Effect of motion on pinning control of time-varying networks

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Abstract. In this paper, the relation between synchronization and control of chaotic nodes connected through a time-varying network is discussed. In particular, the effects of pinning control on a set of moving chaotic agents are investigated showing that the role of system parameters, like agent density, is critical in order to reach the synchronous behavior and also to control the whole network by pinning a reduced set of agents.

Keywords. Complex networks, Chaos control, Pinning control, Time-varying networks.

Introduction

In the last decade, complex networks attracted a lot of attention in different science fields like mathematics, chemistry, neuroscience, physics, biology, but also electrical engineering and social science [1]. This is mainly due to the flexibility of the concept of complex networks which can be used to describe different behaviors and phenomena in many scientific areas. A classical example of complex networks is the human society: the nodes are human beings while links are social relationships between them. Other paradigmatic examples of real complex networks are the World Wide Web and Internet in which the nodes are computers and the links are cables or wireless connections between them.

Furthermore, complex networks can exhibit interesting collective behavior. In particular, great attention has been given to the case in which nodes are nonlinear dynamical units which exchange information through the links reaching a synchronous behavior (we define synchronization as a process wherein some dynamical systems adjust a desired property of their motion to a common behavior, due to coupling). Examples of the onset of synchronous behavior can be found in several contexts: from biology to physics [2,3]. Synchronization on complex networks assumes particular interest if nodes are chaotic systems. In this case, specific strategies to attain the synchronization of the whole network should be defined [4].

In some cases, it is necessary not only to achieve synchronization but also to introduce a control on the entire network. However, controlling each node of a complex net-
work composed by many units may be not only difficult, but also not necessary. Some strategies to regulate and control the dynamical behavior of networks have been proposed in literature, among them *pinning control* attracted more attention. The general idea behind pinning control is to apply a feedback loop only on a small fraction of nodes which propagate the control effect to the rest of the network through the existing links [5].

Another important aspect is connected to the topology of the network. In most research works, in fact, interaction between nodes is time-invariant and, thus, the topology of the network is fixed during time. Aim of this paper is to characterize the effects of a pinning control strategy in time–varying networks, i.e. networks in which links change in time.

In particular, we consider the case of a simple time–varying interaction model such as defined by the interaction network between random walkers based on a metric neighborhood in which the coupling between nodes evolves during time [6]. Moreover, each random walker is associated to a chaotic system, i.e. a Rössler oscillator [7]. The effect of pinning control on the synchronization and control properties of the considered time–varying network is characterized in terms of agents density focusing on the minimum number of pinned nodes needed to reach synchronization and control of the entire network.

The rest of the paper is organized as follows. In Sec. 1 the time–varying interaction network model based on mobile agents is introduced, in Sec. 2 the pinning control technique is described, while in Sec. 3 simulation results showing the effects of pinning control on a network of moving chaotic agents are reported.

### 1. Moving chaotic agents model

Let us consider $N$ mobile agents moving in a bi–dimensional space of size $L$ with density $ho = \frac{N}{L^2}$. The agents in our model are random walkers: each agent moves with velocity $v(t)$, constant in modulus, and with a direction of motion $\theta_i(t)$ that is updated stochastically at each time step. Hence, the position and orientation of our agents in space are updated according to:

\[
\begin{align*}
  y_i(t + \Delta t) &= y_i(t) + v_i(t) \Delta t \\
  \theta_i(t + \Delta t) &= \eta_i(t + \Delta t)
\end{align*}
\]  

(1)

where $y_i(t)$ is the position of the $i$–th agent in the plane at time $t$, $\eta_i(t)$ are $N$ independent random variables chosen at each time with uniform probability in the interval $[-\pi, +\pi]$ and $\Delta t = 0.001$ is both the motion and the dynamics integration step size. Moreover each agent is characterized also by internal state variables $x^j_i(t) \in \mathbb{R}^3$ which evolve according to the following equations describing a Rössler chaotic oscillator:

\[
\begin{align*}
  \dot{x}^1_i &= -x^1_i - x^3_i \\
  \dot{x}^2_i &= x^1_i + ax^2_i \\
  \dot{x}^3_i &= b + x^3_i(x^1_i - c)
\end{align*}
\]  

(2)
with $\dot{x}^i(t) = [\dot{x}_1^i(t) \ \dot{x}_2^i(t) \ \dot{x}_3^i(t)]^T$. Parameter values have been chosen as $a = 0.2$, $b = 0.2$ and $c = 7$ in order to ensure that each oscillator exhibits a chaotic behavior.

As mentioned before each agent is able to connect only to its neighbors, i.e., those agents which are within its interaction radius $r$ at time $t$. When two agents interact, the state equations of each agent are changed to include a diffusive coupling with the neighbor agent, acting on the state variable $x_1^i$. Based on this interaction between neighbors the state dynamics of each agents is described by:

$$\dot{x}^i = f(x^i) + K \sum_{j=1, j \neq i}^{N} a_{ij}(t) \Gamma(x^j - x^i)$$

where $a_{ij}(t)$ is the generic element of the adjacency matrix $A = (a_{ij}) \in R^{N \times N}$, i.e. $a_{ij} = a_{ji} = 1$ if the two agents are neighbors, otherwise $a_{ij} = a_{ji} = 0$. $K$ is the coupling strength, and $\Gamma \in R^{3 \times 3}$ is a constant $0 - 1$ matrix indicating the coupled variables and defined as:

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The degree $k_i$ of node $i$ is defined as the number of its connections:

$$\sum_{j=1, j \neq i}^{N} a_{ij} = \sum_{j=1, j \neq i}^{N} a_{ji} = k_i, \quad i = 1, 2, ..., N$$

where $k$ is the number of neighbors of the $i$-th agent at time $t$. If we define the diagonal elements $a_{ii} = -k_i$, $i = 1, 2, ..., N$, the model can be rewritten in a simpler form:

$$\dot{x}^i = f(x^i) + K \sum_{j=1}^{N} a_{ij}(t) \Gamma x^j$$

2. Pinning control

In this Section we introduce the control strategy. Aim of the control is to stabilize the network onto the homogeneous stationary state: $x^1 = x^2 = ... = x^N = \bar{x}$, with $f(\bar{x}) = 0$. The control acts only on a subset of network nodes, defined as pinned nodes.

Without loss of generality, let us consider the first $l$ nodes as pinned by using a linear feedback controller [8,9]:

$$u^i = -Kd \Gamma_c (x^i - \bar{x}), \quad i = 1, 2, ..., l,$$
where $d^i > 0$ is positive feedback control parameter and $\Gamma_c$ is the identity matrix (in all the simulations $d^i = 100 \forall i = 1, \ldots, N$).

Considering the introduction of pinning control, the dynamic of the oscillators ensemble can be described as:

\[
\begin{align*}
\dot{x}^i &= f(x^i) + K \sum_{j=1}^{N} a_{ij} \Gamma x^j + u^i, \quad i = 1, 2, \ldots, l \\
\dot{x}^i &= f(x^i) + K \sum_{j=1}^{N} a_{ij} \Gamma x^j, \quad i = l + 1, l + 2, \ldots, N 
\end{align*}
\]

The choice of the $l$ pinned nodes can be made on the basis of different considerations related to the network topology [10]. There are two main strategies to choosing the pinned nodes: random pinning, i.e. the $l$ pinned nodes are randomly selected, and selective pinning, i.e. the $l$ pinned nodes are first sorted according to a certain property related to the network topology (the node degree betweenness, or centrality, for instance), then the nodes to be pinned are chosen in that particular order, first pinning the highest node, and then continuing to select and pin the other nodes in decreasing order. In this paper, we study the effect of a pinning control strategy based on a random (and fixed in time) selection of the pinned nodes.

3. Effect of pinning control on moving chaotic agents model

Following the results described in [7], an ensemble of moving chaotic agents can show a synchronous behavior for particular values of the system parameters. In particular, it has been demonstrated that agent density plays a critical role on the onset of synchronization. Increasing the density has an effect similar to increasing the coupling strength, leading the system first to synchronization from a disordered condition and then again through a second bifurcation to a not-synchronized status. In [7] no control has been applied.

When pinning control is applied to the network of moving chaotic agents, different parameters have to be taken into account to investigate the onset of synchronization and the control efficacy. To this aim, two parameters representing average errors, have been defined: $\langle \delta_s \rangle$ which gives indications on network synchronization, and $\langle \delta_c \rangle$ which monitors the effectiveness of the control strategy. In particular, the synchronization error has been defined as follows:

\[
\delta_s(t) = \frac{\sum_{i=l+1}^{N-1} (|x^i_1 - x^N_1| + |x^i_2 - x^N_2| + |x^i_3 - x^N_3|)}{3(N - l - 1)}
\]

and $\langle \delta_s \rangle$ has been defined as the mean of $\delta_s(t)$ over the last 100,000 integration steps. The control error has been defined as:

\[
\delta_c(t) = \frac{\sum_{i=l}^{N} (|x^i_1 - \bar{x}_1| + |x^i_2 - \bar{x}_2| + |x^i_3 - \bar{x}_3|)}{3(N - l)}
\]

and $\langle \delta_c \rangle$ as the mean of $\delta_c(t)$ over the last 100,000 integration steps.
In Fig. 1, the trend of $\langle \delta_s \rangle$ with respect to different densities is reported. It can be observed that exists a density range in which the system without pinning control is synchronized, as already reported in [7]. When pinning is introduced, the synchronization range changes. In order to identify the transition from not synchronized to synchronized behavior, we define $\rho_{s1}$ and $\rho_{s2}$ as the density thresholds for which the condition $\langle \delta_s \rangle < 0.001$ holds, hence for $\rho_{s1} < \rho < \rho_{s2}$ all the not pinned nodes are synchronized. We noticed that introducing pinning control in the system, the synchronization threshold of system is increased, hence the presence of pinned nodes acts as a source of perturbation. In the inset of Fig. 1, the synchronization index $\langle \delta_s \rangle$ is shown for $N = 100$. According to [7] and to the numerical data in the inset, the behavior of the system does not depend on the number of agents, but only on their density.

From the analysis of Fig. 1, it can be shown that the synchronization thresholds in presence of an increasing number of pinned nodes tend to their values $\rho_{s1}$ and $\rho_{s2}$ without pinning. This means that the presence of pinned nodes causes an increasing in the synchronization error, but when the system is fully controlled (and so synchronized), the same density range obtained for observing synchronization in absence of pinning control is recovered. To closely examine this behavior, in Fig. 2 $\rho_{s1}$ is reported with respect to the number of pinned nodes $l$. It can be observed that the critical density decreases monotonically and, when the 50% of nodes are pinned, it reaches the value $\rho_{s1}$ typical of the case in which there is no pinning control.

Focusing on control, we can define two further density thresholds $\rho_{c1}$ and $\rho_{c2}$ so that if $\rho_{c1} < \rho < \rho_{c2}$ the condition $\langle \delta_c \rangle < 0.001$ holds, i.e., the system is fully controlled. Fig. 3 shows the control error $\langle \delta_c \rangle$ vs. the agent density. It can be observed that the density value for which the system is controlled $\rho_{c1}$ is greater than $\rho_{s1}$. This means that
there is a region of density in which the system is synchronized but not controlled. By increasing the number of pinned nodes, it has been observed that $\rho_{c1}$ tends to $\rho_{s1}$.

4. Conclusions

This paper focused on pinning control on a dynamical time-variant complex network representing the interaction among a group of chaotic agents. The possibility of obtaining control of the network by pinning only a fraction of the network nodes has been investigated. It has been shown that global synchronization and stabilization can be achieved by a random (and fixed in time) choice of pinned nodes. We noticed that synchronization and control are related but in a not trivial way. Synchronization as a weaker form of control is obtained, when the network is globally controlled, but there is a region of parameter values in which the system is synchronized but not controlled. Increasing the number of pinned nodes, the density ranges in which the network is controlled tends to be the same in which all the agents are synchronized. Future works will investigate other strategies for pinning control.

References


Figure 3. Control error $\langle \delta_c \rangle$ vs density $\rho$ for $N = 100$ agents. Other parameters are chosen as reported in the caption of Fig. 1.