The solution of linear ODEs with measured right-hand side by the means of the Laplace transform and the inverse problems theory

Richard Andrásík

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Abstract The initial value problem for arbitrary order linear ordinary differential equations with constant coefficients is considered. The right-hand side of the equation and the initial conditions are affected by noise. Moreover, the values of the right-hand side are given only on a sparse set of discrete points. This type of problem arises in applications when the right-hand side or the initial conditions are results of some measurement. Although there are many papers devoted to linear ordinary differential equations, the described problem was not addressed in previous research. We suggested a solution method based on the Laplace transform and the inverse problems theory. The influence of noise level on the performance of the proposed method was studied and numerical experiments were realized to show the computational efficiency of the method. The obtained results were compared with the performance of a commonly used numerical method for ordinary differential equations.

Keywords Ordinary differential equations · Laplace transform · Inverse problems · Measured right-hand side · Uncertain data

1 Introduction

The initial value problem for arbitrary order linear ordinary differential equations with constant coefficients is considered. The right-hand side of the equation and the initial conditions are affected by a noise. Moreover, the values of the right-hand side are given only on a sparse set of discrete points. This type of problem arises when the right-hand side values are results of some measurement. Although there are many papers devoted to linear ordinary differential...
equations, the described problem was not addressed in previous research. Only a similar problem without the noise was solved symbolically in [8].

The exhaustive list of techniques for solving ordinary differential equations is given in [13]. The standard solving method for a linear ordinary differential equation with constant coefficients is to look for solutions in the form of exponential functions. Even though this is a general and straightforward method, finding the roots of a characteristic polynomial is not a trivial problem for equations of higher order. Moreover, it is not clear how to deal with the noisy right-hand side. Apparently, the undetermined coefficients method does not work in this case.

We suggest a solution method based on the Laplace transform (see [1] and [11]) and the inverse problems theory (see [10] and [12]). The proposed solution method uses the Laplace transform of the problem to obtain an algebraic equation. Next, the algebraic equation is solved. Finally, regularization techniques are applied to find the inversion of the Laplace transform.

Numerical experiments show the computational performance of the proposed method. The obtained results were compared with the efficiency of a commonly used numerical method for ordinary differential equations (see section 5).

2 Statement of the problem

We consider the initial value problem for the linear ordinary differential equation with constant coefficients

\[ \sum_{j=0}^{N} a_j y^{(j)}(t) = \tilde{g}(t), \]

\[ y^{(j)}(0) = \tilde{y}_j, \quad \forall j = 0, \ldots, N - 1, \]

where \(a_j \in \mathbb{R}, \forall j = 0, \ldots, N, a_N \neq 0,\) and the order \(N \in \mathbb{N}\) of the equation is arbitrary. The values of function \(\tilde{g}(t)\) and the initial conditions (2) are results of some measurement. It means, that the right-hand side is given only on the finite set \(0 \leq t_1 < t_2 < \cdots < t_k < +\infty, k \in \mathbb{N},\) and the values \(\{\tilde{g}(t_i)\}_{i=1}^{k}\) and \(\{\tilde{y}_j\}_{j=0}^{N-1}\) are noisy. Furthermore, the number of measurements \(k\) cannot be arbitrarily high, because any measurement costs time and money. Due to this fact, it is important to keep \(k\) small (\(k = 16\) is used in numerical examples – see section 5).

We denote

\[ \tilde{g}(t_i) = g(t_i) + \varepsilon_i^g, \quad \forall i = 1, \ldots, k, \]

\[ \tilde{y}_j = y_j + \varepsilon_j^y, \quad \forall j = 0, \ldots, N - 1, \]

where \(g(t_i), i = 1, \ldots, k,\) and \(y_j, j = 0, \ldots, N - 1,\) are unknown exact values. The vectors \(\varepsilon^g = (\varepsilon_1^g, \ldots, \varepsilon_k^g)\) and \(\varepsilon^y = (\varepsilon_0^y, \ldots, \varepsilon_{N-1}^y)\) are the realizations of the measurement noise. We suppose that \(\varepsilon^g\) and \(\varepsilon^y\) are randomly drawn from
a normal distribution with a mean of zero and a given standard deviation \( \sigma^a > 0 \) and \( \sigma^y > 0 \) respectively.

The task is to reconstruct the solution of the following exact problem

\[
\sum_{j=0}^{N} a_j y^{(j)}(t) = g(t),
\]

\[
y^{(j)}(0) = y_j, \quad \forall j = 0, \ldots, N - 1,
\]

from the knowledge of \( \{\tilde{g}(t_i)\}_{i=1}^{k} \) and \( \{y_j\}_{j=0}^{N-1} \).

3 Laplace transform

The Laplace transform is a widely used integral transform represented by the linear operator \( L : g(t) \to G(s) \).

**Definition 1** Let \( g \) be a piecewise continuous function of an exponential order. The **Laplace transform** of \( g \) is given by the following relation

\[
L\{g(t)\}(s) = \int_{0}^{+\infty} g(t)e^{-st}dt.
\]

The method based on the Laplace transform is a classical technique for solving linear ordinary differential equations with constant coefficients (see [13]). This method works well for inhomogeneous equations where the inhomogeneous term is of any form. The differential equation is transformed into an algebraic equation which is easy to solve. The crucial part of this approach is its final step – the inverse Laplace transform, because it is unknown in general how to invert the Laplace transform. We removed this obstacle by the use of regularization techniques (see section 4).

Let \( Y(s) = L\{y\}(s) \) and \( G(s) = L\{\tilde{g}\}(s) \). Because we know only the values \( \{\tilde{g}(t_i)\}_{i=1}^{k} \), we cannot express the Laplace transform of \( \tilde{g} \) analytically. However, the function \( G(s) \) can be calculated numerically using, for example, the trapezoidal rule:

\[
G(s) \approx \frac{t_k}{2k} \left( e^{-s t_1} \tilde{g}(t_1) + 2e^{-s t_2} \tilde{g}(t_2) + \cdots + e^{-s t_k} \tilde{g}(t_k) \right). \tag{6}
\]

Now, we choose the set \( 0 < s_1 < s_2 < \cdots < s_n < +\infty \). The values \( G(s_i), i = 1, \ldots, n, \) are calculated using the formula (6).

The Laplace transform of the problem (1) is

\[
L \left\{ \sum_{j=1}^{N} a_j y^{(j)}(t) \right\}(s) = L\{\tilde{g}\}(s),
\]

\[
\sum_{j=1}^{N} a_j L\{y^{(j)}(t)\}(s) = L\{\tilde{g}\}(s),
\]

\[
q(s)Y(s) - p(s) = G(s),
\]
where

\[ q(s) = \sum_{j=0}^{N} a_j s^{N-j}, \]

\[ p(s) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-i-1} a_j s^{N-j-i-1}, \]

in accordance with the properties of the Laplace transform (see [1] or [11]).

Now, we need to use the inverse Laplace transform to reconstruct \( y(t) \) from the equation

\[ Y(s) = G(s) + p(s) q(s). \] (7)

The inversion of the Laplace transform is hard to calculate even for an analytically given function, because there is no universal method of the inverse Laplace transform (see [7]). It can be done only for special types of \( Y(s) \). In our case, just the finite set \( \{Y(s_i)\}_{i=1}^{n} \) is known. As a consequence, we can compute only \( \{Y(s_i)\}_{i=1}^{n} \) from the equation (7). According to [7], the knowledge of the values \( \{Y(s_i)\}_{i=1}^{n} \) is enough to reconstruct an approximation \( \hat{y} \) of function \( y \).

Therefore, we can use a numerical method to find the values \( \{\hat{y}(T_i)\}_{i=1}^{K} \), where \( 0 < T_1 < T_2 < \cdots < T_K < +\infty \).

3.1 Numerical inversion of the Laplace transform

The numerical inverse Laplace transform is a severely ill-posed problem in the sense of Hadamard (see definition 2). There are many inversion formulas in the case of the exactly given data \( \{Y(s_i)\}_{i=1}^{n} \) (e. g. see [2], [3], [6] and [9]). However, we must take into an account that the values \( \{Y(s_i)\}_{i=1}^{n} \) are affected by an error due to the noisy right-hand side and the initial conditions of the original problem (1). In this case, we face a great problem caused by the ill-posedness of the inverse Laplace transform.

We approximate the integral in equation (5) with the trapezoidal rule as in (6) by the following formula

\[ Y(s) \approx \frac{T_K}{2K} \left( e^{-sT_1} y(T_1) + 2e^{-sT_2} y(T_2) + \cdots + e^{-sT_K} y(T_K) \right). \] (8)

By using formula (8) and denoting \( \hat{m}_i = Y(s_i), \ i = 1, \ldots, n, \) and \( f_j = y(T_j), \ j = 1, \ldots, K, \) we obtain a linear model

\[ \hat{m} = Af, \] (9)

where

\[ A = \frac{t_K}{2K} \begin{pmatrix} e^{-s_1 T_1} & 2e^{-s_1 T_2} & \cdots & 2e^{-s_1 T_{K-1}} & e^{-s_1 T_K} \\ e^{-s_2 T_1} & 2e^{-s_2 T_2} & \cdots & 2e^{-s_2 T_{K-1}} & e^{-s_2 T_K} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{-s_n T_1} & 2e^{-s_n T_2} & \cdots & 2e^{-s_n T_{K-1}} & e^{-s_n T_K} \end{pmatrix}. \] (10)
The linear model (9) is severely ill-posed due to extremely small values (in comparison to the largest value), that appears in matrix (10). Hence, a regularization technique is needed to solve the problem (9).

4 Inverse problems

A direct way of a procedure is to obtain an effect from a cause – direct problem. However, many real-life applications deal with recovering a cause from a known effect – inverse problem. For example, in image processing the direct problem is finding a blurred picture from a given sharp image. The inverse problem is to determine how would the sharp picture look like as we only know how looks the blurred photograph. Another example arises from X-ray tomography. The problem of the inner structure of a human body reconstruction from X-ray images is described in [10].

Inverse problems typically come from incomplete and indirect physical measurements. Theoretical breakthroughs and computational advances in the field of inverse problems are inspired by practical tasks. The theory of inverse problems is tightly connected with the posedness of a problem.

**Definition 2** The problem is called well-posed, according to Hadamard, if the following conditions are satisfied:

- (H\(_1\)) Existence. Exists at least one solution.
- (H\(_2\)) Uniqueness. There is at most one solution.
- (H\(_3\)) Stability. The solution depends continuously on input data.

**Definition 3** If the problem is not well-posed, then it is called ill-posed.

Inverse problems are extremely sensitive to errors and measurement noise. In other words, the characteristic property of inverse problems is the ill-posedness. There are many important inverse problems which are ill-posed. For example a deconvolution problem, backward heat propagation, image de-blurring, X-ray tomography and electrical impedance tomography (see [10]).

The inverse Laplace transform problem (see section 3) is an ill-posed linear inverse problem, therefore the attention was focused on this class of problems.

The linear direct problem is given by the forward map \( \mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{Y} \), where \( \mathcal{D}(\mathcal{A}) \subseteq \mathcal{X} \) is the domain of the continuous linear operator \( \mathcal{A} \). The forward map represents, for example, an integral equation. \( \mathcal{X} \) and \( \mathcal{Y} \) are Banach spaces. The task to calculate \( m \in \mathcal{Y} \) from \( m = \mathcal{A} f \), where \( f \in \mathcal{D}(\mathcal{A}) \), is a linear direct problem.

The linear inverse problem is to recover \( f \in \mathcal{D}(\mathcal{A}) \), at least approximately, from a given noisy measurement

\[
m = \mathcal{A} f + \varepsilon^c,
\]

where \( m \in \mathcal{Y}, \varepsilon^c \in \mathcal{Y} \) is measurement error (noise), which is unknown in general. We have only the information that \( \| \varepsilon^c \|_\mathcal{Y} \leq \delta^c \) and \( \delta^c > 0 \) is known.
The noise have to be taken into account because any measurement is influenced by the measurement error.

Usually, problem (11) is approximated by the finite-dimensional version. The approximation leads to the discrete linear inverse problem in the following form

\[ m = Af + \varepsilon, \quad (12) \]

where \( A \in \mathbb{R}^{k \times n} \) is a matrix, \( m \in \mathbb{R}^k \) and \( f \in \mathbb{R}^n \). The measurement error \( \varepsilon \in \mathbb{R}^k \) has bounded norm, i.e. \( \|\varepsilon\| \leq \delta \) for some known \( \delta > 0 \).

**Remark 1** In practice, the outcome of a realized measurement is \( \tilde{m} = m - \varepsilon \), where \( \varepsilon \) is fixed but still unknown.

In the rest of this section, the main ideas of solution methods for linear inverse problems are explained to obtain the necessary background, which is needed for the solution of problem (9) in general and examples in section 5 in particular. The following solution methods are described in detail in [10]. Further information about the methods can be find in [12].

### 4.1 Naive inversion

It is attractive to solve (11) very simply by the reconstruction

\[ f \approx A^{-1}\tilde{m}. \quad (13) \]

The failure of reconstruction (13) is demonstrated in [10] (section 2). It is caused by the ill-posedness of the inverse problem.

### 4.2 Moore-Penrose regularization

The use of the Moore-Penrose pseudoinverse of \( A \) is recommended in general for linear inverse problems to examine the severity of the ill-posedness of a problem. This method uses the singular value decomposition of \( A \).

**Definition 4** \( L(\tilde{m}) \in \mathbb{R}^n \) is called the **minimum norm solution** of the equation \( Af = \tilde{m} \) if

\[ \|L(\tilde{m})\| = \min \left\{ \|z\| \in \mathbb{R}^n ; z = \arg \min_{w \in \mathbb{R}^n} \|Aw - \tilde{m}\| \right\}. \]

**Theorem 1** Let \( A = UDV^T \) be the singular value decomposition of \( A \in \mathbb{R}^{k \times n} \). The minimum norm solution of \( Af = \tilde{m} \) is given by \( A^+\tilde{m} = VD^+U^T\tilde{m} \), where \( A^+ \) and \( D^+ \) are Moore-Penrose pseudoinverses of \( A \) and \( D \) respectively.

*Proof* see [10] page 54.

The behavior of the method is unstable if \( d_i^{-1} \) is much smaller than \( d_i^{-1} \). That is the reason to use the **truncated singular value decomposition** (see [10]). The idea is to take all \( d_i^{-1} \), for \( i = 1, \ldots, r \) such that \( d_i \) is bellow a chosen level, as zeros.
4.3 Tikhonov regularization

The minimum norm solution is a vector from the set

\[ \left\{ z \in \mathbb{R}^n; \ z = \arg \min_{z \in \mathbb{R}^n} \|Az - \tilde{m}\| \right\}, \]

which has the minimum norm. The Tikhonov approach puts these two minimization procedures in a balance and forms only one minimization problem.

**Definition 5** \( T_\alpha(\tilde{m}) \in \mathbb{R}^n \) is called the Tikhonov regularized solution of the equation \( Af = \tilde{m} \) if

\[ T_\alpha(\tilde{m}) = \arg \min_{z \in \mathbb{R}^n} \left\{ \|Az - \tilde{m}\|^2 + \alpha \|z\|^2 \right\}, \quad (14) \]

where \( \alpha > 0 \) is the regularization parameter.

It is not known in general, how to choose the regularization parameter \( \alpha \) to obtain an ideal solution. The Morozov discrepancy principle (see [4]) or the L-curve method (see [5]) can be used. Although these methods are not universal, they give satisfactory results. In section 5, the L-curve method was used to choose the parameter \( \alpha \).

**Remark 2** When it is expected that \( f \) approximates a smooth function, we consider the following problem

\[ T_\alpha(\tilde{m}) = \arg \min_{z \in \mathbb{R}^n} \left\{ \|Az - \tilde{m}\|^2 + \alpha \|Lz\|^2 \right\}, \quad (15) \]

instead of the problem (14). Matrix \( L \in \mathbb{R}^{n \times n} \) is a discretized differential operator.

**Theorem 2** The Tikhonov regularized solution of the equation \( Af = \tilde{m} \) is given by the following equation

\[ T_\alpha(\tilde{m}) = (A^T A + \alpha L^T L)^{-1} A^T \tilde{m}, \]

where \( L \in \mathbb{R}^{n \times n} \) is an identity matrix or a discretized differential operator.

**Proof** see [10] page 67.

Tikhonov regularization is easy to implement and works well for problems in which reconstruction \( f \) approximates a smooth function (see examples 1 and 2).
4.4 Total variation regularization

Problems like image deblurring and X-ray tomography need a reconstruction with sharp edges. An edge-preserving method is total variation regularization. This method gives better results than Tikhonov regularization for problems in which $f$ approximates a non-smooth function or even a function with jumps. The key idea is to replace $\|\cdot\|_2^2$ by $\|\cdot\|_1$ in the second term of minimization problem (15).

**Definition 6** $T_\alpha(\tilde{m}) \in \mathbb{R}^n$ is called the total variation regularized solution of the equation $Af = \tilde{m}$ if

$$T_\alpha(\tilde{m}) = \arg \min_{z \in \mathbb{R}^n} \left\{ \|Az - \tilde{m}\|_2^2 + \alpha \|Lz\|_1 \right\},$$

where $\alpha > 0$ is the regularization parameter and $L \in \mathbb{R}^{n \times n}$ is an identity matrix or a discretized differential operator.

One of the possible approaches to find the total variation regularized solution is the reformulation of the problem (16) into a quadratic optimization problem (see [10]), which can be solved, for example, in MATLAB using the interior-point algorithm.

5 Numerical experiments

We consider three different model problems to test the performance of the proposed method. The regularization methods mentioned in section 4 were used to solve (9). The proposed approach was numerically implemented in MATLAB. Euler's forward method (see [13]) – as the representative of the most commonly used methods – was also applied to the model problems. The obtained results are described in this section.

To evaluate the accuracy of the methods, the exact solutions have to be known. Therefore, we present the exact setting of each problem and add a noise to the right-hand side. Let the exact right-hand side be identically equal to one in the following examples. The relative error

$$\frac{\|y - \hat{y}\|_2}{\|y\|_2},$$

where $y$ is an exact solution of a given problem, was used to measure the quality of an approximation $\hat{y}$.

The exact solution of a model problem was either easy to find analytically (example 1) or was calculated approximately (examples 2 and 3) from the exact setting of the problem using function `ode45.m` in MATLAB. The exact setting means that the right-hand side of the differential equation is continuous (can be evaluated in any point) and is not affected by a noise. Hence, function `ode45.m` gives almost exact results and the error of an approximation is negligible.
In the following examples we set
\[ K = 128, \]
\[ n = 1024, \]
\[ \Delta t = \frac{t_k}{k-1}, \]
\[ \Delta T = \frac{T_K}{K-1}, \]
\[ \Delta s = \frac{s_n}{n}, \]
where \( k \in \mathbb{N}, t_k > 0, T_K > 0 \) and \( s_n > 0 \) are given in the examples. We set the meshes
\[ t_i = i\Delta t, \quad \forall i = 0, \ldots, k-1, \]
\[ T_i = i\Delta T, \quad \forall i = 0, \ldots, K-1, \]
\[ s_j = j\Delta s, \quad \forall j = 1, \ldots, n. \]

The values \( \tilde{g}(t_i) = 1 + \varepsilon_i^g, \quad i = 1, \ldots, 16, \) are given. Each component of the vector \( \varepsilon^g \in \mathbb{R}^k \) was randomly drawn from a normal distribution having a mean of zero. The standard deviation of the normal distribution was taken as a variable \( \sigma > 0. \) The influence of \( \sigma \in (0\%, 30\%) \) on the relative error (17) was studied and 95\% confidence intervals of the relative error were computed by the use of Monte Carlo approach. Furthermore, we compared the efficiency of the methods depending on the number of the right-hand side evaluations.

**Example 1**
We consider the following problem
\[ y'' + y = \tilde{g}(t), \]
\[ y(0) = 0, \]
\[ y'(0) = 0. \]

Let \( k = 16, \ t_k = T_K = 2\pi \) and \( s_n = 20. \)

Regarding Euler’s method, the results were unsatisfactory with the relative error around 70\% (see Figure 1). It was caused mainly by the low number of the given right-hand side values. The model problem was solved by the proposed approach with the use of all three regularization methods described in section 4. The variant with Tikhonov regularization gave the best results with the mean relative error around or lower than 10\% (see Figure 1). For illustration, Figure 2 shows the fit of reconstructed values for \( \sigma = 0.1. \)

With increasing number \( k, \) we can see that the performance of the proposed method remains stable from \( k = 16. \) At the same time, the relative error of Euler’s method rapidly decreases. Euler’s method performed better than total variation regularization when \( k > 64 \) and even than Tikhonov regularization when \( k > 120 \) (see Figure 3).
Fig. 1 The performance of Euler’s method and the proposed method with three different regularization variants for a problem described in example 1 depending on the standard deviation of the Gaussian noise. Grey areas represent 95% confidence intervals of the relative error and thick solid lines depict mean values of the error.
Fig. 2 The exact solution of a problem described in example 1 and its approximation computed by the use of Euler’s method and the proposed method with three different regularization variants. The noise level $\sigma = 0.1$. 
Example 2  Let \( k = 16, \ t_k = T_K = 15 \) and \( s_n = 20 \). We consider the following problem

\[
y^{(6)} + 6y^{(4)} + 11y'' + 6y = \tilde{g}(t),
\]

\[
y^{(i)}(0) = 0, \ \forall i = 0, \ldots, 5.
\]

Euler’s method gave enormous relative errors. The truncated singular value decomposition and total variation regularization performed better with the mean relative errors were around 45% and 80% respectively. However, these errors are also unsatisfactory. The best fit to the exact function was found by the proposed method with the use of Tikhonov regularization. The mean relative error increased steadily from 20% to 40% as \( \sigma \) goes from zero to thirty percent (see Figure 4). Although the relative error is still high, the behavior of both the approximation and the exact function is almost the same (for example, see Figure 5).

By comparing the performance of the methods for fixed \( \sigma = 0.1 \) and increasing \( k \), we can see that the relative error is approximately constant (around 30%) in the case of Tikhonov regularization and slightly increasing from 80% when total variation regularization was used. Euler’s method becomes comparable with Tikhonov regularization for \( k > 450 \) (see Figure 6).

As a result, the proposed method with Tikhonov regularization having only 16 right-hand side evaluations and Euler’s method with 450 right-hand side evaluations lead to the approximation with the same relative error around 30%.
Fig. 4 The performance of Euler’s method and the proposed method with three different regularization variants for a problem described in example 2 depending on the standard deviation of the Gaussian noise. Grey areas represent 95% confidence intervals of the relative error and thick solid lines depict mean values of the error.
Fig. 5 The exact solution of a problem described in example 2 and its approximation computed by the use of Euler’s method and the proposed method with three different regularization variants. The noise level $\sigma = 0.1$. 
Example 3 We consider the following problem

\[
\sum_{j=0}^{N} \frac{1}{j!} y^{(j)}(t) = \tilde{g}(t),
\]
\[
y^{(j)}(0) = 0, \quad \forall j = 0, \ldots, N - 1,
\]

We set \(N = 3\) in this model example. Let \(k = 16, t_k = T_K = 15\) and \(s_n = 20\).

Euler’s method failed to solve this problem. Also the performance of the truncated singular value decomposition and Tikhonov regularization was not satisfactory (see Figure 8). The best results were obtained by the use of total variation regularization approach with the mean relative error only around 10% (see Figure 7). It is caused by the concept of total variation regularization (the use of \(\|\cdot\|_1\) instead of \(\|\cdot\|_2\)). The norm \(\|\cdot\|_1\) is better suited for functions with sharp edges (see [10] and [12]).

Let \(\sigma = 0.1\) be fixed. The efficiency of Euler’s method was improved rapidly by increasing the number of right-hand side evaluations. The relative error of this method is similar to the relative error of the proposed method with total variation regularization when \(k > 60\) (see Figure 9).

The MATLAB function `ode45.m` failed to solve the presented model problem in the case of \(N \geq 5\). However, the proposed method with the use of total variation regularization remained stable and gave similar results even for high values of \(N\).
Fig. 7 The performance of Euler’s method and the proposed method with three different regularization variants for a problem described in example 3 depending on the standard deviation of the Gaussian noise. Grey areas represent 95% confidence intervals of the relative error and thick solid lines depict mean values of the error.
Fig. 8 The exact solution of a problem described in example 3 and its approximation computed by the use of Euler’s method and the proposed method with three different regularization variants. The noise level $\sigma = 0.1$. 

The solution of linear ODEs with measured right-hand side.
6 Conclusion

The main contribution of this work is the description of a new numerical method for solving the initial value problem for linear ordinary differential equations affected by a noise with constant coefficients. The order of the differential equation can be arbitrarily high. The proposed method is based on the knowledge from two different areas – the Laplace transform and the inverse problems.

The influence of a standard deviation of the Gaussian noise on the relative error of approximate solutions was studied. It was found that the level of the standard deviation has only little impact on the accuracy of approximate solutions (see figures 1, 4 and 7). Furthermore, the results support a statement that the variant with Tikhonov regularization gives the best results in the case of approximating a smooth function (see examples 1 and 2). Considering a function with sharp edges, the variant with total variation regularization performed the best (see example 3).

The proposed method was compared with Euler’s forward method which was taken as the representative of the classical methods. Due to the low number of the given right-hand side values, the proposed method behaved much better than Euler’s method. More refined mesh \( \{t_i\}_{i=0}^{K} \) or the use of a higher order method in place of Euler’s method decreases the relative error. However, the use of a sparse mesh was a part of the problem setting, so the refinement of the mesh cannot be done in general. The same statement holds for higher order methods which usually need more evaluations of a right-hand side of a given differential equation than Euler’s method.
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