Characterization of Inverse-Gaussian and Gamma Distrubutions Through Their Length-Biased Distributions

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Reader Aids —
Purpose: Widen state of the art
Special math needed for explanations: Probability
Special math needed to use results: Same
Results useful to: Reliability theoreticians

Abstract — **Let Y be a length-biased random variable corresponding to a random variable X having an inverse-Gaussian or Gamma distribution. It is shown that Y can be written as a linear combination of X and a chi-square random variable and conversely X can be characterized through this relationship. Finally the Wald distribution is characterized.**

1. INTRODUCTION

Let X be a non-negative random variable having an absolutely continuous pdf $f(\cdot)$. Let X have a finite first moment, say μ; one can define another pdf $h(\cdot)$ as:

$$h(x) = \frac{xf(x)}{\mu}, \; x>0 \tag{1.1}$$

The random variable Y with pdf $h(\cdot)$ is known as the length-biased random variable associated with X. The random variable Y occurs in the studies of lifetime models [1, 2]. Cox [3, page 65] provides the following interpretation of pdf $h(\cdot)$: Consider a sample of failure times with pdf $f(x)$ and let the probability of selecting any individual unit in the population be proportional to its size or length x. Then the failure time selected has the pdf $h(x)$. Also, length-biased distributions arise in various probability-proportional-to-size (pps) sampling procedures [7]. In fact $h(\cdot)$ is simply a special weighted distribution with weight x [5, 6].

This note presents a result characterizing the inverse Gaussian and Gamma distributions. These are common probability distributions in reliability and lifetime models. Finally a characterization of Wald distribution, which is a special case of inverse Gaussian, is given.

2. THE MAIN RESULT

The theorem consists of two characterizations of similar nature and we state these in the following:

Theorem 2.1 Let X be a non-negative random variable with pdf $f(\cdot)$, characteristic function $\phi(\cdot)$ and mean μ. Let Y be the random variable with pdf $h(\cdot)$ as defined in (1.1). If Y can be written as $Y \overset{d}{=} X+Z/\lambda$ where $\lambda>0$ and Z is a chi square random variable, s-independent of X with,

i. a single degree of freedom then X has an inverse gaussian distribution with shape parameter $\lambda\mu^2$ and mean μ respectively.
ii. two degrees of freedom then X has a gamma distribution with shape and scale parameters $\alpha=\lambda\mu/2$ and $\beta=2/\lambda$.

Proof. Let $\psi(t)$ be the characteristic function of $Y \overset{d}{=} X+Z/\lambda$, $\lambda>0$. To prove i, we observe that under the assumptions stated in the theorem and using (1.1),

$$\psi(t) = (i\mu)^{-1}\,\phi'(t) = \phi(t)\left(1-\frac{2it}{\lambda}\right)^{-1/2} \tag{2.1}$$

$$\phi'(t) \equiv \frac{\partial}{\partial t}\,\phi(t).$$

Solve (2.1) and use the fact that $\phi(0)=1$:

$$\phi(t) = \exp\,[\lambda\mu(1-(1-2it/\lambda)^{1/2})]. \tag{2.2}$$

Eq (2.2) after appropriate parameterization, is easily seen to be the characteristic function of an inverse Gaussian r.v. When $\mu=1$, eq (2.2) reduces to the characterization of a Wald distribution [1].

The proof of ii is similar and thus we skip it. As a corollary, the length-biased distribution of a χ^2_n is χ^2_{n+2}. *Q.E.D.*

3. A CHARACTERIZATION OF WALD DISTRIBUTION

In [1], a characterization of the Wald distribution was given. Motivated by [4, lemma 6.1.3], another characterization is given in theorem 3.1.

Theorem 3.1 Let $F(\cdot)$ be the Cdf of a non-negative random variable Y. Let $E(Y^{-1})$ exist, and let the relation

$$\int y^{-1}\exp(ity)dF(y) = \left(1-\frac{2it}{\lambda}\right)^{1/2}\int \exp(ity)dF(y)$$

be valid for some $\lambda>0$ and for all $|t|<\delta$ for some δ, then Y has the length-biased distribution associated with a Wald random variable, with parameter λ (or equivalently Y^{-1} has a Wald distribution).

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Proof: Let

$$A(t) = \int_0^\infty y^{-1} \exp(ity) dF(y)$$

and

$$\phi(t) = \int_0^\infty \exp(ity) dF(y).$$

Thus

$$A'(t) = \frac{\partial A(t)}{\partial t} = i\phi(t)$$

and

$$\frac{A'(t)}{A(t)} = i\left(1 - \frac{2it}{\lambda}\right)^{-1/2} \tag{3.1}$$

Solution of differential equation in (3.1) is

$$A(t) = C\, e^{-\lambda}\left(1 - \frac{2it}{\lambda}\right)^{1/2}$$

for some constant C and thus using the fact that $\phi(0) = 1$, we have

$$\phi(t) = \left(1 - \frac{2it}{\lambda}\right)^{-1/2} \exp\left(\lambda\left[1 - \left(1 - \frac{2it}{\lambda}\right)^{1/2}\right]\right). \tag{3.2}$$

The r.h.s. of (3.2) is the characteristic function of the length-biased distribution associated with a Wald distribution. That Y^{-1} has a Wald distribution follows from [1].

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