The Existence of Positive Solutions for the Sturm-Liouville Boundary Value Problems

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Abstract—For the Sturm-Liouville boundary value problem

\[(p(t)u'(t))' + \lambda f(t, u(t)) = 0, \quad 0 < t < 1,\]
\[\alpha_1 u(0) - \beta_1 p(0)u'(0) = 0,\]
\[\alpha_2 u(1) + \beta_2 p(1)u'(1) = 0,\]

where \(\lambda > 0\), we shall use a fixed point theorem in a cone to obtain the existence of positive solutions for \(\lambda\) on a suitable interval.

Keywords—Sturm-Liouville boundary value problem, Cone, Fixed point theorem.

1. INTRODUCTION

Consider the Sturm-Liouville boundary value problem

\[(p(t)u'(t))' + \lambda f(t, u(t)) = 0, \quad 0 < t < 1,\]
\[\alpha_1 u(0) - \beta_1 p(0)u'(0) = 0,\]
\[\alpha_2 u(1) + \beta_2 p(1)u'(1) = 0,\]

where \(\lambda > 0\), \(\alpha_i, \beta_i \geq 0\) for \(i = 1, 2\) and \(\alpha_1 \alpha_2 + \alpha_1 \beta_2 + \alpha_2 \beta_1 > 0;\)

\(f(t, u) \in C([0, 1] \times [0, \infty), \mathbb{R})\), and there exists a positive constant \(M\) such that \(f(t, u) \geq -M\) for every \(t \in [0, 1], u \geq 0.\)

Recently, Anuradha, Hai and Shivaji [1] have established the following interesting result.

Theorem A. If \((H_1)-(H_3)\) hold, and

\[\lim_{u \to \infty} \frac{f(t, u)}{u} = \infty\]
uniformly on a compact subinterval \([\alpha, \beta]\) of \((0,1)\), then the boundary value problem (BVP) has a positive solution for \(0 < \lambda < \min\{(1/B)\|\bar{\omega}\|, (1/\gamma M)\}\), where
\[ B = \sup_{0 \leq t \leq 1, 0 \leq u \leq 1} (f(t, u) + M) \]
and \(\bar{\omega}, \gamma\) are defined as in Remark 1 below.

The purpose of this paper is to offer the existence of a positive solution to (BVP) for \(\lambda\) on a suitable interval. The obtained results improve Theorem A.

For other related results, we refer to [2–6].

2. MAIN RESULTS

In order to prove our main results, we need the following three useful lemmas. The first is due to \([7,8]\), the other two are due to [1].

**Lemma 1.** (See \([7,8]\).) Let \(K\) be a cone in a Banach space \(E\) and \(\Omega_1, \Omega_2\) be two bounded open sets in \(E\) such that \(\theta \in \Omega_1\) and \(\bar{\Omega}_1 \subset \Omega_2\). Let \(A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K\) be a completely continuous operator. If
\[
\|Ax\| \leq \|x\|, \text{ for all } x \in K \cap \partial \Omega_1 \quad \text{and} \quad \|Ax\| \geq \|x\|, \text{ for all } x \in K \cap \partial \Omega_2 ,
\]
or
\[
\|Ax\| \geq \|x\|, \text{ for all } x \in K \cap \partial \Omega_1 \quad \text{and} \quad \|Ax\| \leq \|x\|, \text{ for all } x \in K \cap \partial \Omega_2 ,
\]
then \(A\) has at least one fixed point in \(K \cap (\bar{\Omega}_2 \setminus \Omega_1)\).

**Lemma 2.** (See \([1]\).) Let \((H_1), (H_2)\) hold and let \(u(t)\) satisfy
\[
(p(t)u'(t))' = -v(t), \quad 0 < t < 1,
\]
\[
\alpha_1 u(0) - \beta_1 p(0)u'(0) = 0,
\]
\[
\alpha_2 u(1) + \beta_2 p(1)u'(1) = 0,
\]
where \(v(t) \in L^1(0,1), v(t) \geq 0\). Then,
\[
u(t) \geq \|u\|q(t), \text{ for } t \in [0,1],
\]
where
\[
q(t) = \min \left( \frac{\beta_1 + \alpha_1 \int_0^t (dr/p(r))}{\beta_1 + \alpha_1 \int_0^1 (dr/p(r))}, \frac{\beta_2 + \alpha_2 \int_0^1 (dr/p(r))}{\beta_2 + \alpha_2 \int_0^1 (dr/p(r))} \right).
\]
Here \(\| \cdot \|\) stands for the sup norm.

**Lemma 3.** (See \([1]\).) Let \((H_1), (H_2)\) hold and let \(\bar{\omega}(t)\) be the solution of following boundary value problem:
\[
(p(t)u'(t))' = -1, \quad 0 < t < 1,
\]
\[
\alpha_1 u(0) - \beta_1 p(0)u'(0) = 0,
\]
\[
\alpha_2 u(1) + \beta_2 p(1)u'(1) = 0.
\]
Then, there exists a positive constant \(\gamma\) such that \(\bar{\omega}(t) \leq \gamma q(t)\) for every \(t \in [0,1]\), where \(q(t)\) is defined in Lemma 2.

**Remark 1.** It follows from the proof of Lemma 3 in [1] that
\[
\bar{\omega}(t) = \rho^{-1} \left[ \left( \beta_2 + \alpha_2 \int_0^1 \frac{dr}{p(r)} \right) \left( \int_0^t \left( \beta_1 + \alpha_1 \int_0^s \frac{dr}{p(r)} \right) ds \right) + \left( \beta_1 + \alpha_1 \int_0^t \frac{dr}{p(r)} \right) \left( \int_0^1 \left( \beta_2 + \alpha_2 \int_0^1 \frac{dr}{p(r)} \right) ds \right) \right] \geq 0,
\]
where
\[ \rho = \alpha_1 \beta_2 + \alpha_1 \alpha_2 \int_0^1 \frac{dr}{p(r)} + \beta_1 \alpha_2. \]

After a simple calculation, we find that
\[ \gamma = \rho^{-1} \left( \beta_1 + \alpha_1 \int_0^1 \frac{dr}{p(r)} \right) \left( \beta_2 + \alpha_2 \int_0^1 \frac{dr}{p(r)} \right). \]

Throughout this paper, we shall denote
\[ \sigma = \min_{t \in [1/4, 3/4]} q(t), \]
\[ Q = \left( \int_{1/4}^{3/4} G \left( \frac{1}{2}, s \right) ds \right)^{-1} \]
and
\[ R = \left( \int_0^1 G(s, s) ds \right)^{-1}, \]
where \( q(t) \) is defined in Lemma 2 and \( G(t, s) \) is the Green's function of the differential equation \( (p(t)u'(t))' = 0 \) with the boundary conditions \( \alpha_1 u(0) - \beta_1 p(0)u'(0) = 0, \alpha_2 u(1) + \beta_2 p(1)u'(1) = 0, \) i.e.,
\[ G(t, s) = \left\{ \begin{array}{ll}
\rho^{-1} \left( \beta_1 + \alpha_1 \int_0^s \frac{dr}{p(r)} \right) \left( \beta_2 + \alpha_2 \int_t^1 \frac{dr}{p(r)} \right), & \text{for } 0 \leq s \leq t \leq 1, \\
\rho^{-1} \left( \beta_1 + \alpha_1 \int_0^t \frac{dr}{p(r)} \right) \left( \beta_2 + \alpha_2 \int_0^1 \frac{dr}{p(r)} \right), & \text{for } 0 \leq t \leq s \leq 1,
\end{array} \right. \]
where \( \rho \) is defined in Remark 1. By the definition of \( G(t, s) \), it is clear that \( G(t, s) \leq G(s, s) \), for all \( t \in [0, 1] \).

**THEOREM 1.** Let (H1)-(H3) hold. Assume that there exist a function \( h : [0, 1] \to \mathbb{R} \) and a positive constant \( k \) such that
\[ f(t, u) + M \geq h(t), \quad \text{for } t \in \left[ \frac{1}{4}, \frac{3}{4} \right], \quad u \in [0, \gamma k(M + 1)], \quad (H_4) \]
\[ \int_{1/4}^{3/4} G \left( \frac{1}{2}, s \right) h(s) ds \geq \gamma (M + 1), \quad (H_5) \]
where \( \gamma \) is defined in Lemma 3. If
\[ \lim_{u \to \infty} \max_{t \in [0, 1]} \frac{f(t, u)}{u} = C_1 \in \left(0, \frac{R}{k}\right). \quad (H_6) \]

Then, (BVP) has at least one positive solution for \( \lambda \in (0, k] \).

**PROOF.** Let \( \lambda \in (0, k] \) be given and \( q(t) \) be as in Lemma 2. Set \( w(t) = \lambda M \bar{w}(t) \), where \( \bar{w}(t) \) is defined in Remark 1. Since \( q(t) \leq 1 \) on \([0, 1]\), it follows from Lemma 3 that
\[ w(t) \leq \lambda \gamma M, \quad \text{for all } t \in [0, 1]. \quad (3) \]

Hence, \( u_1(t) \) is a positive solution of (BVP) if and only if \( \bar{u}(t) = u_1(t) + w(t) \) is a solution of the boundary value problem
\[ (p(t)u')' = -\lambda \bar{g}(t, u - w), \quad 0 < t < 1, \]
\[ \alpha_1 u(0) - \beta_1 p(0)u'(0) = 0, \quad (BVP^*) \]
\[ \alpha_2 u(1) + \beta_2 p(1)u'(1) = 0, \]
with $\tilde{u}(t) > w(t)$ on $(0, 1)$, where

$$g(t, u) = \begin{cases} g(t, u), & \text{for } u \geq 0, \\ g(t, 0), & \text{for } u < 0, \end{cases}$$

with $g(t, u) = f(t, u) + M$ is a nonnegative continuous function on $[0, 1] \times [0, \infty)$.

Let $K = \{ u \in C[0, 1] : u(t) \geq \| u \| q(t), \ t \in [0, 1] \}$. Clearly, $K$ is a cone. If $u(t)$ is a solution of (BVP$^*$), then $u(t)$ satisfies the integral equation

$$u(t) = \lambda \int_0^1 G(t, s)g(s, u(s) - w(s)) \, ds.$$

Now we define the operator $T_\lambda$ on $K$ by

$$T_\lambda u(t) = \lambda \int_0^1 G(t, s)g(s, u(s) - w(s)) \, ds. \tag{4}$$

From Lemma 2 and Ascoli's Lemma, it is easy to show that $T_\lambda : K \to K$ is completely continuous.

Now, we will show that $T_\lambda$ has a fixed point in $K$, for all $\lambda \in (0, k \cdot R)$. Since

$$w(t) = \lambda M \tilde{w}(t) \leq \lambda \gamma M q(t) \leq \frac{\lambda \gamma M}{\| u \|} u(t),$$

it follows that

$$u(t) - w(t) \geq \left(1 - \frac{\lambda \gamma M}{\| u \|}\right) u(t), \quad \text{for } u \in K. \tag{5}$$

By (H$\delta$),

$$\lim_{u \to \infty} \max_{t \in [0, 1]} \frac{f(t, u) + M}{u} = \lim_{u \to \infty} \max_{t \in [0, 1]} \frac{f(t, u)}{u} = C_1 \in \left[0, \frac{R}{k}\right).$$

Taking $\varepsilon = R/k - C_1$, there exists $\xi > 0$ such that

$$\max_{t \in [0, 1]} \frac{g(t, u)}{u} \leq \varepsilon + C_1 = \frac{R}{k}, \quad \text{for } u \in [\xi, \infty). \tag{6}$$

Therefore, for $\eta > \max\{\xi, \lambda \gamma (M + 1)\}$ large enough,

$$g(t, u) \leq \frac{\eta R}{k}, \quad \text{on } [0, 1] \times [0, \eta].$$

By this and (5) and (6), for $u(t) \in K$ and $\| u \| = \eta$, we have

$$\tilde{g}(t, u - w) = g(t, u - w) \leq \frac{\eta R}{k}, \tag{7}$$

for $t \in [0, 1]$. Let

$$\Omega_1 = \{ u \in K : \| u \| < \lambda \gamma (M + 1) \}$$

and

$$\Omega_2 = \{ u \in K : \| u \| < \eta \}.$$

Then, by (2) and (7),

$$T_\lambda u(t) = \lambda \int_0^1 G(t, s)\tilde{g}(s, u(s) - w(s)) \, ds \leq \frac{\lambda \eta R}{k} \int_0^1 G(t, s) \, ds \leq \eta,$$

for $u \in \partial \Omega_2$. Thus

$$\| T_\lambda u \| \leq \| u \|, \quad \text{for } u \in \partial \Omega_2. \tag{8}$$
On the other hand, it follows from Lemma 2 that
\[ \gamma k(M + 1) \geq \lambda \gamma (M + 1) \geq u(t) \geq u(t) - w(t) \geq \|u\|q(t) - \lambda M \bar{w}(t) \geq \lambda \gamma q(t) > 0, \] (9)
for \( u \in \partial \Omega_1 \). Combining (9), (H4), and (H5), we obtain
\[
T_\lambda u \left( \frac{1}{2} \right) = \lambda \int_0^1 G \left( \frac{1}{2}, s \right) \bar{g}(s, u(s) - w(s)) \, ds \\
\geq \lambda \int_{1/4}^{3/4} G \left( \frac{1}{2}, s \right) \bar{g}(s, u(s) - w(s)) \, ds \\
\geq \lambda \int_{1/4}^{3/4} G \left( \frac{1}{2}, s \right) \bar{h}(s) \, ds \geq \lambda \gamma (M + 1).
\]
Thus,
\[ \|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in \partial \Omega_1. \] (10)

It follows from (8), (10), and Lemma 1 that there exists \( \bar{u} \in K \cap (\overline{K} \setminus \Omega_1) \) such that \( T_\lambda \bar{u}(t) = \bar{u}(t) \) and \( \|\bar{u}\| \) is between \( \lambda \gamma (M + 1) \) and \( \eta \). By (3), \( \bar{u}(t) > w(t) \) on \([0,1]\), and so \( u_1(t) = \bar{u}(t) - w(t) \) is a positive solution of (BVP) for \( \lambda \in (0, k] \). This completes the proof.

**Remark 2.** We can take the constant \( k \) in Theorem 1 as
\[ k = \frac{1}{\gamma (M + 1)} \sup \{ \|u\| : f(t, u) + M \geq h(t) \text{ on } [0,1], \text{ and } u \geq 0 \}. \]

**Theorem 2.** Let (H1)-(H3) hold. Assume that there exist a function \( h : [0,1] \to \mathbb{R} \) and a positive constant \( k \) such that
\[ f(t, u) + M \leq h(t), \quad \text{for } t \in [0,1], \quad u \in [0, \gamma k(M + 1)], \] (H7)
\[ \int_0^1 G(s, s) h(s) \, ds \leq \gamma (M + 1), \] (H8)
where \( \gamma \) is defined in Lemma 2.
(a) If
\[ \lim_{u \to \infty} \min_{t \in [1/4,3/4]} \frac{f(t, u)}{u} = \infty, \] (H9)
then (BVP) has at least one positive solution for \( \lambda \in (0, k] \).
(b) If \( k > 1 \) and
\[ \lim_{u \to \infty} \min_{t \in [1/4,3/4]} \frac{f(t, u)}{u} = C_2 \in \left( \frac{Q}{\sigma}, \infty \right), \] (H10)
then (BVP) has at least one positive solution for \( \lambda \in [1, k] \).

**Proof.** Let \( w(t), \bar{g}(t, u), \bar{g}(t, u), K, \) and \( T_\lambda \) be as in the proof of Theorem 1. Thus (3), (5), and (9) hold. Moreover, \( u_1(t) \) is a positive solution of (BVP) if and only if \( \bar{u}(t) = u_1(t) + w(t) \) is a solution of the boundary value problem (BVP*). If we can prove that \( T_\lambda \) has a fixed point \( \bar{u}(t) \in K \) with \( \bar{u}(t) > w(t) \) on \([0,1]\), then \( u(t) = \bar{u}(t) - w(t) \) is a positive solution of (BVP).

**Case (a).** Let \( \lambda \in (0, k] \). By (H9),
\[ \lim_{u \to \infty} \min_{t \in [1/4,3/4]} \frac{f(t, u) + M}{u} = \lim_{u \to \infty} \min_{t \in [1/4,3/4]} \frac{f(t, u)}{u} = \infty. \] (11)
Let $\tilde{M} = 2Q/\lambda \sigma$. Using (5) and (11), we see that, for $\eta > \max\{2\lambda \gamma M, \lambda \gamma (M + 1)\}$ large enough and $u \in K$ with $\|u\| = \eta$,

$$u(t) - w(t) \geq \frac{1}{2} u(t) \geq \frac{1}{2} \eta \sigma$$

and

$$\tilde{g}(t, u - w) = g(t, u - w) \geq \tilde{M} (u(t) - w(t)) \geq \frac{\tilde{M} \eta \sigma}{2} = \eta Q,$$

for $t \in [1/4, 3/4]$. Again, let $\Omega_1 = \{u \in K : \|u\| < \lambda \gamma (M + 1)\}$ and $\Omega_2 = \{u \in K : \|u\| < \eta\}$. Then, by (1) and (12),

$$T_\lambda u \left( \frac{1}{2} \right) = \lambda \int_0^{1/4} G \left( \frac{1}{2}, s \right) \tilde{g}(s, u(s) - w(s)) \, ds$$

$$\geq \lambda \int_{1/4}^{3/4} G \left( \frac{1}{2}, s \right) \tilde{g}(s, u(s) - w(s)) \, ds$$

$$\geq \eta Q \int_{1/4}^{3/4} G \left( \frac{1}{2}, s \right) \, ds = \eta,$$

for $u \in \partial \Omega_2$. Thus,

$$\|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in \partial \Omega_2. \quad (13)$$

On the other hand, by (9), (H1), and (H8),

$$T_\lambda u(t) = \lambda \int_0^t G(t, s) \tilde{g}(s, u(s) - w(s)) \, ds$$

$$\leq \lambda \int_0^t G(s, s) h(s) \, ds \leq \lambda \gamma (M + 1),$$

for $u \in \partial \Omega_1$. Hence,

$$\|T_\lambda u\| \leq \|u\|, \quad \text{for } u \in \partial \Omega_1. \quad (14)$$

It follows from (13), (14), and Lemma 1 that there exists $\bar{u} \in K \cap (\Omega_2 \setminus \Omega_1)$ such that $T_\lambda \bar{u}(t) = \bar{u}(t)$ and $\|\bar{u}\|$ is between $\lambda \gamma (M + 1)$ and $\eta$. By (3), we obtain that $\bar{u}(t) > w(t)$ on $[0, 1]$, and so $u_1(t) = \bar{u}(t) - w(t)$ is a positive solution of (BVP) for $\lambda \in (0, k]$. This completes the proof of Case (a).

**Case (b).** Let $\lambda \in [1, k]$. By (H10),

$$\lim_{u \to \infty} \min_{t \in [1/4, 3/4]} f(t, u) + \frac{M}{u} = \lim_{u \to \infty} \min_{t \in [1/4, 3/4]} \frac{f(t, u)}{u} = C_2 \in \left( \frac{Q}{\sigma}, \infty \right). \quad (15)$$

Taking $\epsilon = C_2 - Q/\sigma$, there exists $\eta > \gamma (M + 1)$ large enough such that

$$\min_{t \in [1/4, 3/4]} \frac{g(t, u)}{u} \geq -\epsilon + C_2 = \frac{Q}{\sigma}, \quad \text{for } u \in [\delta \eta, \infty),$$

where $\delta \in (0, \sigma) \subset [0, 1]$. Hence, for $u(t) \in K$ and $\|u\| = \eta$,

$$g(t, u(t)) \geq \frac{Q}{\sigma} u(t) \geq \frac{Q}{\sigma} \|u\| q(t) \geq \eta Q, \quad \text{on } \left[ \frac{1}{4}, \frac{3}{4} \right] \times [\delta \eta, \eta]. \quad (16)$$

Since (5) holds and $\delta \in (0, \sigma) \subset (0, 1)$, we can choose $\eta$ so large that, for $\|u\| = \eta$,

$$u(t) - w(t) \geq \left( 1 - \frac{\gamma (M + 1)}{\|u\|} \right) \sigma \|u\| \geq \delta \|u\| = \delta \eta.$$
This and (16) imply
\[ g(t, u(t) - w(t)) = g(t, u(t) - w(t)) \geq \eta Q, \] (17)
for \( t \in [1/4, 3/4], u \in \mathcal{K}, \) and \( \|u\| = \eta. \) Let \( \Omega_1 \) and \( \Omega_2 \) be as in Case (a) for \( \lambda \in [1, k]. \) Then, by (1) and (17),
\[
T_\lambda u \left(\frac{1}{2}\right) = \lambda \int_0^1 G \left(\frac{1}{2}, s\right) g(s, u(s) - w(s)) \, ds \\
\geq \int_{1/4}^{3/4} G \left(\frac{1}{2}, s\right) g(s, u(s) - w(s)) \, ds \\
\geq \eta Q \int_{1/4}^{3/4} G \left(\frac{1}{2}, s\right) \, ds = \eta,
\]
for \( u \in \partial \Omega_2. \) Thus,
\[
\|T_\lambda u\| \geq \|u\|, \quad \text{for } u \in \partial \Omega_2. \tag{18}
\]
On the other hand, it follows from (9), (H_7), and (H_8) that (14) holds for \( u \in \partial \Omega_1. \) By (14), (18), and Lemma 1, there exists \( \bar{u} \in \mathcal{K} \cap (\Omega_2 \setminus \Omega_1) \) such that \( T_\lambda \bar{u}(t) = \bar{u}(t) \) and \( \|\bar{u}\| \) is between \( \lambda \gamma (M+1) \) and \( \eta. \) By (3), we obtain \( \bar{u}(t) > w(t) \) on \([0,1],\) and so \( u(t) = \bar{u}(t) - w(t) \) is a positive solution of (BVP) for \( \lambda \in [1, k]. \) This completes the proof.

Remark 3. The constant \( k \) in Theorem 2 can be taken as
\[
k = \frac{1}{\gamma(M+1)} \sup\{\|u\| : f(t, u) + M \leq h(t) \text{ on } [0,1], \text{ and } u \geq 0\}.
\]

Example 1. Consider the boundary value problem
\[
\begin{align*}
u''(t) + \lambda f(t, u(t)) &= 0, \quad \text{for } 0 < t < 1, \\
u(0) &= u(1) = 0, \tag{19}
\end{align*}
\]
where
\[ f(t, u) = 214 t e^{1000-u} - 9t \cos u \geq -9 = -M, \quad \text{for } t \in [0,1] \text{ and } u \geq 0. \]
Clearly, \( \bar{u}(t) = 1/2 t(1-t), \gamma = 1. \) Since \( f(t, u) \) satisfies
\[
\lim_{u \to \infty} \frac{f(t, u)}{u} = 0 \text{ uniformly on each compact subset of } (0,1),
\]
Theorem A cannot be applied to (19). However, if we take \( h(t) = 214t, \) then \( k = 100, \gamma k(M+1) = 1000. \) Therefore,
\[
f(t, u) + M \geq 214 t e^{1000-u} \geq 214t = h(t), \quad \text{on } [0,1] \times [0,1000].
\]
Since the Green's function of (19) is
\[
G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases}
\]
we have
\[
\int_{1/4}^{3/4} G \left(\frac{1}{2}, s\right) h(s) \, ds = \frac{321}{32} \geq 10 = \gamma (M+1)
\]
and
\[
\lim_{u \to \infty} \max_{t \in [0,1]} \frac{f(t, u)}{u} = 0.
\]
Hence \((H_4)-(H_6)\) hold. Thus, by Theorem 1, we see that (19) has at least one positive solution for all \(\lambda \in (0, 100]\).

**Example 2.** Consider the boundary value problem

\[
\begin{align*}
  u''(t) + \lambda f(t, u(t)) &= 0, \quad \text{for } 0 < t < 1, \\
  u(0) = u(1) &= 0,
\end{align*}
\]

where \(f(t, u) = t^{10}u^{3/2} - 9t \cos u \geq -9 = -M, \quad \text{for } t \in [0, 1] \text{ and } u \geq 0.\)

Clearly, \(\bar{w}(t) = (1/2)t(1-t), \quad \gamma = 1.\) Since \(f(t, u)\) satisfies

\[
\lim_{u \to \infty} \frac{f(t, u)}{u} = \infty \quad \text{uniformly on a compact subset of } (0, 1),
\]

by Theorem A, we find that (20) has a nonnegative solution for

\[
0 < \lambda < \min \left\{ \frac{1}{B\|\bar{w}\|}, \frac{1}{\gamma M} \right\} \leq \frac{1}{\gamma M} = \frac{1}{9}.
\]

However, if we take \(h(t) = 1000t^{10} + 18, \) then \(k = 10, \quad \gamma k(M+1) = 100\) and

\[
f(t, u) + M \leq t^{10}u^{3/2} + 18 \leq 1000t^{10} + 18 = h(t), \quad \text{on } [0, 1] \times [0, 100].
\]

Since the Green’s function of (20) is the same as in Example 1, it is easy to see that

\[
\int_0^1 G(s, s)h(s) \, ds = \int_0^1 s(1-s)(1000s^{10} + 18) \, ds = \frac{1468}{156} < 10 = \gamma(M+1)
\]

and

\[
\lim_{u \to \infty} \min_{t \in [1/4, 3/4]} \frac{f(t, u)}{u} = \infty.
\]

Hence \((H_7)-(H_9)\) hold. Thus, by Theorem 2(a), we see that (20) has at least one positive solution for all \(\lambda \in (0, 10].\)

**References**


