On Monotonic Inclusion Interval Uninorms

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In this paper, some results about interval uninorms with the additional property of monotonicity inclusion are introduced; e.g construction of interval uninorms from usual uninorms and constructions of interval t-norms and t-conorms from interval uninorms. It is also shown that the neutral element of this type of interval uninorm must be a degenerate interval.

Keywords: Uninorms; Neutral Element; Interval Fuzzy.

Introduction

A uninorm operator (concept introduced in\(^1\)) is an aggregation function \(U : [0, 1] \times [0, 1] \rightarrow [0, 1]\), that is commutative, associative, monotonic and has a neutral element. The difference between uninorms and t-norms/t-conorms is that the neutral element can be any number \(e \in [0, 1]\). Of course, if \(e = 1\), then \(U\) is a t-norm and if \(e = 0\), then \(U\) is a t-conorm. The uninorms has applications in several areas (see\(^3\)). For example, in multicriteria decision making, an uninorm can be used to aggregate the evaluation of alternatives, where the neutral element represents a level of satisfaction (see\(^3\)). Some aggregation functions on \([0, 1]\) have been used in fuzzy logic.
as logical connectives (t-norms are conjunctions and t-conorms are disjunctions). Thinking about applications of fuzzy logic, we can realize that determination of membership degrees could have some kind of uncertainty, since different specialists can assign different degrees. The interval fuzzy logic (or equivalently intuitionistic fuzzy logic, see\(^4\)) introduced independently in\(^7-10\) deal with this uncertainty by using intervals (pairs of real numbers) as membership degrees. Interval Uninorms were introduced in\(^11\), but in that paper, only the monotonicity by product order (or Kulish-Miranker order, see\(^12\)) is required. In this paper, we show some natural ways to obtain an interval uninorm from usual ones, and we note that this interval uninorms also are monotonic with respect to \(\subseteq\). So, we only consider uninorms that are \(\subseteq\)-monotonic and follow the ideas of\(^13,14\). We show that in this kind of uninorm, the neutral element is a degenerate interval. We also show a way to obtain interval \(\subseteq\)-monotonic t-norms and t-conorms from interval uninorms.

1. Interval Uninorms

The set of all intervals will be denoted by \(\mathbb{I}\) and the set of all intervals whose the ends lies on the unit interval \([0, 1]\) by \(\mathbb{I}[0, 1]\). If \(X \in \mathbb{I}\), we use the notation \(X = [x, \pi]\). We use the interval operations introduced in\(^15\), the inclusion order \(\subseteq\) and the Kulschk-Miranker order (or product order) \(\leq_{km}\) defined by \([x, \pi] \leq_{km} [y, \eta] \iff x \leq y\) and \(\pi \leq \eta\).

**Definition 1.1.** A function \(U : \mathbb{I}[0, 1] \times \mathbb{I}[0, 1] \rightarrow \mathbb{I}[0, 1]\) is a \(\subseteq\)-monotonic interval uninorm (i-uninorm for short) if satisfies:

1. \(U(X, Y) = U(Y, X)\); (Commutativity)
2. \(U(X, U(Y, Z)) = U(U(X, Y), Z)\); (Associativity)
3. \(U(X, Y) \leq_{km} U(Z, W)\) if \(X \leq_{km} Z\) and \(Y \leq_{km} W\); (\(\leq_{km}\)-Monotonicity)
4. \(U(X, Y) \subseteq U(Z, W)\) if \(X \subseteq Z\) and \(Y \subseteq W\); (\(\subseteq\)-Monotonicity)
5. There exists an element \(E \in [0, 1]\) such that \(U(E, X) = X\), for all \(X \in [0, 1]\). (Neutral Element)

**Theorem 1.1.** If \(U_1\) and \(U_2\) are two uninorms such that \(U_1(x, y) \leq U_2(x, y)\), for all \(x, y \in [0, 1]\) and it’s neutral elements are equals\(^a\) have

\(^a\)This requirement allows that the neutral element be well defined. Since \(U_1(x, y) \leq U_2(x, y)\), if \(e_1\) and \(e_2\) are the neutral elements of \(U_1\) and \(U_2\), we must have \(U_1(e_1, e_2) = e_2\) and \(U_2(e_1, e_2) = e_1\), so \(e_2 \leq e_1\) and as will be seen, we must have \(e_1 \leq e_2\).
seen then the function $U : I[0, 1] \times I[0, 1] \rightarrow I[0, 1]$ defined by $U(X, Y) = [U_1(x, y), U_2(x, y)]$ is a i-uninorm.

Proof. The first two conditions of the i-uninorm definition are immediate. For the third condition, suppose that $X \leq_{km} Z$ and $Y \leq_{km} W$, ie, $\underline{x} \leq \underline{z}$, $\underline{x} \leq \underline{z}$, $\underline{y} \leq \underline{w}$ and $\underline{y} \leq \underline{w}$. By the monotonicity of the usual uninorms, we have $U_1(\underline{x}, \underline{y}) \leq U_1(\underline{z}, \underline{w})$ and $U_2(\underline{x}, \underline{y}) \leq U_2(\underline{z}, \underline{w})$, so $[U_1(\underline{x}, \underline{y}), U_2(\underline{x}, \underline{y})] \leq_{km} [U_1(\underline{z}, \underline{w}), U_2(\underline{z}, \underline{w})]$ which means that $U(X, Z) \leq_{km} U(Y, W)$. The $\subseteq$-monotonicity (condition 4) is proved analogously. For the last condition, take $E = [e, e]$ such that $e$ is the common neutral element of $U_1$ and $U_2$. Thus, for every $X \in I[0, 1]$, we have $U(E, X) = [U_1(e, x), U_2(e, x)] = [x, x] = X$. 

In this last theorem, we consider that the neutral elements of $U_1$ and $U_2$ are the same. With this assumption the i-uninorms have degenerate intervals as neutral elements. In the next results we prove that every i-uninorm have this property.

Lemma 1.2. Let $U : I[0, 1] \times I[0, 1] \rightarrow I[0, 1]$ be a i-uninorm with neutral element $E$.

1. If $E \leq_{km} X$, then $Y \leq_{km} U(X, Y)$, for all $Y \in I[0, 1]$;
2. If $X \leq_{km} E$, then $U(X, Y) \leq_{km} Y$, for all $Y \in I[0, 1]$;
3. If $E \subseteq X$, then $Y \subseteq U(X, Y)$, for all $Y \in I[0, 1]$;
4. If $X \subseteq E$, then $U(X, Y) \subseteq Y$, for all $Y \in I[0, 1]$.

Proof.

1. Since $E \leq_{km} X$, we have $Y = U(E, Y) \leq_{km} U(X, Y)$, so $Y \leq_{km} U(X, Y)$;
2. Since $X \leq_{km} E$, we have $U(X, Y) \leq_{km} U(E, Y) = Y$, so $Y \leq_{km} U(X, Y)$;
3. Since $E \subseteq X$, we have $Y = U(E, Y) \subseteq U(X, Y)$, so $Y \subseteq U(X, Y)$;
4. Since $X \subseteq E$, we have $U(X, Y) \subseteq U(E, Y) = Y$, so $U(X, Y) \subseteq Y$. 

Lemma 1.3. Let $U : I[0, 1] \times I[0, 1] \rightarrow I[0, 1]$ be a i-uninorm with neutral element $E = [\underline{x}, \underline{y}]$.

1. $U([\underline{x}, \underline{x}], [x, x]) = [x, x]$, for all $x \in [0, 1]$;
2. $U([\underline{y}, \underline{y}], [x, x]) = [x, x]$, for all $x \in [0, 1]$.

Proof.
(1) Since \([\overline{r}, \overline{r}] \subseteq [\overline{e}, \overline{e}]\), the above lemma guarantees that \(U([\overline{r}, \overline{r}], [x, x]) \subseteq U(E, [x, x]) = [x, x]\). Thus, \(U([\overline{r}, \overline{r}], [x, x])\) is an interval and also a subset of \([x, x]\), so \(U([\overline{r}, \overline{r}], [x, x]) = [x, x]\).

(2) Since \([\overline{e}, \overline{e}] \subseteq [\overline{e}, \overline{e}]\), we have \(U([\overline{e}, \overline{e}], [x, x]) \subseteq U(E, [x, x]) = [x, x]\). Thus, \(U([\overline{e}, \overline{e}], [x, x]) = [x, x]\).

Theorem 1.4. If \(U : [0, 1] \times [0, 1] \rightarrow [0, 1]\) is a \(i\)-uninorm, then its neutral element must to be a degenerated interval.

Proof. Suppose that \(E = [\overline{e}, \overline{e}]\). Follows from the lemma 1.3 \(U([\overline{e}, \overline{e}], [\overline{e}, \overline{e}] = [\overline{e}, \overline{e}]\) and \(U([\overline{e}, \overline{e}], [\overline{e}, \overline{e}] = [\overline{e}, \overline{e}]\). Thus, we have \([\overline{e}, \overline{e}] = [\overline{e}, \overline{e}]\), i.e., \(\overline{e} = \overline{e}\).

2. i-t-Norms and i-t-Conorms from i-Uninorms

We consider the definitions of interval t-norms (i-t-norms) in\(^{14}\) and interval t-conorms (i-t-conorms) in\(^{16}\). In this definitions, the \(\subseteq\)-monotonicity is also required. With respect to interval operations, for intervals \(X, Y, Z, [e, e] (e > 0)\) and \([1, 1]\) the following results are valids:

1. \(E \cdot X = [eX, eX]\);
2. \(X \cdot [e, e] = [\frac{X}{e}, \frac{X}{e}]\);
3. \([1, 1] \cdot X = X\);
4. \(0 \not\in [1, 1] = [e, e]\).

The next results status that i-t-norms and i-t-conorms can be obtained from i-uninorms\(^{b}\).

Theorem 2.1. Let \(U : [0, 1] \times [0, 1] \rightarrow [0, 1]\) be a \(i\)-uninorm with neutral element \(E = [e, e]\), where \(e \neq 0\). The function \(T : [0, 1] \times [0, 1] \rightarrow [0, 1]\) defined by:

\[
T(X, Y) = \frac{U(E \cdot X, E \cdot Y)}{E}
\]

is a \(i\)-T-norm.

Proof. The commutativity and the associativity of \(T\) are straightforward. Also, is easy to see that \([1, 1]\) is, in fact, the neutral element of \(T\). We must check the two monotonicities.

\(^{b}\)This is a generalization of the ideas exposed in\(^{17}\)
For $\leq_{km}$-Monotonicity, suppose that $X \leq_{km} Y$ and $Z \leq_{km} W$, ie, $x \leq y, \pi \leq \eta, z \leq w$ and $\pi \leq \eta$. Thus, we have $e_x \leq e_y, e_\pi \leq e_\eta, e_z \leq e_w$ and $e_\pi \leq e_\eta$, which implies $E \cdot X \leq_{km} E \cdot Y$ and $E \cdot Z \leq_{km} E \cdot W$, so $U(E \cdot X, E \cdot Z) \leq_{km} U(E \cdot Y, E \cdot W)$, which implies $E \cdot X \subseteq E \cdot Y$ and $E \cdot Z \subseteq E \cdot W$, so $U(E \cdot X, E \cdot Z) \subseteq U(E \cdot Y, E \cdot W)$.

Finally, for $\subseteq$-Monotonicity, suppose that $X \subseteq Y$ and $Z \subseteq W$, ie, $z \geq y, \pi \leq \eta, z \geq w$ and $\pi \leq \eta$. Thus, we have $e_x \geq e_y, e_\pi \leq e_\eta, e_z \geq e_w$ and $e_\pi \leq e_\eta$, which implies $E \cdot X \subseteq E \cdot Y$ and $E \cdot Z \subseteq E \cdot W$, so $U(E \cdot X, E \cdot Z) \subseteq U(E \cdot Y, E \cdot W)$.

**Theorem 2.2.** Let $U : \mathbb{I}[0, 1] \times \mathbb{I}[0, 1] \rightarrow \mathbb{I}[0, 1]$ be a $i$-uninorm with neutral element $E = [e, e]$, where $e \neq 1$. The function $S : \mathbb{I}[0, 1] \times \mathbb{I}[0, 1] \rightarrow \mathbb{I}[0, 1]$ defined by:

$$S(X, Y) = \frac{U(E + [1 - e, 1 - e] \cdot X, E + [1 - e, 1 - e] \cdot Y) - E}{[1 - e, 1 - e]}$$

is a $i$-$T$-conorm.

**Proof.** Again, the commutativity and the associativity of $S$ and the fact that $[0, 0]$ is the neutral element of $S$ are straightforward.

Suppose that $X \leq_{km} Y$ and $Z \leq_{km} W$. It follows immediately from the properties of interval arithmetic that $E + [1 - e, 1 - e] \cdot X \leq_{km} E + [1 - e, 1 - e] \cdot Y$, so $U(E + [1 - e, 1 - e] \cdot X, E + [1 - e, 1 - e] \cdot Y) - E \leq_{km} U(E + [1 - e, 1 - e] \cdot X, E + [1 - e, 1 - e] \cdot Z) - E$, ie, $S(X, Z) \leq_{km} S(Y, W)$.

The proof of $\subseteq$-Monotonicity is analogous to the proof of $\leq_{km}$-Monotonicity.

**3. Final Remarks**

In this paper we introduced some results about interval uninorms that are $\subseteq$-monotonic. Also, we show how to obtain interval $t$-norms and interval $t$-conorms from such interval uninorms and from given two uninorms (satisfying some conditions) we can obtain an interval uninorm. In future works, we will study some other conceptions between interval uninorms and others interval fuzzy connectives and use the Canonical Interval Representation to construct an interval uninorm from an usual uninorm.
References