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Abstract

This paper presents a new approach to the analysis of asymptotic stability of artificial neural networks (ANN) with multiple time-varying delays subject to polytope-bounded uncertainties. This approach is based on the Lyapunov–Krasovskii stability theory for functional differential equations and the linear matrix inequality (LMI) technique with the use of a recent Leibniz–Newton model based transformation without including any additional dynamics.

Three examples with numerical simulations are used to illustrate the effectiveness of the proposed method. The first example considers the neural network with multiple time-varying delays, which may be seen as a particular case of the second example where it is subject to uncertainties and multiple time-varying delays. Finally, the third example analyzes the stability of the neural network with higher numbers of neurons subject to a single time-delay. The Hopf bifurcation theory is used to verify the stability of the system when the origin falls into instability in the bifurcation point.

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1. Introduction

Since the seminal work of Hopfield [1] in 1982, which showed the relation between recurrent autoassociative neural networks and physical systems, this issue has been extensively studied. Particularly, as discussed in [2–4], in the hardware implementation of neural networks, when communication and response of neurons happens, time-delays may occur. Moreover, the presence of time-delays may influence the stability of the neural network creating oscillatory phenomena or instability. Also, physical systems usually suffer from uncertainties which arise because of variations in the system parameters, modelling errors or some ignored factors.

A problem in any dynamic system, such as analogical neural networks, is relative to requirements of stability. In the literature, there are two concepts concerning the stability of systems with time-delays. The first one is concerned with the delay-independent stability condition which does not include any information about the size of the time-delay [5–20]. The second is concerned with the delay-dependent stability condition, in which the size of the time-delay is taken

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explicitly in the formulation [4,21–25]. As the information on the length of delays is used by delay-dependent criteria, they may be less conservative than delay-independent ones. Notice that most of the cited approaches for the stability analysis of artificial neural networks with time-delay are based on the Lyapunov–Krasovskii functional method as introduced in [26–28].

In [7], has been introduced the first study of the influence of a single time-delay in the neural network. Soon after, Gopalsamy and He [9] considered a modification incorporating different delays in different communication channels. In [10], a new term was added so that besides the delay propagation of the signs, the model of the neural network includes information about the instantaneous propagation of the signs, being in this way a more general model. Moreover in [22–30], the asymptotic stability of uncertain neural networks with time-delay were studied. In this way, some sufficient conditions arise that are viable, but they have the disadvantage of not being capable of finding the largest allowable time-delay.

The main contribution of this paper is to state a delay-dependent condition for the asymptotic stability analysis of neural networks with multiple constant or time-varying delays subject to polytope-bounded uncertainties. A linear matrix inequality (LMI) based method is proposed, using as the starting point a recent technique reported in [31], in which additional free weighting matrices are introduced to express the influence of the terms in the standard Leibniz–Newton formula, without increasing the dynamics and being capable of identifying the largest time-delay. This feature allows the proposed approach to be applied to both multiple time-delays and polytopic models, which is not a trivial task, with the aforementioned works. Finally, the effectiveness of the proposed method is verified through three examples. In this paper the notation “∗” is used instead of writing terms in the symmetric matrix, the superscript “T” represents the transpose, $M > 0$ ($< 0$) means that the matrix $M$ is positive (negative) definite. diag{·} denotes a diagonal matrix.

2. Preliminaries

Hopfield in [1] shows that it is possible to relate recurrent autoassociative neural networks to physical systems, where each $i$th neuron is represented by a linear circuit consisting of a resistor $R_i$, a capacitor $C_i$, a source of current $I_i$ and a nonlinear sigmoidal activation function $g_i(·)$. Between a channel of communication of a $j$th neuron and an $i$th neuron there is a conductance $1/R_{ji}$ and the neurons are connected with each other via additive junctions of current. Thus, the physical realization of neural networks can be represented through differential equations.

In this paper, the Hopfield’s model is extended to a larger class of neural networks. Specifically, the following delay differential equation that represents the neural networks endowed with multiple time-varying delays is considered, such as studied in [22]. This class has a dynamic behavior much more complex due to the incorporation of $K$ multiple time-varying delays as given below:

$$\dot{v}(t) = -Av(t) + W_0g(v(t)) + \sum_{k=1}^{K} W_k g(v(t - d_k(t))) + I,$$  \hspace{1cm} (1)

where $v(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^T$ is the neuron state vector (or, is the voltage on the input of the neuron), $A = \text{diag}(a_1, a_2, \ldots, a_n)$ is a positive diagonal matrix (where each term is $a_i = 1/(C_i R_i)$, $i = 1, \ldots, n$), $W_0 = (w_{ij}^0)_{n \times n}$ (where $w_{ij}^0 = 1/(C_i R_{ij}^0)$) and $W_k = (w_{ij}^k)_{n \times n}$ (where $w_{ij}^k = 1/(C_i R_{ij}^k)$) are the connection matrices, $g(v(t)) = [g_1(v_1(t)), g_2(v_2(t)), \ldots, g_n(v_n(t))]^T$ denotes the neuron activation function with $g(0) = 0$ and $I = [I_1, I_2, \ldots, I_n]^T$ is a constant vector.

Moreover, the $d_k(t)$ are differentiable time-varying bounded delays satisfying:

$$\begin{align*}
(1) \text{ } 0 & \leq d_k(t) \leq \tau_k < \infty, \quad \forall t \geq 0, \quad \text{and} \quad \text{ (2) } 0 \leq \dot{d}_k(t) \leq \mu_k \leq 1, \quad \forall t \geq 0, \\
\end{align*}$$  \hspace{1cm} (2)

with

$$\tau_{\max} = \max_k \tau_k, \quad \mu_{\max} = \max_k \mu_k.$$  

Notice that $\tau_k$ and $\mu_k$, $k = 1, \ldots, K$, are constant scalars.

The system (1) can be modified to take into account the fixed point for the origin, to study the stability of the system at the origin, through an alteration in the fixed point $v^*$ taken to the origin, i.e., $x^* = v - v^*$. Therefore, the system
(1) becomes:
\[
\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + \sum_{k=1}^K W_k f(x(t - d_k(t))),
\]
where \( x = [x_1, x_2, \ldots, x_n]^T \) is the state vector of the transformed system, with \( f(x) = [f_1(x_1), f_2(x_2), \ldots, f_n(x_n)]^T \) and \( f_i(x_i) = g_i(x_i + u_i^s) \) for any \( i = 1, 2, \ldots, n \).

In the field of neural networks, a typical assumption is that the activation functions are continuous, differentiable, monotonically increasing and bounded, such as with the sigmoid type function. Therefore, the activation function of each neuron in (3), \( f_j(\cdot), j = 1, \ldots, n \), satisfies the following assumption.

**Assumption (II).** The activation function \( f(x) \) is bounded and satisfies,
\[
0 \leq \frac{f_j(x) - f_j(y)}{x - y} \leq \sigma_j, \quad j = 1, 2, \ldots, n
\]
for any \( x, y \in \mathbb{R}, x \neq y \), where \( \sigma_j \in \mathbb{R}, 0 < \sigma_j < \infty \), for \( i = 1, 2, \ldots, n \).

The initial conditions of (3) are defined by:
\[
x(t) = \phi(t), \quad \forall t \in [-\tau_{\text{max}}, 0),
\]
where \( \phi(t) \) is a given initial vector function that is differentiable on the segment \([-\tau_{\text{max}}, 0)\).

For simplicity, the following notation is used:
\[
x^d(t) \triangleq [x^T(t - d_1) \ x^T(t - d_2) \ldots x^T(t - d_K)]^T,
\]
\[
W_d \triangleq [W_1 \ W_2 \ldots W_K].
\]

Then the system (3) can be rewritten as:
\[
\begin{align*}
\dot{x}(t) &= -Ax(t) + W_0 f(x(t)) + W_d f(x^d(t)), \\
x(t) &= \phi(t), \quad \forall t \in [-\tau_{\text{max}}, 0). 
\end{align*}
\]

In order to obtain delay-dependent conditions, the Newton–Leibniz formula is usually used:
\[
\int_{t-d(t)}^t \dot{x}(s)ds = x(t) - x(t - d(t)).
\]

The relationship between the state and the delayed state as well as the delay-dependence can be explicitly taken into account in the stability analysis considering the Newton–Leibniz identity above together with slack matrix variables. This strategy is used for reducing conservativeness in the resulting LMIs, without introducing any extra dynamics in the approach as suggested in [31]. This simple trick is stated in the next lemma which is not explicitly presented in [31].

**Lemma 1.** For any appropriately dimensioned matrices \( Y \) and \( T \), the null term
\[
2 \left[ x^T(t)Y + x^T(t - d(t))T \right] \times \left[ x(t) - \int_{t-d(t)}^t \dot{x}(s)ds - x(t - d(t)) \right] = 0
\]
is equivalent to
\[
\xi^T(t) \Gamma \xi(t) - \int_{t-d(t)}^t \xi^T(s) T \xi(t)ds = 0,
\]
where
\[
\xi(t) \triangleq \begin{bmatrix} x(t) \\ x(t - d(t)) \end{bmatrix}, \quad \xi(t, s) \triangleq \begin{bmatrix} x(t) \\ x(t - d(t)) \\ \dot{x}(s) \end{bmatrix}
\]
and
\[
\Gamma \triangleq \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \ast & \Gamma_{22} \end{bmatrix}, \quad T \triangleq \begin{bmatrix} 0 & 0 & -Y \\ 0 & 0 & -T \\ -Y^T & -T^T & 0 \end{bmatrix}
\] (7)

with \( \Gamma_{11} \triangleq Y + Y^T, \Gamma_{12} \triangleq -Y + T^T \) and \( \Gamma_{22} \triangleq -T - T^T \).

**Lemma 2** ([31]). For any appropriately dimensioned semi-positive definite matrix
\[
X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{bmatrix} \geq 0,
\]
the following holds:
\[
\tau \xi^T(t)X\xi(t) - \int_{t-d(t)}^{t} \xi^T(t)X\xi(t)ds \geq 0,
\] (8)

where \( \xi(t) \triangleq [x^T(t) \ x^T(t - d(t))]^T \).

Since one of the main objectives of this paper is also to handle multiple time-delays, Lemmas 1 and 2 are adequately rewritten in the next corollaries.

**Corollary 1.** For any appropriately dimensioned matrices \( Y_k \) and \( T_k \), \( k = 1, \ldots, K \), the null term
\[
\sum_{k=1}^{K} 2 \left[ x^T(t)Y_k + x^T(t - d_k(t))T_k \right] \times \left[ x(t) - \int_{t-d_k(t)}^{t} \dot{x}(s)ds - x(t - d_k(t)) \right] = 0
\] (9)
is equivalent to
\[
\xi^T_d(t)\tilde{\Gamma}\xi_d(t) - \sum_{k=1}^{K} \int_{t-d_k(t)}^{t} \xi_k(t, s)^T Y_k \xi_k(t, s)ds = 0,
\] (10)

where
\[
\tilde{\Gamma} \triangleq \begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} \\ \ast & \tilde{\Gamma}_{22} \end{bmatrix}, \quad \begin{cases} 
\tilde{\Gamma}_{11} \triangleq \sum_{k=1}^{K} Y_k + Y_k^T, \\
\tilde{\Gamma}_{12} \triangleq \left[ -Y_1 + T_1^T - Y_2 + T_2^T \ldots - Y_K + T_K^T \right], \\
\tilde{\Gamma}_{22} \triangleq \text{diag}(-T_k - T_k^T), \quad k = 1, \ldots, K, 
\end{cases}
\]
(11)

\[
T_k \triangleq \begin{bmatrix} 0 & 0 & -Y_k \\ \ast & 0 & -T_k \\ \ast & \ast & 0 \end{bmatrix}
\]
(12)

and \( \xi_d(t) \triangleq [x^T(t) \ x^T(t - d(t))]^T, \xi_k(t, s) \triangleq [x^T(t) \ x^T(t - d_k(t)) \ 
\dot{x}^T(s)]^T \).

**Corollary 2.** For any appropriately dimensioned semi-positive definite matrix
\[
X_k = \begin{bmatrix} \hat{X}_k & \hat{X}_k \\ \ast & \hat{X}_k \end{bmatrix} \geq 0
\] (13)

together with the condition
\[
\eta_k(t) \triangleq \begin{bmatrix} x(t) \\
\dot{x}(t - d_k(t)) \end{bmatrix},
\]
the following holds
\[
\sum_{k=1}^{K} \tau_k \eta_k^T(t)X_k \eta_k(t) - \int_{t-d_k(t)}^{t} \eta_k^T(t)X_k \eta_k(t)ds \geq 0,
\]
or, in an equivalent form
\[
\xi_d^T(t)X_d(t) - \sum_{k=1}^{K} \int_{t-d_k(t)}^{t} \eta_k^T(s) X_k \eta_k(s) \, ds \geq 0,
\]
(14)
where
\[
X \triangleq \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix},
\]
\[
\begin{cases}
X_{11} \triangleq \sum_{k=1}^{K} \tau_k \hat{X}_k, \\
X_{12} \triangleq \begin{bmatrix} \tau_1 \hat{X}_1, \tau_2 \hat{X}_2, \ldots, \tau_K \hat{X}_K \end{bmatrix}, \\
X_{22} \triangleq \text{diag} \{ \tau_k \bar{X}_k \}, \quad k = 1, \ldots, K.
\end{cases}
\]

3. Main results

In this section, the asymptotic stability for neural networks with multiple time-varying delays is considered. The case of guaranteeing stability when the system (5) is uncertain is also analyzed. The main strategy is to add the null term in Corollary 1 to the derivative of the Lyapunov functional to obtain a delay-dependent condition with slack matrix variables. This simple strategy enables us to get a less conservative approach for stability analysis. The first main result is stated in the following theorem.

**Theorem 1.** Let \( \tau_k > 0 \) and \( \mu_k < 1 \), \( k = 1, \ldots, K \), be given scalars for the size of the time-delays and their rates of variations, respectively, and suppose that Assumption (H) holds. The system (5) is asymptotically stable if there exist symmetric positive definite matrices \( P, Q_k, Z_k \), semi-positive definite matrices \( X_k \) as in (13) and \( \chi \) as in (15) and any matrices \( Y_k, T_k \) of appropriate dimensions, such that the following LMIs are satisfied:

\[
\Xi \triangleq V_1 + V_2 + V_3 + \bar{\Gamma} + \chi < 0 
\]
and
\[
\Psi_k \triangleq Y_k + X_k + Z_k \geq 0, \quad k = 1, \ldots, K,
\]
where
\[
V_1 \triangleq \begin{bmatrix} -PA - AP + PW_0 \Sigma_\sigma + \Sigma_\sigma W_0^T P W_d \Sigma_\sigma & PW_0 \Sigma_\sigma \\ * & 0 \end{bmatrix},
\]
(18)
\[
V_2 \triangleq \begin{bmatrix} \sum_{k=1}^{K} Q_k & 0 \\ * & \text{diag} \{ (1 - \mu_k) Q_k \} \end{bmatrix}, \quad k = 1, \ldots, K,
\]
(19)
\[
V_3 \triangleq \sum_{k=1}^{K} \tau_k \begin{bmatrix} (-A + \Sigma_\sigma W_0^T) Z_k (-A + W_0 \Sigma_\sigma) \quad (-A - \Sigma_\sigma W_0^T) Z_k (W_d \Sigma_\sigma) \\ * & (\Sigma_\sigma W_d^T) Z_k (W_d \Sigma_\sigma) \end{bmatrix},
\]
(20)
\[
\bar{\Gamma} \quad \text{and} \quad \chi \quad \text{are defined in (11) and (15), respectively.} \quad Y_k \quad \text{and} \quad X_k \quad \text{are defined in (12) and (13), respectively, and}
\]
\[
Z_k \triangleq \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & Z_k \end{bmatrix},
\]
(21)
\[
\Sigma_\sigma \triangleq \text{diag} \{ \sigma_j \}, \quad j = 1, \ldots, n
\]
(22)
where \( \sigma_j \) are the bounds in (4) in Assumption (H).

**Proof.** Select the following Lyapunov–Krasovskii functional candidate as in [32,33]:
\[
V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t),
\]
(23)
where:

\[ V_1(x_t) \triangleq x^T(t) P x(t), \]

\[ V_2(x_t) \triangleq \sum_{k=1}^{K} \int_{t-d_k(t)}^{t} x^T(s) Q_k x(s) ds, \]

\[ V_3(x_t) \triangleq \sum_{k=1}^{K} \int_{-t_k}^{t} \int_{t+	heta}^{t} \dot{x}^T(s) Z_k \dot{x}(s) ds d\theta \]

and \( P = P^T > 0, Q_k = Q_k^T > 0, Z_k = Z_k^T > 0, k = 1, \ldots, K. \) Taking the time derivative of \( V(x_t) \) in (23) along of the state trajectories, \( x(t) \), in (5) yields

\[ \dot{V}(x_t) = \dot{V}_1(x_t) + \dot{V}_2(x_t) + \dot{V}_3(x_t), \]

where

\[
\dot{V}_1(x_t) = x^T(t) P [-Ax(t) + W_0 f(x(t)) + W_d f(x^d(t))] \\
+ [-x^T(t) A + f^T(x(t)) W_0^T + f^T(x^d(t)) W_d^T] P x(t) \\
\leq x^T(t) P [-Ax(t) + W_0 \Sigma_\sigma(x(t)) + W_d \Sigma_\sigma x^d(t)] \\
+ [-x^T(t) A + \Sigma_\sigma x^T(t) W_0^T + \Sigma_\sigma x^d(t) W_d^T] P x(t) \\
= x^T(t) [-PA - AP + PW_0 \Sigma_\sigma + \Sigma_\sigma W_0 P] x(t) + x^T(t) (PW_d \Sigma_\sigma) x^d(t) + x^dT(t) (\Sigma_\sigma W_d^T P) x(t) \\
= \xi_d^T(t) \Lambda_1 \xi_d(t), \tag{25} \]

\[
\dot{V}_2(x_t) = \sum_{k=1}^{K} \{ -(1 - \dot{k}(t)) x^T(t - d_k(t)) Q_k x(t - d_k(t)) + x^T(t) Q_k x(t) \} \\
\leq \sum_{k=1}^{K} \{ -(1 - \mu_k) x^T(t - d_k(t)) Q_k x(t - d_k(t)) + x^T(t) Q_k x(t) \} \\
= \xi_d^T(t) \Lambda_2 \xi_d(t), \tag{26} \]

\[
\dot{V}_3(x_t) = \sum_{k=1}^{K} \int_{-t_k}^{0} \dot{x}^T(t) Z_k \dot{x}(t) d\theta - \sum_{k=1}^{K} \int_{-t_k}^{0} \dot{x}^T(t + \theta) Z_k \dot{x}(t + \theta) d\theta \\
= \sum_{k=1}^{K} \left\{ t_k \dot{x}^T(t) Z_k \dot{x}(t) - \int_{t-t_k}^{t} \dot{x}^T(s) Z_k \dot{x}(s) ds \right\} \\
= \sum_{k=1}^{K} t_k [-x^T(t) A + f^T(x(t)) W_0^T + f^T(x^d(t)) W_d^T] \\
\times Z_k [-Ax(t) + W_0 f(x(t)) + W_d f(x^d(t))] - \sum_{k=1}^{K} \int_{t-t_k}^{t} \dot{x}^T(s) Z_k \dot{x}(s) ds \\
\leq \sum_{k=1}^{K} t_k [-x^T(t) A + \Sigma_\sigma x^T(t) W_0^T + \Sigma_\sigma x^d(t) W_d^T] \\
\times Z_k [-Ax(t) + W_0 \Sigma_\sigma x(t) + W_d \Sigma_\sigma x^d(t)] - \sum_{k=1}^{K} \int_{t-t_k}^{t} \dot{x}^T(s) Z_k \dot{x}(s) ds \\
\leq \sum_{k=1}^{K} t_k [x^T(t)(-A + \Sigma_\sigma W_0^T) + x^dT(t) \Sigma_\sigma W_d^T] \times Z_k [(-A + W_0 \Sigma_\sigma) + W_d \Sigma_\sigma x^d(t)] \\
- \sum_{k=1}^{K} \int_{t-d_k(t)}^{t} \dot{x}^T(s) Z_k \dot{x}(s) ds \]
\[ \dot{V}(x_t) \leq \tilde{V}_1(x_t) + \tilde{V}_2(x_t) + \sum_{k=1}^{K} 2[x^T(t)Y_k + x^T(t - d_k(t))\tau_k] \]

\[ \times \left[ x(t) - \int_{t-d_k(t)}^{t} \dot{x}(s)ds - x(t - d_k(t)) \right] + \xi_d^T(t)X_k^T(t) - \sum_{k=1}^{K} \left[ \int_{t-d_k(t)}^{t} \eta_k^T(t)X_k\eta_k(t)ds \right. \]

\[ = \xi_d^T(t)\Xi_d(t) - \sum_{k=1}^{K} \left[ \int_{t-d_k(t)}^{t} \xi_d^T(t, s)\Psi_k(t, s)\xi_d(t, s)ds \right. \] 

where the matrices \( \Xi \) and \( \Psi_k \) are defined by (16) and (17), respectively. If \( \Xi < 0, \Psi_k \geq 0, k = 1, \ldots, K \), then \( \dot{V}(x_t) < 0 \) for any \( \xi(t) \neq 0 \). Then, the origin of system (5) is asymptotically stable in the context of the Lyapunov–Krasovskii theory.

### 3.1. Robust stability analysis

Notice that Theorem 1 can be extended to the case of robust stability analysis when the system matrices are not exactly known (see, for instance, [31–33] and references therein).

Suppose that the system matrices in (5) are assembled in the matrix

\[ \mathcal{S} \triangleq \begin{bmatrix} A & W_0 & W_d \end{bmatrix}. \] 

Also consider that the uncertainty belongs to a polytope type uncertain domain \( \mathcal{P} \) defined as the set of all matrices obtained with the convex combination of its vertices:

\[ \mathcal{P}(\gamma) \triangleq \left\{ \mathcal{S} : \mathcal{S} = \sum_{i=1}^{N} \gamma_i\mathcal{S}_i; \gamma \in \Omega \right\}, \]

\[ \Omega \triangleq \left\{ \gamma : \gamma \geq 0, \sum_{i=1}^{N} \gamma_i = 1 \right\}. \]

where \( \mathcal{S}_i, i = 1, \ldots, N, \) are the polytope vertices and the vector \( \gamma = [\gamma_1 \ldots \gamma_N]^T \) parameterizes the polytope.

The robust stability analysis problem is to verify if the system (5) is stable for all \( \mathcal{S} \in \mathcal{P} \). In this case, the easiest sufficient robust stability condition is based on the concept of quadratic stability, where a single Lyapunov–Krasovskii functional is used to prove the stability for the entire polytope. By quadratic stability, the stability of a polytope of matrices can be attested by means of a feasibility test of a set of LMIs involving only the vertices of the uncertainty domain. Then the following theorem states a sufficient condition for the robust stability analysis of the artificial neural networks (ANN) in terms of LMIs.
Figs. 1–3 is stable and any matrices $Y$ holds. The uncertain system is quadratically stable if there exist symmetric definite positive matrices $P, Q_k, Z_k$, semi-definite positive matrices $X_k$ as in (13) and $X$ as in (15) and any matrices $Y_k$ and $T_k$ of appropriate dimensions satisfying the LMIs (16)–(17), where $A, W_0, W_d$ are taken with the superscript $i$, $\forall i = 1, \ldots, N$, where $i$ denotes the vertices of the polytope.

4. Numerical illustrative examples

Example 1. Babcock and Westervelt [3] considered the following simple two-neuron network with two delays:

$$\begin{align*}
\frac{dx_1}{dt} &= -x_1(t) + a_1 \tanh[x_2(t - \tau_1)], \\
\frac{dx_2}{dt} &= -x_2(t) + a_2 \tanh[x_1(t - \tau_2)],
\end{align*}$$

(33)

where $a_1$, $a_2$, $\tau_1$ and $\tau_2$ are positive constants.

In [3] was showed that when $a_1a_2 < -1$ and $\tau_1 + \tau_2$ is sufficiently small, the origin of system (33) is stable and the transients spiral toward it. When the total delay increases through a critical value, the origin becomes unstable and the input and output voltages oscillate in a limit cycle.

Wei and Ruan [4] confirmed their analysis and mathematically proved that the bifurcating periodic solutions are orbitally asymptotically stable considering system (33) with $a_1 = 2, a_2 = -1.5$ and that $\tau_1 + \tau_2 < 0.8$.

Taking the same values for $a_1 = 2$ and $a_2 = -1.5$ and considering $\tau_1 \neq \tau_2$, the largest time-delays are sought when considering the neural network (5) with the following matrices:

$$A = \Sigma_\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_0 = 0, \quad W_1 = \begin{bmatrix} 0 & 0 \\ -1.5 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$  

Applying Theorem 1 one can obtain the data in Table 1, in which the values of $\tau_1$ are related to values of $\tau_2$ and the values of $(\tau_1)$ are related to values of $(\tau_2)$.

The Figs. 1–3 show the diagrams of bifurcation of the neural network for some time-delays picked out in Table 1, the vertical value indicates the maximum upper bound to the time-delays, when fixing one other time-delay, such that the neural network is stable. It is easy to notice that the Hopf bifurcation occurs very near to the largest time-delay achieved. In this case one can verify the effectiveness of the method proposed if the case in that the neural network is subject to multiple time-delays.

Example 2. Consider the same neural network as in (33), but with the parameters $a_1$ and $a_2$ subject to the following variations:

$$1.4 \leq a_1 \leq 2.6,$$

$$-1.515 \leq a_2 \leq -1.48.$$

Then the largest time-delays ($\tau_1$ and $\tau_2$) are sought when considering the uncertain neural network (5) with the following matrices denoting the vertices:

$$A^{(1)} = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 1.3 & 0 \\ 0 & 0.97 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 0.7 & 0 \\ 0 & 1.03 \end{bmatrix}, \quad A^{(4)} = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.03 \end{bmatrix},$$

$$W_1^{(1)} = \begin{bmatrix} 0 & 1.4 \\ 0 & 0 \end{bmatrix}, \quad W_1^{(2)} = \begin{bmatrix} 0 & 2.6 \\ 0 & 0 \end{bmatrix}, \quad W_2^{(1)} = \begin{bmatrix} 0 & 0 \\ -1.515 & 0 \end{bmatrix}, \quad W_2^{(2)} = \begin{bmatrix} 0 & 0 \\ -1.48 & 0 \end{bmatrix}.$$
Fig. 1. Bifurcation diagram with $\tau_1 = 0.1$ fixed and with $\tau_2$ as the control parameter. The vertical line indicates $\tau_2 = 0.7328$.

Fig. 2. Bifurcation diagram with $\tau_1 = 0.3$ fixed and with $\tau_2$ as the control parameter. The vertical line indicates $\tau_2 = 0.5529$.

Fig. 3. Bifurcation diagram with $\tau_1 = 0.4283$ fixed and with $\tau_2$ as the control parameter. The vertical line indicates $\tau_2 = 0.4284$. 
Table 2
Allowable largest time-delays for $\mu = 0$

<table>
<thead>
<tr>
<th>$\tau_1(\tau_2)$</th>
<th>0.05 (0.3515)</th>
<th>0.10 (0.3027)</th>
<th>0.15 (0.2533)</th>
<th>0.20 (0.2034)</th>
<th>0.2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_2(\tau_1)$</td>
<td>0.3515 (0.05)</td>
<td>0.3027 (0.10)</td>
<td>0.2533 (0.15)</td>
<td>0.2034 (0.20)</td>
<td>0.2017</td>
</tr>
</tbody>
</table>

Fig. 4. Bifurcation diagram with $\tau$ as the control parameter. The vertical line indicates $\tau = 1.6409$.

Applying Theorem 2 one can obtain the data in Table 2, in which the values of $\tau_1$ are related to values of $\tau_2$ and the values of $(\tau_1)$ are related to values of $(\tau_2)$.

**Example 3.** Consider a neural network with a higher number of neurons, say, 10 neurons, without self-connection and subject to a single time-delay that can be represented like the neural network (5) with $f(\cdot) = \tanh(\cdot)$ and with the matrices:

$$
A = \Sigma_0 = I_{10} \text{(identity matrix of dimension = 10)}, \quad W_0 = 0,
$$

$$
W_1 = \begin{bmatrix}
0 & -1 & -1 & 0.1 & -0.1 & 0.1 & -0.1 & 0.2 & 0.1 & 0.1 \\
0.1 & 0 & 0.2 & -0.3 & -0.4 & 0.5 & 0.6 & 0.1 & -0.3 & -0.2 \\
0.2 & 0.3 & 0 & 0.1 & 0.3 & 0.1 & 0.1 & 0.2 & 0.3 & 0.1 \\
-0.2 & 0.1 & -0.4 & 0.5 & 0 & 0.1 & 0.1 & 0.2 & 0.3 & 1 \\
0.1 & 0 & 0.6 & 0.3 & 0.4 & 0.1 & 0 & 0.5 & -0.1 & -0.2 & -0.1 \\
-0.2 & 0.3 & -0.4 & 0.1 & 0.6 & 0.5 & 0 & 0.1 & 0.1 & 0.5 \\
0.1 & 0.2 & -0.3 & 0.4 & 0.1 & 0.3 & -0.5 & 0 & -0.1 & -0.1 \\
0.7 & -0.1 & 0.1 & 0.1 & -0.2 & 0.1 & 0.1 & 0.3 & 0 & 1 \\
0.1 & 0.1 & -0.2 & 0.3 & 0.1 & 0.1 & -0.6 & 0.1 & -0.4 & 0
\end{bmatrix}.
$$

Applying Theorem 1 one can obtain the largest time-delay, $\tau = 1.6409$, such that the artificial neural network (ANN) is stable.

Figs. 4–8 show the diagrams of bifurcation of the neural network for all the neurons of this ANN. Through these diagrams, one can verify that the proposed method is efficient and, e.g., the diagram of bifurcation in the Fig. 6 shows that the maximum time-delay, $\tau = 1.6409$, is very close to the original Hopf bifurcation.

**5. Conclusions**

The problem of asymptotic stability analysis of artificial neural networks with multiple time-varying delays subject to polytope-bounded uncertainties has been discussed. An LMI based approach to the asymptotic stability analysis was
Fig. 5. Bifurcation diagram with $\tau$ as the control parameter. The vertical line indicates $\tau = 1.6409$.

Fig. 6. Bifurcation diagram with $\tau$ as the control parameter. The vertical line indicates $\tau = 1.6409$.

Fig. 7. Bifurcation diagram with $\tau$ as the control parameter. The vertical line indicates $\tau = 1.6409$.
Fig. 8. Bifurcation diagram with $\tau$ as the control parameter. The vertical line indicates $\tau = 1.6409$.

derived through the selection of an appropriate Lyapunov–Krasovskii functional and the selection of slack matrices that express the influence of the Newton–Leibniz condition. The results obtained, as illustrated by the examples, suggest that the proposed approach is a good option to check stability of artificial neural networks with multiple time-delays.

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References