Delay-Dependent Robust $\mathcal{H}_\infty$ Control of Uncertain Linear Systems with Time-Varying Delays

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(Received January 2004; revised and accepted April 2005)

Abstract—This paper proposes an optimization-based approach to the robust $\mathcal{H}_\infty$ control problem of uncertain continuous or discrete-time linear time-invariant systems with different time-varying delays in the state vector and control input of the dynamic equation and controlled output. Sufficient delay-dependent conditions are derived for the control stabilization problem, where both the size of the time-varying delay and the size of its time derivative (in the continuous-time case) play a crucial role for the closed-loop stability with a guaranteed $\mathcal{H}_\infty$ performance index. The solutions that are proposed for the $\mathcal{H}_\infty$ control problem are similar or less conservative, when compared to other recent approaches © 2005 Elsevier Ltd All rights reserved

Keywords—Time delay, Robust stability, Optimization

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The authors are grateful to the Brazilian agencies CNPq and CAPES, and to the anonymous reviewers for their comments and suggestions that helped them improve the presentation of the paper.
1. INTRODUCTION

The control design problem for systems subject to time delay in the state vector and/or control input has received a lot of attention in the last decades. The increasing interest about this topic can be understood by the fact that time delays appear as an important source of instability or performance degradation in a great number of important engineering problems involving material, information or energy transportation. In the context of basic sciences, it is not hard to find mathematical (non)linear models of real phenomena with time delay in circuits theory, economics, biology and mechanics, or more specifically, in chemical process, hydraulic, rolling mill, or computer controlled systems, and more recently Internet (or in a more general setting, time delays due to the signal communications in communication channels).

Most of the approaches in literature concentrate on dealing with constant time delays. However, in some practical systems, time delays are varying in consequence of rapid (or even random) variations in transmission delays or motion of separated systems of master-slave type. In telecommunications systems, the presence of transmission delay variations is usually unavoidable, e.g., congestion-control algorithms, internet rerouting, internet-based telerobotic systems, or network traffic [1-5]. Other classes of problems involving time-varying delays are biomedical and robotic systems [6] and cellular neural networks [7].

Besides the stability analysis of systems with time delay itself, many other subjects have been incorporated to the control design problem: parameter uncertainties, \( H_\infty \) performance index, multiple time delays, delay-independent conditions, delay-dependent conditions, lumped and distributed delay cases, and optimization techniques involving LMI descriptions. Commonly, most of the approaches employ only the traditional delay-independent condition, in which the controller design is provided irrespective of the size of the time delay. On the other hand, it is well known that for systems in which the stability issue depends explicitly on the time delay, a delay-independent condition may not work. In fact, in order to overcome this difficulty, it is necessary to evidence the time delay in the control design to obtain a delay-dependent condition.

The reader can find a great number of references about this topic in literature. The following references systemize the main ideas, [8-20].

In the scenario of continuous-time \( H_\infty \) control, [10,16,21] deal with linear systems with constant time delays in the state. For systems with time-varying delays, [17,22] consider time delays in the state and control input. Also, [10,16,17,21] are delay-dependent approaches whereas [12,22,23] are delay-independent approaches where no explicit information regarding the size of the time delay but only a fixed upper bound on the time derivative appears in the LMIs.

Unlike the continuous-time case, there exist a few papers handling the robust \( H_\infty \) control problem of discrete-time systems with time-varying delays in the state [24-26]. The LMIs approaches derived in [24-26] are of delay-dependent type and based on standard Lyapunov-Krasovskii functionals for discrete-time systems. A delay-independent approach for discrete-time systems is presented in [12].

In this paper, the main contribution is to state LMI sufficient delay-dependent conditions for the robust state-feedback control stabilization design, which guarantees an \( H_\infty \) level of disturbance attenuation for both uncertain continuous or discrete-time state- and control-delayed systems with different time-varying delays. The parameter uncertainties dealt with are of polytopic type, which allows to extend the LMI results obtained for precisely known systems for a set where the vertices are elements of LMI type. The main results for continuous or discrete-time systems take as initial point a recent work [11], which is based on a new upper bound for the inner product of two vectors, and deals with the stabilization problem for systems with constant time delays. An early short version of this paper has appeared in [27] without addressing the problem of different time delays entering the state and control input vectors. Finally, for the continuous-time case one example is presented and the results are compared with recent approaches [10,16,17,19,21].
Regarding the discrete-time case, one example is also handled and the results are compared with [24,25].

The notation used in this paper is as follows. \(\dot{x}(t)\) indicates \(\dot{x}(t)\) for continuous-time systems or \(x(t + 1)\) for discrete-time systems. \(L_2\) denotes both the spaces \(L_2\) (set of all square integrable and Lebesgue measurable functions defined on a given interval) or \(l_2\) (the discrete-time context of sequences). The boldface characters \(I\) and \(0\) denote, respectively, the identity and the null matrices of convenient sizes. \(\tau_{1,2}(t)\) is used to denote \(\tau_1(t)\) and \(\tau_2(t)\), the time-varying delays of the system.

2. DELAY-DEPENDENT CONTINUOUS-TIME CASE

Consider the following linear time-invariant dynamic system with time delay,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_d x(t - \tau_1(t)) + Bu(t) + B_d u(t - \tau_2(t)) + Ew(t), \\
z(t) &= Cx(t) + C_d x(t - \tau_1(t)) + Du(t) + D_d u(t - \tau_2(t)) + Fw(t),
\end{align*}
\]

where \(x(t) : \mathbb{R} \to \mathbb{R}^n\) is the state vector, \(u(t) : \mathbb{R} \to \mathbb{R}^m\) is the control input vector, \(w(t) : \mathbb{R} \to \mathbb{R}^p\) is the exogenous disturbance vector and \(z(t) : \mathbb{R} \to \mathbb{R}^q\) is the controlled output. \(\phi(t)\) is a given initial vector function which is continuous on the segment \([-\bar{\tau},0)\), with \(\bar{\tau}\) defined as \(\max[\tau_1(t),\tau_2(t)]\), and \(\tau_1(t)\) and \(\tau_2(t)\) are the time-varying delays of the system. Assume perfect state measurement. Moreover, the time-varying delays satisfy the following conditions for continuous-time,

\[
0 \leq \tau_{1,2}(t) \leq \bar{\tau}_{1,2} < \infty, \quad 0 \leq \bar{\tau}_{1,2}(t) \leq \xi_{1,2} < 1, \quad \forall t \in \mathbb{R},
\]

or the following one for discrete-time,

\[
0 \leq \tau_{1,2}(t) \leq \bar{\tau}_{1,2} < \infty, \quad \forall t \in \mathbb{Z}.
\]

For control purposes, we consider the memoryless state-feedback control law,

\[u(t) = Kx(t)\]

Then, the closed-loop state-delayed system is given by

\[
\begin{align*}
\dot{x}(t) &= \tilde{A}x(t) + A_d x(t - \tau_1(t)) + B_d Kx(t - \tau_2(t)) + Ew(t), \\
z(t) &= \tilde{C}x(t) + C_d x(t - \tau_1(t)) + D_d Kx(t - \tau_2(t)) + Fw(t),
\end{align*}
\]

with \(\tilde{A} = A + BK\) and \(\tilde{C} = C + DK\).

The main control problem to be addressed in this section is stated below.

\((P_{\infty})\) — THE ROBUST \(H_\infty\) CONTROL PROBLEM WITH TIME DELAY. Given scalars \(\bar{\tau}_{1,2}\) and \(\xi_{1,2}\) satisfying (2) (or (3) for discrete-time systems) and scalar \(\gamma > 0\), determine a controller gain \(K\) such that the closed-loop state-delayed system (4) is robustly stable and ensures a prescribed \(H_\infty\) disturbance attenuation for any time-varying delays satisfying (2) (or (3) for discrete-time systems), namely, under zero initial conditions and for any nonzero \(w \in L_2\),

\[
\|z\|_{L_2} \leq \gamma \|w\|_{L_2}, \quad \forall (A,A_d,B,B_d,C,C_d,D,D_d,E,F) \in \mathcal{P},
\]

where \(\mathcal{P}\) is a polytopic set described by \(\kappa\) vertices,

\[
\mathcal{P} \triangleq \left\{ (A_1,\ldots,F) \mid (A_1,\ldots,F) = \sum_{i=1}^{\kappa} \xi_i (A_{i1},\ldots,F_i) ; \xi_i \geq 0; \sum_{i=1}^{\kappa} \xi_i = 1 \right\}.
\]

In this situation, the closed-loop system is said to be robustly stable with \(\gamma\) disturbance attenuation level.

The above notion of robust stability for uncertain system means that (4) with \(w(t) \equiv 0\) is robustly stable if its trivial solution \(x(t) \equiv 0\) is globally uniformly asymptotically stable for all admissible uncertainties and the time delays \(\tau_{1,2}(t)\) satisfy (2) or (3).
3. ANALYSIS AND SYNTHESIS FOR CONTINUOUS-TIME SYSTEMS

In the following are presented the delay-dependent LMI robust \( H_\infty \) performance analysis and control design for continuous-time systems subject to different time delays in the state and control input.

**THEOREM 1.** Consider the closed-loop system (4) and let \( \tau_{1,2}, \xi_{1,2}, \) and \( \gamma > 0 \) be given scalars, with \( \tau_{1,2} \) and \( \xi_{1,2} \) satisfying (2). If there exist symmetric matrices \( X > 0, H_{1,2} > 0, Q_{1,2} > 0, \) and matrices \( V_{1,2} \) satisfying

\[
\begin{bmatrix}
\gamma & \gamma & \gamma & \gamma & \gamma \\
X & 0 & 0 & 0 & 0 \\
0 & c_1 A_c^T Z_1 & 0 & 0 & 0 \\
0 & 0 & c_2 K^T B_{d1} Z_1 & 0 & 0 \\
0 & 0 & 0 & c_2 K^T B_{d2} Z_2 & 0 \\
0 & 0 & 0 & 0 & c_2 K^T D_{d1} \\
\end{bmatrix} < 0, \quad \forall i = 1, \ldots, \kappa, (6)
\]

\[
\begin{bmatrix}
H_1 & V_1 \\
V_1^T & Z_1 \\
\end{bmatrix} \succeq 0, \quad \begin{bmatrix}
H_2 & V_2 \\
V_2^T & Z_2 \\
\end{bmatrix} \succeq 0,
\]

where

\[
\begin{align*}
T_{111} & \triangleq X A_1 + A_1^T X + \tau_1 H_1 + \tau_2 H_2 + V_1 + V_1^T + V_2 + V_2^T + \frac{1}{1 - \xi_{1}} Q_1 + \frac{1}{1 - \xi_{2}} Q_2, \\
T_{12} & \triangleq X A_d - V_1, \\
T_{13} & \triangleq X B_{d1} K - V_2, \\
e_1 & \triangleq \frac{\tau_1}{1 - \xi_{1}}, \\
e_2 & \triangleq \frac{\tau_2}{1 - \xi_{2}}.
\end{align*}
\]

Then, the closed-loop system is robustly stable with disturbance attenuation \( \gamma \) for any time-varying delay satisfying (2).

**PROOF** Consider the Leibniz-Newton identity,

\[
\int_{a}^{b} \dot{\nu}(t) \, dt = \nu(b) - \nu(a). \quad (8)
\]

Then, system (4) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= (\bar{A} + A_d + B_d K) x(t) - A_d \int_{t-\tau_1(t)}^{t} \dot{x}(\alpha) \, d\alpha - B_d K \int_{t-\tau_2(t)}^{t} \dot{x}(\alpha) \, d\alpha + E \omega(t), \\
z(t) &= \bar{C} x(t) + C_d x(t - \tau_1(t)) + D_d K x(t - \tau_2(t)) + F \omega(t).
\end{align*}
\]

Let the Lyapunov-Krasovskii functional be

\[
V(x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), t) = V_1 + V_2 + V_3 + V_4 + V_5, \quad (10)
\]
with

\[ V_1 = x^T(t) X x(t), \]
\[ V_2 = \frac{1}{1 - \tau_1(t)} \int_{-\tau_1(t)}^0 \int_{t+\beta}^t \dot{x}^T(\alpha) Z_1 \dot{x}(\alpha) \, d\alpha \, d\beta, \]
\[ V_3 = \frac{1}{1 - \tau_1(t)} \int_{-\tau_1(t)}^t x^T(\alpha) Q_1 x(\alpha) \, d\alpha, \]
\[ V_4 = \frac{1}{1 - \tau_2(t)} \int_{-\tau_2(t)}^0 \int_{t+\beta}^t \dot{x}^T(\alpha) Z_2 \dot{x}(\alpha) \, d\alpha \, d\beta, \]
\[ V_5 = \frac{1}{1 - \tau_2(t)} \int_{-\tau_2(t)}^t x^T(\alpha) Q_2 x(\alpha) \, d\alpha. \]

Taking the time-derivative of the functional (10), it follows that

\[ \dot{V}(\cdot) = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 + \dot{V}_5, \] (11)

with

\[ \dot{V}_1 = 2x^T(t) X (\dot{A} + Ad + BdK) x(t) + 2x^T(t) X E w(t) \]
\[ - 2x^T(t) X A_d \int_{t-\tau_1(t)}^t \dot{x}(\alpha) \, d\alpha \] (12)
\[ - 2x^T(t) X B_d K \int_{t-\tau_1(t)}^t \dot{x}(\alpha) \, d\alpha, \] (13)
\[ \dot{V}_2 = \frac{\tau_1(t)}{1 - \tau_1(t)} \dot{x}^T(t) Z_1 \dot{x}(t) - \int_{t-\tau_1(t)}^t \dot{x}^T(\alpha) Z_1 \dot{x}(\alpha) \, d\alpha \]
\[ \leq \frac{\tau_1(t)}{1 - \tau_1(t)} \dot{x}^T(t) Z_1 \dot{x}(t) \] (14)
\[ - \int_{t-\tau_1} \dot{x}^T(\alpha) Z_1 x(\alpha) \, d\alpha, \] (15)
\[ \dot{V}_3 = \frac{1}{1 - \tau_1(t)} x^T(t) Q_1 x(t) - x^T(t - \tau_1(t)) Q_1 x(t - \tau_1(t)), \]
\[ \dot{V}_4 = \frac{\tau_2(t)}{1 - \tau_2(t)} \dot{x}^T(t) Z_2 \dot{x}(t) - \int_{t-\tau_2(t)}^t \dot{x}^T(\alpha) Z_2 \dot{x}(\alpha) \, d\alpha \]
\[ \leq \frac{\tau_2(t)}{1 - \tau_2(t)} \dot{x}^T(t) Z_2 \dot{x}(t) \] (16)
\[ - \int_{t-\tau_2} \dot{x}^T(\alpha) Z_2 x(\alpha) \, d\alpha, \] (17)
\[ \dot{V}_5 = \frac{1}{1 - \tau_2(t)} x^T(t) Q_2 x(t) - x^T(t - \tau_2(t)) Q_2 x(t - \tau_2(t)). \]

Using the upper bound of the inner product of two vectors introduced in [11] for (12) and (13), it follows that

\[ (12) \leq \tau_1(t) x^T(t) H_1 x(t) \]
\[ + 2x^T(t)(V_1 - X A_d) \int_{t-\tau_1(t)}^t x(\alpha) \, d\alpha \] (18)
\[ + \int_{t-\tau_1} \dot{x}^T(\alpha) Z_1 x(\alpha) \, d\alpha, \]
\[ (13) \leq \tau_2(t) x^T(t) H_2 x(t) \]
\[ + 2x^T(t) (V_2 - XB_d K) \int_{t-\tau_2(t)}^{t} \dot{x}(\alpha) \, d\alpha \]
\[ + \int_{t-\tau_2}^{t} \dot{x}^T(\alpha) Z_2 \dot{x}(\alpha) \, d\alpha. \]

The above upper bounds hold if the matrices \( H_{1,2}, V_{1,2}, \) and \( Z_{1,2} \) satisfy the constraints,
\[
\begin{bmatrix} H_1 & V_1 \\ V_1^T & Z_1 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} H_2 & V_2 \\ V_2^T & Z_2 \end{bmatrix} \succeq 0, \tag{22}
\]

After straightforward manipulations of terms (14), (16), (18), and (20), elimination of terms (15), (17), (19), and (21), and taking into account \( \tau_{1,2} \) and \( \varsigma_{1,2} \), the upper bounds to the size of the time delay and its variation rate, respectively, one may rewrite (11) as
\[
\dot{V}(\cdot) \leq \begin{bmatrix} x(t) \\ x(t-\tau_1(t)) \\ x(t-\tau_2(t)) \\ w(t) \end{bmatrix}^T A_c \begin{bmatrix} x(t) \\ x(t-\tau_1(t)) \\ x(t-\tau_2(t)) \\ w(t) \end{bmatrix}, \tag{23}
\]
where
\[
A_c \triangleq \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & \Pi_{34} \\ \Pi_{41} & \Pi_{42} & \Pi_{43} & \Pi_{44} \end{bmatrix},
\]
\[
\Pi_{11} \triangleq X\tilde{A} + \tilde{A}^T X + \tau_1 H_1 + \tau_2 H_2 + V_1 + V_1^T + V_2 + V_2^T + \frac{1}{1-\varsigma_1} Q_1 + \frac{1}{1-\varsigma_2} Q_2 + e_1 \tilde{A}^T Z_1 \tilde{A} + e_2 \tilde{A}^T Z_2 \tilde{A},
\]
\[
\Pi_{12} \triangleq X A_d - V_1 + e_1 \tilde{A}^T Z_1 A_d + e_2 \tilde{A}^T Z_2 A_d,
\]
\[
\Pi_{13} \triangleq X B_d K - V_2 + e_1 \tilde{A}^T Z_1 B_d K + e_2 \tilde{A}^T Z_2 B_d K,
\]
\[
\Pi_{14} \triangleq X E + e_1 \tilde{A}^T Z_1 E + e_2 \tilde{A}^T Z_2 E,
\]
\[
\Pi_{22} \triangleq -Q_1 + e_1 A_d^T Z_1 A_d + e_2 A_d^T Z_2 A_d,
\]
\[
\Pi_{23} \triangleq e_1 A_d^T Z_1 B_d K + e_2 A_d^T Z_2 B_d K,
\]
\[
\Pi_{24} \triangleq e_1 A_d^T Z_1 E + e_2 A_d^T Z_2 E,
\]
\[
\Pi_{33} \triangleq -Q_2 + e_1 K^T B_d^T Z_1 B_d K + e_2 K^T B_d^T Z_2 B_d K,
\]
\[
\Pi_{34} \triangleq e_1 K^T B_d^T Z_1 E + e_2 K^T B_d^T Z_2 E,
\]
\[
\Pi_{44} \triangleq e_1 E^T Z_1 E + e_2 E^T Z_2 E,
\]
\[
e_1 \triangleq \frac{\tau_1}{1-\varsigma_1},
\]
\[
e_2 \triangleq \frac{\tau_2}{1-\varsigma_2},
\]
when \( H_{1,2}, V_{1,2}, \) and \( Z_{1,2} \) satisfy (22).

Now, considering the \( H_\infty \) performance index,
\[
J = \int_0^\infty [z^T(t) z(t) - \gamma^2 w^T(t) w(t)] \, dt,
\]
and assuming, without loss of generality, zero initial conditions to (4) where \( V(\cdot)|_{t=0} = 0 \) and \( V(\cdot)|_{t=\infty} \to 0 \), the above index may be rewritten as
\[
J \leq \int_0^\infty [z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \hat{V}(\cdot)] \, dt
\]
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and

\[
J \leq \int_0^\infty \begin{bmatrix} x(t) \\ x(t - \tau_1(t)) \\ x(t - \tau_2(t)) \\ w(t) \end{bmatrix}^T \Delta_c \begin{bmatrix} x(t) \\ x(t - \tau_1(t)) \\ x(t - \tau_2(t)) \\ w(t) \end{bmatrix} dt,
\]

(26)

where

\[
\Delta_c \equiv \Delta_c + \begin{bmatrix} \bar{C}^T C & \bar{C}^T C_d & \bar{C}^T D_d K & \bar{C}^T F \\ \ast & C_d^T C_d & C_d^T D_d K & C_d^T F \\ \ast & \ast & K^T D_d^T D_d K & K^T D_d^T F \\ \ast & \ast & \ast & -\gamma^2 I + F^T F \end{bmatrix},
\]

which is equivalent to the inequality below, after applying the Schur’s complement,

\[
\begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 & X & E & e_1 \bar{A}^T Z_1 & e_2 \bar{A}^T Z_2 & \bar{C}^T \\ \ast & -Q_1 & 0 & 0 & e_1 A_d^T Z_1 & e_2 A_d^T Z_2 & C_d^T \\ \ast & \ast & -Q_2 & 0 & e_1 KT B_d^T Z_1 & e_2 KT B_d^T Z_2 & K^T D_d^T \\ \ast & \ast & \ast & -\gamma^2 I & e_1 E^T Z_1 & e_2 E^T Z_2 & F^T \\ \ast & \ast & \ast & \ast & -e_1 Z_1 & 0 & 0 \\ \ast & \ast & \ast & \ast & -e_2 Z_2 & 0 & 0 \\ \ast & \ast & \ast & \ast & \ast & \ast & -I \\ \end{bmatrix} < 0,
\]

\[
\begin{align*}
\Upsilon_1 & \equiv X \bar{A} + \bar{A}^T X + \bar{\tau}_1 H_1 + \bar{\tau}_2 H_2 + V_1 + V_1^T + V_2 + V_2^T + \frac{1}{1 - \varsigma_1} Q_1 + \frac{1}{1 - \varsigma_2} Q_2, \\
\Upsilon_2 & \equiv X A_d - V_1, \\
\Upsilon_3 & \equiv X B_d K - V_2, \\
e_1 & \equiv \frac{\bar{\tau}_1}{1 - \varsigma_1}, \\
e_2 & \equiv \frac{\bar{\tau}_2}{1 - \varsigma_2},
\end{align*}
\]

when \( H_{1,2}, V_{1,2}, \) and \( Z_{1,2} \) satisfy (22).

Considering that the LMIs (6) ensure that \( \Delta_c < 0 \) for the entire uncertain domain \( \mathcal{P} \), one can conclude that \( J < 0 \) for all nonzero \( w(t) \in \mathcal{L}_2 \). Thus, system (4) is guaranteed to be robustly stable with \( \mathcal{H}_\infty \) disturbance attenuation level \( \gamma \) for any time delay condition that satisfy (2).

Based on the \( \mathcal{H}_\infty \) stability analysis developed above the next theorem can be derived. It must be noticed that the \( \mathcal{H}_\infty \) analysis result presented in Theorem 1 is an extension of the new stability analysis result derived in [11].

**THEOREM 2.** Consider system (1) and let \( \bar{\tau}_{1,2}, \varsigma_{1,2} \) and \( \gamma > 0 \) be given scalars, with \( \bar{\tau}_{1,2} \) and \( \varsigma_{1,2} \) satisfying (2). If there exist symmetric matrices \( Y > 0, M_{1,2} > 0, W_{1,2} > 0, R_{1,2} > 0, \) and matrices \( L, N_{1,2}, \) satisfying

\[
\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & E_3 & e_1 \Psi_{15} & e_2 \Psi_{16} & \Psi_{17} \\ * & -W_1 & 0 & 0 & e_1 YA_{d1} & e_2 YA_{d1} & Y C_{d1} \\ * & -W_2 & 0 & 0 & e_1 L^T B_{d1} & e_2 L^T B_{d1} & L^T D_{d1} \\ * & \ast & \ast & -\gamma^2 I & e_1 E_3 & e_2 E_3 & F^T \\ * & \ast & \ast & -e_1 R_1 & 0 & 0 \\ * & \ast & \ast & -e_2 R_2 & 0 & 0 \\ * & \ast & \ast & \ast & \ast & \ast & -I \\ \end{bmatrix} < 0, \quad \forall i = 1, \ldots, \kappa,
\]

(27)

\[
\begin{bmatrix} M_1 & N_1 \\ N_1^T & Y R_1^{-1} Y \end{bmatrix} \succeq 0, \quad \begin{bmatrix} M_2 & N_2 \\ N_2^T & Y R_2^{-1} Y \end{bmatrix} \succeq 0,
\]

(28)
where
\[ \Psi_{11} \triangleq A_i Y + Y A_i^T + B_i L + L^T \tilde{B}_i^T + \tau_1 M_1 + \tau_2 M_2 + N_1 + N_1^T + N_2 + N_2^T + \frac{1}{1 - \varsigma_1} W_1 + \frac{1}{1 - \varsigma_2} W_2, \]
\[ \Psi_{12} \triangleq A_d Y - N_1, \]
\[ \Psi_{13} \triangleq B_d L - N_2, \]
\[ \Psi_{15} \triangleq Y A_i^T + L^T \tilde{B}_i^T, \]
\[ \Psi_{16} \triangleq Y A_i^T + L^T \tilde{B}_i^T, \]
\[ \Psi_{17} \triangleq Y C_i^T + L^T D_i^T, \]
\[ e_1 \triangleq \frac{\tau_1}{1 - \varsigma_1}, \]
\[ e_2 \triangleq \frac{\tau_2}{1 - \varsigma_2}, \]

then problem \((P_d)\) is solvable for any time-varying delay satisfying (2), with the control gain \( K = L Y^{-1} \).

**Proof.** Pre- and post-multiplying the LMI (6) by
\[ \text{diag} [X^{-1}, X^{-1}, X^{-1}, I, Z_1^{-1}, Z_2^{-1}, I], \]
with \( \tilde{A}_i \) and \( \tilde{C}_i \) replaced by \( A_i + B_i K \) and \( C_i + D_i K \), respectively, one gets

\[
\begin{bmatrix}
\tilde{T}_{11} & \tilde{T}_{12} & \tilde{T}_{13} & E_i \\
\ast & -X^{-1}Q_1X^{-1} & 0 & 0 \\
\ast & \ast & -X^{-1}Q_2X^{-1} & 0 \\
\ast & \ast & \ast & -\gamma^2 I \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{bmatrix}
\]

\[ \begin{bmatrix}
e_1(X^{-1}A_i^T + X^{-1}K^T \tilde{B}_i^T)
& e_2(X^{-1}A_i^T + X^{-1}K^T \tilde{B}_i^T)
& X^{-1}C_i^T + X^{-1}K^T D_i^T \\
e_1X^{-1}A_{di}
& e_2X^{-1}A_{di}
& X^{-1}C_{di}
& X^{-1}K^T D_{di} \\
e_1X^{-1}K^T \tilde{B}_{di}
& e_2X^{-1}K^T \tilde{B}_{di}
& X^{-1}K^T D_{di} \\
e_1E_i^T
& e_2E_i^T
& F_i^T
& 0 \\
e_1Z_1^{-1}
& -e_2Z_1^{-1}
& 0
& -I \\
\ast & \ast & \ast & \ast
\end{bmatrix} < 0, \]

where
\[
\tilde{T}_{11} \triangleq A_i X^{-1} + X^{-1}A_i^T + B_i K X^{-1} + X^{-1}K^T \tilde{B}_i^T + \tau_1 X^{-1} H_1 X^{-1} + \tau_2 X^{-1} H_2 X^{-1} + X^{-1}V_1 X^{-1} + X^{-1}V_1^T X^{-1} + X^{-1}V_2 X^{-1} + \frac{1}{1 - \varsigma_1} V_1 X^{-1} + \frac{1}{1 - \varsigma_2} V_2 X^{-1},
\]
\[
\tilde{T}_{12} \triangleq A_d X^{-1} - X^{-1} V_1 X^{-1},
\]
\[
\tilde{T}_{13} \triangleq B_d K X^{-1} - X^{-1} V_2 X^{-1},
\]
\[ e_1 \triangleq \frac{\tau_1}{1 - \varsigma_1}, \]
\[ e_2 \triangleq \frac{\tau_2}{1 - \varsigma_2}. \]
Pre- and post-multiplying (7) by \( \text{diag}[X^{-1}, X^{-1}] \),

it follows

\[
\begin{bmatrix}
X^{-1}H_2X^{-1} & X^{-1}V_{1}X^{-1} \\
X^{-1}V_{1}^{\top}X^{-1} & X^{-1}Z_1X^{-1}
\end{bmatrix} \succeq 0,
\begin{bmatrix}
X^{-1}H_2X^{-1} & X^{-1}V_{2}X^{-1} \\
X^{-1}V_{2}^{\top}X^{-1} & X^{-1}Z_2X^{-1}
\end{bmatrix} \succeq 0.
\]

Introducing the change of variables: \( Y = X^{-1}, M_{1,2} = X^{-1}H_{1,2}X^{-1}, N_{1,2} = X^{-1}V_{1,2}X^{-1}, W_{1,2} = X^{-1}Q_{1,2}X^{-1}, R_{1,2} = Z_{1,2} \), and \( L = KK^{-1} \), the inequalities (27) and (28) are obtained.

The last theorem ensures sufficient conditions for obtaining a robust \( \mathcal{H}_\infty \) controller. It must be noticed that the matrix inequalities (27),(28) are not LMIs, in fact Theorem 2 states a nonconvex problem. In order to overcome this one can either do the simple linearization \( R = Y \), which turns (27),(28) into LMI conditions, but is somehow conservative, or choose to proceed with the same cone complementarity linearization algorithm, proposed in [28,29] (and similar to [11]) replacing (28) with

\[
\begin{bmatrix}
M_{1,2} & N_{1,2} \\
* & S_{1,2}
\end{bmatrix} \succeq 0,
\begin{bmatrix}
\hat{S}_{1,2} & \hat{Y}_{1,2} \\
* & \hat{R}_{1,2}
\end{bmatrix} \succeq 0,
\begin{bmatrix}
S_{1,2} & I \\
* & \hat{S}_{1,2}
\end{bmatrix} \succeq 0,
\begin{bmatrix}
Y & I \\
* & \hat{Y}_{1,2}
\end{bmatrix} \succeq 0,
\begin{bmatrix}
R_{1,2} & I \\
* & \hat{R}_{1,2}
\end{bmatrix} \succeq 0.
\]

In this context, the following algorithm is proposed.

1. For \( \tau_{1,2}, \zeta_{1,2} \) and a disturbance attenuation level \( \gamma > 0 \) given, with \( \tau_{1,2} \) and \( \zeta_{1,2} \) satisfying (2). Set \( k = 0 \). Find a feasible set of matrices \( (S_{1,2}^0, Y^0, R_{1,2}^0, \hat{S}_{1,2}^0, \hat{Y}_{1,2}^0, \hat{R}_{1,2}^0) \) satisfying (27) and (33).

2. Solve the problem,

\[
\min_{M_{1,2}, W_{1,2}, N_{1,2}, L, S_{1,2}, Y, R_{1,2}, \hat{S}_{1,2}, \hat{Y}_{1,2}, \hat{R}_{1,2} \atop \text{subject to} \ (27) \ and \ (33),} \ \text{Trace} \ \{f^k\}
\]

where

\[
f^k \equiv \left( \hat{S}_{1}^k S_{1} + S_{1}^k \hat{S}_{1} + \hat{Y}_{1}^k Y + Y^k \hat{Y}_{1} + \hat{R}_{1}^k R_{1} + R_{1}^k \hat{R}_{1} \right) + \left( \hat{S}_{2}^k S_{2} + S_{2}^k \hat{S}_{2} + \hat{Y}_{2}^k Y + Y^k \hat{Y}_{2} + \hat{R}_{2}^k R_{2} + R_{2}^k \hat{R}_{2} \right)
\]

3. If \( \text{Trace} \ \{f^k\} \rightarrow \{ (3n_1) + (3n) \} \) and condition (28) holds, then there exists an \( \mathcal{H}_\infty \) controller, \( K = LY^{-1} \), which ensures the disturbance attenuation level \( \gamma \). If (28) is not verified, set \( \tau_{1,2}^{k+1} = \hat{\tau}_{1,2}, \zeta_{1,2}^{k+1} = S_{1,2}, \hat{Y}_{1,2}^{k+1} = \hat{Y}_{1,2}, Y_{1,2}^{k+1} = Y, \hat{R}_{1,2}^{k+1} = R_{1,2}, \hat{S}_{1,2}^{k+1} = \hat{S}_{1,2}, \hat{R}_{1,2}^{k+1} = \hat{R}_{1,2}, \hat{S}_{2,2}^{k+1} = \hat{S}_{2,2}, \hat{R}_{2,2}^{k+1} = \hat{R}_{2,2} \) solutions of the optimization problem in Step 2. Set \( k = k + 1 \), if \( k < k_{\text{max}} \) (where \( k_{\text{max}} \) is the number of maximum iterations) return to Step 2, otherwise stop.

Related to this algorithm, two paths can be implemented. The first one is concerned with finding the maximum time delay \( \bar{\tau} \). For that, an additional information must be introduced at Step 3, namely, if the condition (28) holds, then the time delay \( \bar{\tau} \) can be increased returning to Step 2. The second one deals with the minimization of the disturbance attenuation level, \( \gamma \), i.e., for \( \bar{\tau} \) fixed, one can find the minimum \( \gamma \) by implementing any line search algorithm and proceeding in the same way as indicated at Steps 2 and 3. In all these situations, the upper bound to the time-derivative of the time-varying delay, \( \zeta \), is fixed.

Moreover, if an initial \( \gamma \) is required, one can compute the minimum disturbance attenuation level, \( \gamma \), by taking (27),(28) with the linear change of variables \( R = Y \), which is an LMI problem for \( \bar{\tau} \) and \( \zeta \) given, with \( \bar{\tau} \) and \( \zeta \) satisfying (2). Thus, solving this linearized problem can be seen as a starting point for the above algorithm. Nevertheless, it must be noted that this linearized problem could be infeasible for increasing time delay, say \( \tau_{\text{m}} > \bar{\tau} \), but for that same \( \tau_{\text{m}} \), the algorithm above can work
4. ANALYSIS AND SYNTHESIS
FOR DISCRETE-TIME SYSTEMS

In the following are presented the delay-dependent LMI robust $\mathcal{H}_\infty$ performance analysis and control design for discrete-time systems subject to different time delays in the state and control input.

**Theorem 3.** Consider the closed-loop system (4) and let $\bar{\tau}, \gamma > 0$ be given scalars, with $\bar{\tau}$ satisfying (3). If there exist symmetric matrices $X > 0$, $H_1, H_2 > 0$, $Q_1, Q_2 > 0$, and matrices $V_1, V_2$ satisfying

$$
\begin{bmatrix}
\Gamma_1 & -V_1 & -V_2 \\
* & -Q_1 & 0 \\
* & * & -Q_2
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_1 & Z_1 \\
\tilde{A}_2 & Z_2
\end{bmatrix}
\begin{bmatrix}
\tilde{G}_1 \\
\tilde{G}_2
\end{bmatrix}
< 0,
$$

where

$$
\Gamma_1 = -X + \tau_1 H_1 + \tau_2 H_2 + V_1 + V_1^T + V_2 + V_2^T + Q_1 + Q_2,
$$

$$
\Gamma_2 = (\tilde{A}_1 - I)^T,
$$

$$
\Gamma_3 = (D_d K)^T,
$$

Then, the closed-loop system is robustly stable with disturbance attenuation $\gamma$ for any time-varying delay satisfying (3).

**Proof.** Consider the identity,

$$
\sum_{k=t-\tau(t)}^{t-1} \Delta x(k) = x(t) - x(t - \tau(t)),
$$

where $\Delta x(k) = x(k + 1) - x(k)$.

System (4) can be rewritten as

$$
x(t + 1) = (\tilde{A} + A_d + B_d K) x(t) - A_d \sum_{k=t-\tau_1(t)}^{t-1} \Delta x(k) - B_d K \sum_{k=t-\tau_2(t)}^{t-1} \Delta x(k) + E w(t),
$$

$$
z(t) = C x(t) + C_d x(t - \tau_1(t)) + D_d K x(t - \tau_2(t)) + F w(t).
$$

Let the Lyapunov-Krasovskii functional be

$$
V(x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), w(t), t) = V_1 + V_2 + V_3 + V_4 + V_5,
$$

with

$$
V_1 = x^T(t) X x(t),
$$

$$
V_2 = \sum_{s=-\tau(t)}^{t-1} \sum_{k=t+s}^{t-1} \Delta x^T(k) Z_1 \Delta x(k),
$$
\[ V_3 = \sum_{s=t-r_2(t)}^{t-1} \sum_{k=t+s}^{t-1} \Delta x^T(k) Z_2 \Delta x(k), \]
\[ V_4 = \sum_{k=t-r_1(t)}^{t-1} x^T(k) Q_1 x(k), \]
\[ V_5 = \sum_{k=t-r_2(t)}^{t-1} x^T(k) Q_2 x(k). \]

Taking the difference of the functional (38) it follows that
\[ \Delta V(\cdot) = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4 + \Delta V_5, \] (39)
with
\[ \Delta V_1 = x^T(t) \left[ (\tilde{A} + A_d + B_d K)^T X (\tilde{A} + A_d + B_d K) - X \right] x(t) \]
\[ - 2x^T(t) (\tilde{A} + A_d + B_d K)^T X A_d \sum_{k=t-r_1(t)}^{t-1} \Delta x(k) \]
\[ - 2x^T(t) (\tilde{A} + A_d + B_d K)^T X B_d K \sum_{k=t-r_2(t)}^{t-1} \Delta x(k) \]
\[ + \left( \sum_{k=t-r_1(t)}^{t-1} \Delta x^T(k) \right) A_d^T X A_d \left( \sum_{k=t-r_1(t)}^{t-1} \Delta x(k) \right) \]
\[ + \left( \sum_{k=t-r_2(t)}^{t-1} \Delta x^T(k) \right) (B_d K)^T X (B_d K) \left( \sum_{k=t-r_2(t)}^{t-1} \Delta x(k) \right) \]
\[ + 2 \left( \sum_{k=t-r_1(t)}^{t-1} \Delta x^T(k) \right) A_d^T X (B_d K) \left( \sum_{k=t-r_2(t)}^{t-1} \Delta x(k) \right) \]
\[ - 2 \left[ \sum_{k=t-r_1(t)}^{t-1} \Delta x^T(k) A_d^T + \sum_{k=t-r_2(t)}^{t-1} \Delta x^T(k) (B_d K)^T \right] X E w(t) \]
\[ + 2x^T(t) (\tilde{A} + A_d + B_d K)^T X E w(t) + w^T(t) E^T X E w(t), \] (46)
\[ \Delta V_2 = \tau_1(t) \Delta x^T(t) Z_1 \Delta x(t) - \sum_{k=t-r_1(t)}^{t-1} \Delta x^T(k) Z_1 \Delta x(k) \]
\[ \leq \tau_1(t) \Delta x^T(t) Z_1 \Delta x(t), \] (47)
\[ - \sum_{k=t-r_1(t)}^{t-1} \Delta x^T(k) Z_1 \Delta x(k), \] (48)
\[ \Delta V_3 = \tau_2(t) \Delta x^T(t) Z_2 \Delta x(t) - \sum_{k=t-r_2(t)}^{t-1} \Delta x^T(k) Z_2 \Delta x(k) \]
\[ \leq \tau_2(t) \Delta x^T(t) Z_2 \Delta x(t), \] (49)
\[ - \sum_{k=t-r_2(t)}^{t-1} \Delta x^T(k) Z_2 \Delta x(k), \] (50)
\[ \Delta V_4 = x^T(t) Q_1 x(t) - x^T(t - r_1(t)) Q_1 x(t - r_1(t)), \]
\[ \Delta V_5 = x^T(t) Q_2 x(t) - x^T(t - r_2(t)) Q_2 x(t - r_2(t)). \]
Using the upper bound of the inner product of two vectors introduced in [11] for (40) and (41) it follows that

\[(40) \leq \tau_1(t) x^T(t) H_1 x(t) + 2x^T(t) \left[ V_1 - \left( \tilde{A} + A_d + B_d K \right)^T X A_d \right] \sum_{k=t-\tau_1(t)}^{t-1} \Delta x(k) + \sum_{k=t-\tau_1}^{t-1} \Delta x^T(k) Z_1 \Delta x(k), \]

\[(41) \leq \tau_2(t) x^T(t) H_2 x(t) + 2x^T(t) \left[ V_2 - \left( \tilde{A} + A_d + B_d K \right)^T X (B_d K) \right] \sum_{k=t-\tau_2(t)}^{t-1} \Delta x(k) + \sum_{k=t-\tau_2}^{t-1} \Delta x^T(k) Z_2 \Delta x(k). \]

The above upper bounds hold if the matrices $H_{1,2}$, $V_{1,2}$, and $Z_{1,2}$ satisfy the constraints

\[
\begin{bmatrix}
H_1 & V_1 \\
V_1^T & Z_1
\end{bmatrix} \succeq 0, \quad \begin{bmatrix}
H_2 & V_2 \\
V_2^T & Z_2
\end{bmatrix} \succeq 0
\]

After straightforward manipulations of terms (42)-(47), (49), (51), and (53), elimination of terms (48), (50), (52), and (54), and taking into account $\tau_{1,2}$, the upper bounds to the size of the time delay, one may rewrite (39) as

\[
\Delta V(\cdot) \leq \begin{bmatrix}
x(t) \\
x(t-\tau_1(t)) \\
x(t-\tau_2(t)) \\
w(t)
\end{bmatrix}^T \Lambda_d \begin{bmatrix}
x(t) \\
x(t-\tau_1(t)) \\
x(t-\tau_2(t)) \\
w(t)
\end{bmatrix},
\]

where

\[
\Lambda_d \triangleq \begin{bmatrix}
\Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} \\
\ast & \Xi_{22} & \Xi_{23} & \Xi_{24} \\
\ast & \ast & \Xi_{33} & \Xi_{34} \\
\ast & \ast & \ast & \Xi_{44}
\end{bmatrix},
\]

\[
\Xi_{ij} \triangleq \tilde{A}^T X \tilde{A} - \tilde{A} + \tau_1 H_1 + \tau_2 H_2 + V_1 + V_1^T + V_2 + V_2^T + Q_1 + Q_2 + \tau_1 \left( \tilde{A} - I \right)^T Z_1 \left( \tilde{A} - I \right) + \tau_2 \left( \tilde{A} - I \right)^T Z_2 \left( \tilde{A} - I \right),
\]

\[
\Xi_{12} \triangleq -V_1 + \tilde{A}^T X A_d + \tau_1 \left( \tilde{A} - I \right)^T Z_1 A_d + \tau_2 \left( \tilde{A} - I \right)^T Z_2 A_d,
\]

\[
\Xi_{13} \triangleq -V_2 + \tilde{A}^T X B_d K + \tau_1 \left( \tilde{A} - I \right)^T Z_1 B_d K + \tau_2 \left( \tilde{A} - I \right)^T Z_2 B_d K,
\]

\[
\Xi_{14} \triangleq \tilde{A}^T X E + \tau_1 \left( \tilde{A} - I \right)^T Z_1 E + \tau_2 \left( \tilde{A} - I \right)^T Z_2 E,
\]

\[
\Xi_{22} \triangleq -Q_1 + A_d^T X A_d + \tau_1 A_d^T Z_1 A_d + \tau_2 A_d^T Z_2 A_d,
\]

\[
\Xi_{23} \triangleq A_d^T X B_d K + \tau_1 A_d^T Z_1 B_d K + \tau_2 A_d^T Z_2 B_d K,
\]

\[
\Xi_{24} \triangleq A_d^T X E + \tau_1 A_d^T Z_1 E + \tau_2 A_d^T Z_2 E,
\]

\[
\Xi_{33} \triangleq -Q_2 + (B_d K)^T X B_d K + \tau_1 (B_d K)^T Z_1 B_d K + \tau_2 (B_d K)^T Z_2 B_d K,
\]

\[
\Xi_{34} \triangleq (B_d K)^T X E + \tau_1 (B_d K)^T Z_1 E + \tau_2 (B_d K)^T Z_2 E,
\]

\[
\Xi_{44} \triangleq E^T X E + \tau_1 E^T Z_1 E + \tau_2 E^T Z_2 E,
\]
when \( H_{1,2}, V_{1,2} \in Z_{1,2} \) satisfy (22). Now, considering the \( \mathcal{H}_\infty \) performance index,

\[
\mathcal{J} = \sum_{t=0}^{\infty} \left[ z^T(t) z(t) - \gamma^2 w^T(t) w(t) \right],
\]

and assuming, without loss of generality, zero initial conditions and stability to system (4), it implies \( V(\cdot)|_{t=0} = 0 \) and \( V(\cdot)|_{t=\infty} \to \epsilon \), with \( \epsilon \to 0 \) if \( w(t) = 0 \) or \( \epsilon < \infty \) if \( w(t) \neq 0 \). In this way, the above index may be rewritten as

\[
\mathcal{J} \leq \sum_{t=0}^{\infty} \left[ z^T(t) z(t) - \gamma^2 w^T(t) w(t) + \Delta V(\cdot) \right],
\]

moreover,

\[
\mathcal{J} \leq \sum_{t=0}^{\infty} \left[ \begin{array}{c}
x(t) \\
x(t - \tau_1(t)) \\
x(t - \tau_2(t)) \\
w(t)
\end{array} \right]^T \Delta_d \left[ \begin{array}{c}
x(t) \\
x(t - \tau_1(t)) \\
x(t - \tau_2(t)) \\
w(t)
\end{array} \right],
\]

where

\[
\Delta_d \triangleq \Delta_d + \begin{bmatrix}
\bar{C}^T C & \bar{C}^T D d K & \bar{C}^T F \\
* & C_d^T C_d & C_d^T D d K & C_d^T F \\
* & * & K_d^T D d K & K_d^T D_d F \\
* & & * & -\gamma^2 I + F^T F
\end{bmatrix},
\]

which is equivalent to the inequality below, after applying the Schur’s complement,

\[
\Delta_d \triangleq \begin{bmatrix}
\Gamma_1 & -V_1 & -V_2 & 0 & \bar{A}^T X & \bar{r}_1 \Gamma_2 Z_1 & \bar{r}_2 \Gamma_2 Z_2 & \bar{C}^T \\
* & -Q_1 & 0 & 0 & \Gamma_1 A_d Z_1 & \bar{r}_1 A_d Z_2 & \bar{C}_d^T \\
* & * & -Q_2 & 0 & \Gamma_3 X & \bar{r}_2 \Gamma_3 Z_1 & \bar{r}_2 \Gamma_3 Z_2 & \Gamma_4 \\
* & * & * & -\gamma^2 I & E^T X & \bar{r}_1 E^T Z_1 & \bar{r}_2 E^T Z_2 & F^T \\
* & * & * & * & -X & 0 & 0 & 0 \\
* & * & * & * & * & -\bar{r}_1 Z_1 & 0 & 0 \\
* & * & * & * & * & * & -\bar{r}_2 Z_2 & 0 \\
* & * & * & * & * & * & * & -I
\end{bmatrix},
\]

with

\[
\Gamma_1 \triangleq -X + \bar{r}_1 H_1 + \bar{r}_2 H_2 + V_1 + V_1^T + V_2 + V_2^T + Q_1 + Q_2,
\]

\[
\Gamma_2 \triangleq (\bar{A} - I)^T,
\]

\[
\Gamma_3 \triangleq (B_d K)^T,
\]

\[
\Gamma_4 \triangleq (D_d K)^T,
\]

when \( H_{1,2}, V_{1,2}, \) and \( Z_{1,2} \) satisfy (22).

Considering that the LMIs (34) ensure that \( \Delta_d < 0 \) for the entire uncertain domain \( \mathcal{P} \), one can conclude that \( \mathcal{J} < 0 \) for all nonzero \( w(t) \in \ell_2 \). Thus, system (4) is guaranteed to be robustly stable with \( \mathcal{H}_\infty \) disturbance attenuation level \( \gamma \) for any time delay condition that satisfy (3).

Next, the discrete-time synthesis version is presented and the same cone algorithm indicated in the last section can be used.

**THEOREM 4.** Consider system (1) and let \( \bar{r}_{1,2} \) and \( \gamma > 0 \) be given scalars, with \( \bar{r}_{1,2} \) satisfying (3). If there exist symmetric matrices \( Y > 0 \), \( M_{1,2} > 0 \), \( W_{1,2} > 0 \), \( R_{1,2} > 0 \), and matrices \( L, N_{1,2}, \)
satisfying
\[
\begin{bmatrix}
\Phi_{11} & -N_1 & -N_2 & 0 & & & \\
* & -W_1 & 0 & 0 & & & YC_{d_1} \\
* & * & -W_2 & 0 & & & YC_{d_1} \\
* & * & * & -\gamma I & \tilde{E}_1^T & \tilde{E}_2^T & \tilde{F}_1^T \\
* & * & * & * & -\gamma I & \tilde{E}_1^T & \tilde{F}_1^T \\
* & * & * & * & * & -\gamma I & \tilde{F}_1^T \\
* & * & * & * & * & * & -I
\end{bmatrix}
\prec 0, \quad \forall i = 1, \ldots, \kappa, \quad (60)
\]

where
\[
\Phi_{11} \triangleq -Y + \tau_1 M_1 + \tau_2 M_2 + N_1 + N_1^T + N_2 + N_2^T + W_1 + W_2,
\]
\[
\Phi_{151} \triangleq YA_i^T + L^T B_i^T,
\]
\[
\Phi_{161} \triangleq YA_i^T + L^T B_i^T - Y,
\]
\[
\Phi_{171} \triangleq YA_i^T + L^T B_i^T - Y,
\]
\[
\Phi_{181} \triangleq YC_i^T + L^T D_i^T,
\]
\[
\Phi_{21} \triangleq YA_i^T,
\]
\[
\Phi_{23} \triangleq L^T B_i^T,
\]
then problem \((P_{d_1}^L)\) is solvable for any time-varying delay satisfying \((3)\), with the control gain \(K = LY^{-1}\).

**Proof.** Pre- and post-multiplying the LMI (34) by
\[
\text{diag} \begin{bmatrix} X^{-1}, X^{-1}, X^{-1}, I, X^{-1}, Z_1^{-1}, Z_2^{-1}, I \end{bmatrix},
\]
with \(\tilde{A}_i\) and \(\tilde{C}_i\) replaced by \(A_i + B_i K\) and \(C_i + D_i K\), respectively, one gets
\[
\begin{bmatrix}
\tilde{r}_1 & -X^{-1}V_1 X^{-1} & -X^{-1}V_2 X^{-1} & 0 & \tilde{r}_2, \\
* & -X^{-1}Q_1 X^{-1} & 0 & 0 & X^{-1}A_{d_1}^T \\
* & * & -X^{-1}Q_2 X^{-1} & 0 & X^{-1}K^T B_{d_1}^T \\
* & * & * & -\gamma I & E_1^T \\
* & * & * & * & -X^{-1} \\
* & * & * & * & * \\
* & * & * & * & *
\end{bmatrix}
\prec 0,
\]

\[
\begin{bmatrix}
\tilde{r}_1 (\tilde{r}_2 - X^{-1}) & \tilde{r}_2 (\tilde{r}_2 - X^{-1}) & X^{-1} C_i^T + X^{-1} K^T D_i^T \\
\tau_1 X^{-1} A_{d_1}^T & \tau_2 X^{-1} A_{d_1}^T & X^{-1} C_i^T \\
\tau_1 X^{-1} K^T B_{d_1}^T & \tau_2 X^{-1} K^T B_{d_1}^T & X^{-1} K^T D_i^T \\
\tau_1 E_1^T & \tau_2 E_1^T & F_1^T \\
0 & 0 & 0 \\
-\gamma I & 0 & 0 \\
* & -\gamma I & 0 \\
* & * & -I
\end{bmatrix}
\prec 0,
\]

\[
\begin{bmatrix}
M_1 & N_1 \\
N_1^T & Y R_1^{-1} Y
\end{bmatrix} \succeq 0, \quad \begin{bmatrix}
M_2 & N_2 \\
N_2^T & Y R_2^{-1} Y
\end{bmatrix} \succeq 0,
\]

\[
(M_1 N_1^T - Y R_1^{-1} Y) \succeq 0, \quad (61)
\]
where
\[
\hat{\Gamma}_1 \triangleq -X^{-1} + \tau_1 X^{-1} H_1 X^{-1} + \tau_2 X^{-1} H_2 X^{-1} + X^{-1} V_1 X^{-1} + X^{-1} V_1^T X^{-1} + X^{-1} V_2 X^{-1} + X^{-1} V_2^T X^{-1} + X^{-1} Q_1 X^{-1} + X^{-1} Q_2 X^{-1},
\]
\[
\hat{\Gamma}_2 \triangleq X^{-1} A_i + X^{-1} B_i^T B_i^T.
\]

Pre- and post-multiplying (35) by
\[
\text{diag} [X^{-1}, X^{-1}],
\]
it follows that
\[
\begin{bmatrix}
X^{-1} H_1 X^{-1} & X^{-1} V_1 X^{-1} \\
X^{-1} V_1^T X^{-1} & X^{-1} Z_1 X^{-1}
\end{bmatrix} \succeq 0,
\]
\[
\begin{bmatrix}
X^{-1} H_2 X^{-1} & X^{-1} V_2 X^{-1} \\
X^{-1} V_2^T X^{-1} & X^{-1} Z_2 X^{-1}
\end{bmatrix} \succeq 0
\]

Introducing the change of variables, \( Y \equiv X^{-1}, M_{1,2} \equiv X^{-1} H_{1,2} X^{-1}, N_{1,2} \equiv X^{-1} V_{1,2} X^{-1}, W_{1,2} \equiv X^{-1} Q_{1,2} X^{-1}, R_{1,2} \equiv Z_{1,2}^{-1}, \) and \( L \equiv K X^{-1}, \) the inequalities (60) and (61) are obtained.

5. EXAMPLES

**EXAMPLE 1.** Consider the same example as shown in [16,17,19,21], with the following matrices for system (1).

\[
A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]
\[
C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D = 0.1, \quad D_d = 0, \quad F = 0.
\]

For this example, the time-derivative of the time delay is \( \dot{\tau}_1 \equiv 0, \) so \( \zeta_1 = 0, \) as exactly considered in the above references, where the LMIs approaches had also been developed to deal with linear systems with time-varying delays.

Searching for the maximum time delay for which there exists an \( \mathcal{H}_\infty \) stabilizing controller, in [16], the maximum delay achieved is \( \tau_{16} = 1.28 \text{ s}, \) with \( \gamma_{16} = 0 \) and \( K_{16} = \begin{bmatrix} 0 & -130.38 \end{bmatrix}. \)

Now, in [17] (using Theorem 5), one can find as maximal time delay \( \tau_{17} = 1.408 \text{ s} \) and \( \gamma_{17} = 106.1506, \)

with
\[
K_{17} = \begin{bmatrix} -156.36 & -1439.66 \end{bmatrix}.
\]

In [19] (using Theorem 2), one can find the same results as presented in [17]

On the other hand, applying the approach presented here, even for the time delay \( \tau = 5.0 \text{ s} \) one is able to find a stabilizing controller,
\[
K_{\text{Nonconvex}} = \begin{bmatrix} -102.4825 & -136.66 \end{bmatrix},
\]

with \( \gamma = 22 \) (although some computational effort is required). However, as can be seen, increasing the value of the maximum time delay implies a degradation of the disturbance attenuation level.
Figure 1 Evolution of the state variables of the closed-loop system in Example 1, for $\tau = 5.0\,s$, considering the control $K_{\text{Nonconvex}}$.

Figure 2 State evolution of the closed-loop system on Example 1 with $K_{[19]}$ and simulated for the time-delay $\tau = 5.0\,s$.

Figure 3 State evolution of the closed-loop system in Example 1, simulated for the time-delay $\tau = 1.408\,s$ with $K_{[19]}$ (solid line), and $K_{\text{Nonconvex}}$ (dashdot line) which has been obtained for $\tau = 5.0\,s$.

Figure 1 depicts the time-response of the system we considered above, for the control gain $K_{\text{Nonconvex}}$, closing the loop, and the time delay $\tau = 5.0\,s$.

As a second illustration, Figure 2 depicts the time-response of the closed-loop system with the state-feedback control gain $K_{[19]}$, when considering the time-delay $\tau = 5.0\,s$. Notice that these controllers do not ensure the closed-loop stability for that size of time-delay.

Finally, Figure 3 depicts the time-response of the closed-loop system for two cases. The first one takes into account the control gain $K_{[19]}$ obtained for the maximal time delay $\bar{\tau}_{[19]} = 1.408\,s$, which is represented by the solid line for the two state variables. The second one considers the control gain $K_{\text{Nonconvex}}$, obtained for the time delay $\tau = 5.0\,s$, however simulated for the same $\tau = 1.408\,s$, and represented by the dashdot line. It is clear that better performance was obtained with $K_{\text{Nonconvex}}$. 
EXAMPLE 2. Consider the same example as in [25], with matrices $C_d = 0$ and $F = 0$, where the discrete-time system with time-varying delay is given by

$$x(t + 1) = \begin{bmatrix} -0.5 & 1 \\ 0 & 0.2 \end{bmatrix} x(t) + \begin{bmatrix} -0.05 \\ 0 \end{bmatrix} x(t - \tau(t))$$

$$+ \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0.3 \end{bmatrix} w(t),$$

$$z(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \end{bmatrix} u(t).$$

Figure 4 depicts the histogram of the time-varying delay, and let $\bar{\tau} = 5$ s be the upper bound for the time-varying delay.

With $\tau$ fixed, Table 1 presents the minimum disturbance attenuation level, $\gamma$, obtained from the approach proposed in this paper, and two other ones in [24,25]. From Table 1 one can note the improvement in terms of the disturbance attenuation level $\mathcal{H}_\infty$, for $\bar{\tau}$ fixed.

<table>
<thead>
<tr>
<th>Approach</th>
<th>$\gamma$</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Proposed)</td>
<td>1.9703</td>
<td>$[0.3703 - 1.1163]$</td>
</tr>
<tr>
<td>[25]</td>
<td>2.2388</td>
<td>$[0.4046 - 0.1762]$</td>
</tr>
<tr>
<td>[24]</td>
<td>2.3999</td>
<td>$[0.4044 - 0.1762]$</td>
</tr>
</tbody>
</table>

The simulation results for the proposed approach are depicted in Figures 5 and 6 with zero initial conditions and the varying-time delay evolution in Figure 4. The disturbance signal is
Figure 5 State evolution of the closed loop system, Example 2.

Figure 6 Evolution of the disturbance signal, $w$ (dotted line), and the controlled output signal, $z$ (solid line), Example 2.

defined as

$$w(t) = \begin{cases} 
2, & \text{if } 10 \leq t \leq 30, \\
-2, & \text{if } 60 \leq t \leq 80, \\
0, & \text{otherwise}.
\end{cases}$$

The state evolution is presented in Figure 5, and Figure 6 illustrates the relation between the disturbance signal and the controlled output signal. It is easy to note that $\gamma = 1.9703$ holds, for this disturbance signal.
On the other hand, suppose that the disturbance attenuation level, $\gamma$, is fixed, say $\gamma = 5$, and one looks for the maximum upper bound to the time-varying delay. Table 2 shows the maximum upper bound, $\bar{\tau}$, allowed when considering the proposed approach and two other ones in [24,25]. It is clear that the proposed approach relaxed the upper bound

<table>
<thead>
<tr>
<th>Approach</th>
<th>$\bar{\tau}$ (s)</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Proposed)</td>
<td>$1 \times 10^6$</td>
<td>$[-0.9850, -0.8813]$</td>
</tr>
<tr>
<td>[25]</td>
<td>172</td>
<td>$[0.7492, 1.0387]$</td>
</tr>
<tr>
<td>[24]</td>
<td>172</td>
<td>$[0.7485, 1.0373]$</td>
</tr>
</tbody>
</table>

6. CONCLUSIONS

This paper addressed the robust $\mathcal{H}_\infty$ control problem for uncertain linear systems with different time-varying delays in the state and input vectors. Both the size of the time-varying delay as well as its time-derivative were considered in the approach. Further, a delay-dependent sufficient condition was presented allowing to handle the robust control problem in an LMI-based iterative algorithm. As a particular characteristic of the proposed approach, the original system was not increased as other technique in the literature called descriptor system approach [16]. The behavior and efficiency of the control design approach had been illustrated by means of two examples for continuous and discrete-time systems.

REFERENCES

17. E. Fridman and U. Shaked, Stability and $\mathcal{H}_\infty$ control of systems with time-varying delays, *Proceedings of the 15th IFAC Triennial World Congress*, Barcelona, Spain, (July 2002)


