbor is one of $u_1, \ldots, u_a$, $v_1, \ldots, v_b$. But one easily verifies that $p(D') < p(D^*)$ and (3.3) is strict.

We now finish the proof by using induction on the number $m$ of vertices. We need only consider the case where the digraph $D^*$ resulting from $D$ has the form of the digraph $D_1$ in Fig. 4 and not all the arcs on the undirected path between $e$ and $f$ are directed from $e$ towards $f$. Let $z$ be the first vertex of $D_1$ at or below $e$ which has outdegree equal to 0. Let $D_1^{(1)}$ (respectively, $D_1^{(2)}$) be the digraph obtained from $D_1$ by removing all vertices below (respectively, above) $z$. Let $D_1^{(1)}$ have $h \geq 2$ vertices so that $D_1^{(2)}$ has $m - h + 1 \geq 2$ vertices. Then $p(D_1) = p(D_1^{(1)}) + p(D_1^{(2)}) - 1$, and using the inductive hypothesis we obtain

$$P(D) = P(D_1^{(1)}) + P(D_1^{(2)}) - 1, \quad \text{and using the inductive hypothesis we obtain}$$

$$p(D) \leq p(D_1) \leq \left\lfloor \frac{h - 1}{2} \right\rfloor \left\lfloor \frac{h - 1}{2} \right\rfloor + \left\lfloor \frac{m - h}{2} \right\rfloor \left\lfloor \frac{m - h}{2} \right\rfloor + 1$$

$$< \left\lfloor \frac{m - 1}{2} \right\rfloor \left\lfloor \frac{m - 1}{2} \right\rfloor + 1.$$

Hence (3.3) holds with strict inequality, and the lemma follows by induction.

Lemma 3.2. Let $D$ be a digraph which is obtained by directing the edges of an acyclic graph with $m \geq 2$ vertices none of which is isolated. Then

$$p(D) \geq \begin{cases} 1 & \text{if } m = 2, 3, 4, 5 \\ - \left\lfloor \frac{m - 4}{2} \right\rfloor \left\lfloor \frac{m - 4}{2} \right\rfloor + 2 & \text{if } m \geq 6. \end{cases} \quad (3.6)$$

Proof. As in the proof of Lemma 3.1, except that now we remove and add arcs to decrease $p(D)$, we obtain that $p(D) \geq p(D^*)$, where $D^*$ is one of the digraphs shown in Fig. 4. Since $p(D_2) = 2$ and $p(D_3) = p(D_4) = 1$, (3.6) holds when $D^*$ is one of $D_2$, $D_3$, or $D_4$. Now suppose $D^*$ has the form $D_1$. When $m < 6$, (3.6) is readily verified. Thus we assume $m \geq 6$. We first consider the case where the arcs on the undirected path between $e$ and $f$ are all directed from $e$ towards $f$. When $k$ is odd, (3.4) implies that (3.6) holds. Let $k$ be even. Then by (3.5),

$$p(D_1) = -(a - \frac{1}{2})(b - \frac{1}{2}) + \frac{2m + 1}{2},$$

which is minimum when $k = 2$, and $a$ and $b$ are as nearly equal as possible. It
follows that
\[ p(D_1) \geq -\left\lfloor \frac{m-4}{2} \right\rfloor \left\lceil \frac{m-4}{2} \right\rceil + 2. \]

We now complete the proof by using induction on \( m \). We need only treat the case where \( D^* \) has the form \( D_1 \) where not all the arcs on the undirected path, between \( e \) and \( f \) are directed from \( e \) towards \( f \). Let \( D_1^{(1)}, D_1^{(2)}, \) and \( h \) be defined as in the corresponding part of the proof of Lemma 3.1. We have the following four cases:

**Case 1.** \( h \leq 5, m-h+1 \leq 5 \):

\[ p(D_1) = p(D_1^{(1)}) + p(D_1^{(2)}) - 1 \]
\[ \geq 1 + 1 - 1 \geq -\left\lfloor \frac{m-4}{2} \right\rfloor \left\lceil \frac{m-4}{2} \right\rceil + 2. \]

**Case 2.** \( h \leq 5, m-h+1 \geq 6 \):

\[ p(D_1) \geq 1 + \left( -\left\lfloor \frac{m-h-3}{2} \right\rfloor \left\lceil \frac{m-h-3}{2} \right\rceil + 2 \right) - 1 \]
\[ \geq -\left\lfloor \frac{m-4}{2} \right\rfloor \left\lceil \frac{m-4}{2} \right\rceil + 2. \]

**Case 3.** \( h \geq 6, m-h+1 \leq 5 \):

\[ p(D_1) \geq \left( -\left\lfloor \frac{h-4}{2} \right\rfloor \left\lceil \frac{h-4}{2} \right\rceil + 2 \right) + 1 - 1 \]
\[ \geq -\left\lfloor \frac{m-4}{2} \right\rfloor \left\lceil \frac{m-4}{2} \right\rceil + 2. \]

**Case 4.** \( h \geq 6, m-h+1 \geq 6 \):

\[ p(D_1) \geq \left( -\left\lfloor \frac{h-4}{2} \right\rfloor \left\lceil \frac{h-4}{2} \right\rceil + 2 \right) + \left( -\left\lfloor \frac{m-h-3}{2} \right\rfloor \left\lceil \frac{m-h-3}{2} \right\rceil + 2 \right) - 1 \]
\[ \geq -\left\lfloor \frac{m-4}{2} \right\rfloor \left\lceil \frac{m-4}{2} \right\rceil + 2. \]

Since \( p(D) \geq p(D_1) \), the lemma holds by induction. \( \square \)

Combining Lemmas 3.1 and 3.2 we easily obtain the following.

**Theorem 3.3.** Let \( D \) be a digraph which is obtained by directing the edges of an acyclic graph with \( m \geq 2 \) vertices none of which is isolated. Then

\[ |p(D)| \leq \left\lfloor \frac{m-1}{2} \right\rfloor \left\lceil \frac{m-1}{2} \right\rceil + 1, \]

with equality if and only if \( D \) has the form shown in Fig. 3.
We now come to the first main result of this section.

**Theorem 3.4.** For $n \geq 2$, let $A$ be an $n \times n$ complementary acyclic $(0, 1)$-matrix with $\rho_0(A) = n$. Suppose there is a 0 in $A$ whose row and column contains no other 0. Then

$$|\det A| \leq \begin{cases} n - 1 & \text{if } n \leq 6 \\ \left\lceil \frac{n - 2}{2} \right\rceil + 1 & \text{if } n \geq 6. \end{cases} \quad (3.7)$$

Equality holds in (3.7) if and only if $A$ or $A^t$ is permutation equivalent to $J_n - I_n$ (for $n \leq 6$) or

\[
\begin{bmatrix}
0 & 1 & \cdots & \cdots & 1 \\
1 & J_{[(n-2)/2]} - I_{[(n-2)/2]} & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & \cdots & 1 \\
1 & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \quad (n \geq 6). \quad (3.8)
\]

**Proof.** Without loss of generality we may assume that $A$ is nonsingular and that $A$ has the form (3.1) where $v$ is a vector of all 1's. Let $B$, $C$, and $X$ be derived from $A$ as in the beginning of this section. Then as before, $\det A = (-1)^n \det B$ and $D(X)$ is obtained by directing the edges of an acyclic graph with $n - 1$ vertices. Since $C$ is nonsingular, there is a unique row vector $w = (w_2, \ldots, w_n)$ such that $wC = e$, where $e$ is the row vector of $(n - 1)$ 1's. Subtracting $w_i$ times row $i$ from row 1 of $B$ for $i = 2, \ldots, n$, we obtain a triangular matrix $B'$ whose main diagonal entries are $-\sum_{i=2}^n w_i$, 1, $\ldots$, 1. Hence $\det A = (-1)^{n+1} \sum_{i=2}^n w_i$. Since $X$ has 0's on and above its main diagonal,

$$C^{-1} = (I_{n-1} + X)^{-1} = I_{n-1} - X + \cdots + (-1)^{n-2}X^{n-2},$$

and hence

$$w = eC^{-1} = e - eX + \cdots + (-1)^{n-2}eX^{n-2}.$$ 

The $i$th coordinate $(eX^k)_i$ of $eX^k$ counts the number of directed paths of length $k$ of $D(X)$ which terminate at its $i$th vertex. Hence $\sum_{i=2}^n (eX^k)_i$ counts the number of directed paths of length $k$ of $D(X)$. It follows that

$$\sum_{i=2}^n w_i = p(D(X)),$$

so that

$$\det A = (-1)^{n+1}p(D(X)). \quad (3.9)$$
Let \( t \) be the number of isolated vertices of \( D(X) \). If \( t = n - 1 \), then \( X = 0 \) and it follows that \( A \) is permutation equivalent to \( J_n - I_n \) and \( |\det A| = n - 1 \). Now suppose \( t \neq n - 1 \) so that \( 0 \leq t \leq n - 3 \). Applying Theorem 3.3 we obtain

\[
|\det A| = |p(D(X))| \leq t + \left\lfloor \frac{n-t-2}{2} \right\rfloor \left\lfloor \frac{n-t-2}{2} \right\rfloor + 1.
\]

For \( n = 2 \), \( |\det A| = 1 \) and \( A \) is permutation equivalent to \( J_2 - I_2 \). Let \( 3 \leq n \leq 5 \). Then

\[
t + \left\lfloor \frac{n-t-2}{2} \right\rfloor \left\lfloor \frac{n-t-2}{2} \right\rfloor + 1 < n - 1,
\]

and hence \( |\det A| \leq n - 1 \) with equality if and only if \( A \) is permutation equivalent to \( J_n - I_n \). Now let \( n \geq 6 \). Then

\[
n - 1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 1,
\]

with equality if and only if \( n = 6 \). Moreover,

\[
t + \left\lfloor \frac{n-t-2}{2} \right\rfloor \left\lfloor \frac{n-t-2}{2} \right\rfloor + 1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 1,
\]

with equality if and only if \( t = 0 \). Hence

\[
|\det A| \leq \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 1.
\] (3.10)

Let \( n \geq 7 \). Then equality holds in (3.10) if and only if \( D(X) \) has no isolated vertices and \( |p(D(X))| = \left\lfloor (n-2)/2 \right\rfloor \left\lfloor (n-2)/2 \right\rfloor + 1 \). Hence by Theorem 3.3 equality holds in (3.10) if and only if \( D(X) \) has the form shown in Fig. 3 (with \( m = n - 1 \)), that is, if and only if \( A \) or \( A^t \) is permutation equivalent to (3.8). For \( n = 6 \), equality holds if and only if \( A \) or \( A^t \) is permutation equivalent to \( J_6 - I_6 \) or (3.8).

This completes the proof. \( \square \)

We now obtain the second main result of this section which removes the restriction that there is a 0 whose row and column contains no other 0.

**Theorem 3.5.** For \( n \geq 2 \), let \( A \) be an \( n \times n \) complementary acyclic \((0, 1)\)-matrix with \( \rho_0(A) = n \). Then

\[
|\det A| \leq \begin{cases} n - 1 & \text{if } n \leq 5 \\ \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor & \text{if } n \geq 6. \end{cases}
\] (3.11)

Equality holds in (3.11) if and only if \( A \) or \( A^t \) is permutation equivalent to \( J_n - I_n \).
Proof. First consider the case where there is a 0 in $A$ whose row and column contains no other 0. We then invoke Theorem 3.4. For $n \leq 5$, $|\det A| \leq n - 1$ with equality if and only if $A$ is permutation equivalent to $J_n - I_n$. For $n \geq 6$, 

$$|\det A| \leq \left[ \frac{n-2}{2} \right] + 1 < \left[ \frac{n-1}{2} \right].$$

We now consider the case where there is no 0 in $A$ which is isolated both in its row and column. We may again assume that $A$ has the form (3.1). Define two $(n + 1) \times (n + 1)$ complementary acyclic matrices by

$$A^*(u) = \begin{bmatrix}
  u & 1 & \cdots & 1 \\
  1 &  & \cdots & \\
  \vdots & & \ddots & A \\
  1 & & & 1
\end{bmatrix} \quad (u = 0 \text{ and } 1).$$

Then

$$\det A = \det A^*(1) - \det A^*(0).$$

(3.13)

It follows from the proof of Lemma 2.1 that

$$\det A^*(1) = (-1)^n.$$

(3.14)

As in the proof of Theorem 3.4 (see Eq. (3.9)),

$$\det A^*(0) = (-1)^{n+2}p(D).$$

(3.15)

Here $D$ is a digraph on $n$ vertices satisfying the hypotheses of Lemmas 3.1 and 3.2. Hence by eqns. (3.3) and (3.2),

$$\left\{ \begin{array}{ll}
  1 & \text{if } n = 2, 3, 4, 5 \\
  - \left[ \frac{n-4}{2} \right] - 1 & \text{if } n \geq 6
\end{array} \right.$$
Hence in this case

$$|\det A| \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \quad (n \leq 2),$$

(3.17)

where, by Lemma 3.1, equality holds if and only if $D$ has the form shown in Fig 3 with $m = n$. We now combine the two cases. Since for $n = 2, 3, \text{ or } 4, \left\lfloor \frac{(n-1)/2}{2} \right\rfloor \left\lfloor \frac{(n-1)/2}{2} \right\rfloor < n - 1$ we have $|\det A| \leq n - 1$ with equality if and only if $A$ is permutation equivalent to $J_n - I_n$. For $n \geq 6$,

$$|\det A| \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor,$$

with equality if and only if $D$ has the form shown in Fig. 3 with $m = n$, that is, if and only if $A$ or $A^t$ is permutation equivalent to (3.12). For $n = 5$, $\left\lfloor \frac{(n-1)/2}{2} \right\rfloor \left\lfloor \frac{(n-1)/2}{2} \right\rfloor = n - 1$ and hence

$$|\det A| \leq n - 1$$

with equality as given in the statement of the theorem. This completes the proof. \(\Box\)

Since the bound in Theorem 3.5 is bigger than the bound in Theorem 2.5, (3.11) gives the maximum absolute value of determinants of $n \times n$ complementary acyclic $(0, 1)$-matrices.

References


