APPROXIMATION AND IMBEDDING THEOREMS FOR WEIGHTED SOBOLEV SPACES ASSOCIATED WITH LIPSCHITZ CONTINUOUS VECTOR FIELDS

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Abstract. In this paper, we prove the density of regular functions in weighted Sobolev spaces associated with a family of Lipschitz vector fields. Moreover, compact imbedding theorems for these function spaces are proved, together with some regularity results for degenerate elliptic equations in divergence form.

Introduction

In the previous paper [FSSC] the authors considered the so-called Lavrentiev phenomenon for a class of variational functionals associated with a family of vector fields $X_1 \dots , X_m$ in an open set $\Omega \subseteq R^n$. Roughly speaking, we say that Lavrentiev phenomenon occurs for a functional of the Calculus of Variations if its infimum in a natural class of functions (e.g. Sobolev type spaces) does not coincide with its infimum on regular functions. The functionals we considered can be exemplified by the energy of the generalized p-Laplace operator

$$\int_\Omega (\sum_{j=1}^m |X_j u|^2)^{p/2} dx \qquad (p > 1) \tag{1}$$

and by the generalized area functional

$$\int_\Omega (1 + \sum_{j=1}^m |X_j u|^2)^{1/2} dx. \tag{2}$$

A crucial step in order to prove the absence of Lavrentiev phenomenon for this kind of functionals is a Meyers-Serrin type density theorem. In other words, we have to prove that a function $u \in L^p(\Omega)$ such that $X_j u \in L^p(\Omega)$ for $j = 1, \dots , m$ can be approximated in the natural norm $\|u\|_{\mathcal{W}^{1,p}_X(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_j \|X_j u\|_{L^p(\Omega)}$

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by a sequence of smooth functions. The problem is not trivial because standard convolution regularizations do not fit the geometry of the problem, mainly since the vector fields are not translation-invariant like ordinary partial derivatives in R^n. Nevertheless, it is possible to show following Friedrichs idea in [Fr] that these approximations still converge. The interest in functionals of type (1) and (2) is originated by the fact that, under suitable geometric assumptions on $X_1, \dots, X_m$, the differential operators $\sum_j X_j^* X_j$ share many properties of elliptic operators, like hypoellipticity, Harnack's inequality and Hölder continuity of weak solutions. On the other hand, a larger class of degenerate elliptic operators enjoy the same properties, namely the class of weighted operators of the form $\mathcal{L} := \sum_{i,j=1}^n \partial_i \left(a_{ij}(x)\partial_j\right)$, where

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \sim w(x) \sum_{l=1}^{m} \langle X_l(x), \xi\rangle^2$$

for any $\xi \in R^n$, w being a weight function (i.e. a nonnegative locally summable function) belonging to suitable classes (such as, for instance, A_p classes of Muckenhoupt).

In this paper, we prove Meyers-Serrin theorems for function spaces associated with a family of vector fields in the presence of a weight function. Typically, our results can be applied to the study of the Lavrentev phenomenon for functionals of the form

$$\int_\Omega f(x, X_1(x), \dots, X_m(x))w(x)dx + \{\text{appropriate conditions}\}$$

where

$$f = f(x, \eta) : \Omega \times R^m \longrightarrow [0, +\infty)$$

is a Carathéodory function (measurable in x, continuous and convex in η) such that

$$c_1|\eta|^p \le f(x, \eta) \le c_2(1 + |\eta|)^p$$

for some suitable $p > 1$, for a.e. $x \in \Omega$ and for any $\eta \in R^m$, and the weight w satisfies suitable condition we will list below.

In Section 1, we consider the general situation of an arbitrary family of Lipschitz continuous vector fields $X = (X_1, \dots, X_m)$ and we prove a density theorem for weighted Sobolev spaces associated with X when the weight function belongs to the standard A_p class (see below for precise definitions). This proof follows basically the scheme of the proof of the corresponding result in [Fr] and in [FSSC]. We are indebted with N. Garofalo who raised our attention on Friedrichs paper during a meeting in Ischia in June 1995.

In Section 2, we consider the case when it is possible to associate with the vector fields $X_1, \dots, X_m$ a natural metric ρ by mean of sub-unit curves as in [FP] or [NSW]. In this case we are still able to prove a density theorem if the weight function belongs to an A_p class with respect to the metric ρ as in [C]. However, in this case the proof is completely different from the one in the preceeding section; basically, problems arise since the mollifiers in [FSSC] are shaped on Euclidean balls, so that they cannot be bounded by the maximal function associated with the metric, which in turn is a natural tool to deal with A_p classes. We get rid of this difficulty following a different approach, as given in [SC], which relies on the Poincaré inequality or, more precisely, on a suitable representation formula for regular functions with zero average on a ball.

Finally in the last section, these results are applied to prove some 'abstract' regularity results for degenerate elliptic equations. In the same spirit, we prove a Rellich's type compact imbedding theorem for weighted spaces associated with a family of vector fields.

The results of sections 2 and 3 extend previous density results proved in [CPSC], both for the presence of the vector fields and for the limit case $p = 1$ is allowed in some cases. Related results can be found in [BH], [FS], [D], [CDG1], [CDG2], [GN], [H1], [H2], [H3], [L1], [L2].

1. Approximation through convolutions

1.1 Notations and Definitions.

Through this paper $\Omega \subset R^n$ is a fixed open set. If $v, w \in R^n$, we denote by $|v|$ and $\langle v, w\rangle$ the Euclidean norm and the scalar product, respectively. If $x \in R^n$ and $E, F \subset R^n$ then $\mathrm{dist}(x, E) = \inf\{|x-y| : y \in E\}$ and $\mathrm{dist}(E, F) = \inf\{|x-y| : y \in E,\ x \in F\}$. If Ω and Ω' are subsets of R^n then $\Omega' \subset\subset \Omega$ means that Ω' is compactly contained in Ω. Moreover, $B(x, r)$ is the open Euclidean ball of radius r centered at x. If $A \subset R^n$ then χ_A is the characteristic function of A, $|A|$ is its n-dimensional Lebesgue measure. More generally, if w is a weight function (i.e. a nonnegative locally integrable function), we denote by $L^p(\Omega, w)$ the L^p-space with respect to the measure $dw = w(x)dx$ (we denote by $L^p(\Omega)$ the L^p-space with respect to the Lebesgue measure) and we will put $w(A) = \int_A w(x)dx$. Thus, if $f \in L^1(A, w)$ (respectively if $f \in L^1(A)$) we denote its w-average by

$$\fint_A f(x)\, dw = \frac{1}{w(A)} \int_A f(x)\, dw$$

and

$$\fint_A f(x)dx = \frac{1}{|A|} \int_A f(x)dx$$

the average of f on A with respect to Lebesgue measure.

$\mathcal{C}^k(\Omega; R^m)$ is the space of R^m-valued functions k times continuously differentiable: $\mathrm{Lip}(\Omega; R^m)$ is the space of R^m-valued Lipschitz continuous functions and we set $\mathcal{C}_0^k(\Omega; R^m) = \{f \in \mathcal{C}^k(\Omega; R^m) : \mathrm{supp}\, f \subset\subset \Omega\}$ and $\mathrm{Lip}_0(\Omega; R^m) = \{f \in \mathrm{Lip}(\Omega; R^m) : \mathrm{supp}\, f \subset\subset \Omega\}$. Moreover, for sake of brevity, we write $\mathcal{C}^k(\Omega)$ and $\mathcal{C}_0^k(\Omega)$ if $m = 1$. Finally we use the letters $c, C, c_1, c_2, \ldots$ for constants not necessarily the same at each occurrence.

Let $X_1, \ldots, X_m$ be a family of Lipschitz continuous vector fields in Ω, where $X_j = (c_{j1}, \ldots, c_{jn})$ for $j = 1, \ldots, m$. We identify each vector field X_j with the first order differential operator (still denoted by X_j) $\sum_i c_{ji}(x)\partial_i$. Moreover, we put $X = (X_1, \ldots, X_m)$ and $|Xf|^2 = |X_1 f|^2 + \ldots + |X_m f|^2$ for any $f \in L^1_{\mathrm{loc}}(\Omega)$ such that $X_j f \in L^1_{\mathrm{loc}}(\Omega)$ for $j = 1, \ldots, m$.

If w is a weight function, that is if $w \geq 0$ and $w \in L^1_{\mathrm{loc}}(R^n)$ we say that $w \in A_p = A_p(R^n,$ Euclidean metric, Lebesgue measure) if there is a constant $c > 0$

such that for all Euclidean balls $B = B(x,r) = \{y \in R^n : |y-x| < r\}$

$$\text{(i)} \qquad \fint_B w\,dx\,(\fint_B w^{-1/(p-1)}dx)^{p-1} \le c \qquad \text{when} \quad 1 < p < \infty$$

$$\text{(ii)} \qquad \fint_B w\,dx \le c \quad \operatorname*{ess\,inf}_B w \qquad \text{when } p = 1.$$

The smallest constant for which (i) or (ii) holds is the A_p bound of w. Note that if $w \in A_p$ then $L^p(\Omega, w) \subseteq L^1_{\text{loc}}(\Omega)$.

Definition 1.1. If $p \in [1,\infty)$ and if w is a weight function, we define

$$\begin{aligned} W^{1,p}_X(\Omega,w) &:= \{f \in L^p(\Omega,w) :\ X_j f \in L^p(\Omega,w) \text{ for } j = 1,\dots,m\} \\ H^{1,p}_X(\Omega,w) &= \text{ closure of } C^\infty(\Omega) \cap W^{1,p}_X(\Omega,w) \text{ in } W^{1,p}_X(\Omega,w). \end{aligned}$$

where for any $f \in L^1_{\text{loc}}(\Omega)$

$$X_j f := \sum_{i=1}^n \partial_i(c_{ji} f) - \left(\sum_{i=1}^n \partial_i c_{ji} \right) f.$$

Endowed with the norm

$$\|f\|_{W^{1,p}_X(\Omega,w)} := \|f\|_{L^p(\Omega,w)} + \sum_j \|X_j f\|_{L^p(\Omega,w)},$$

$W^{1,p}_X(\Omega,w)$ is a Banach space, reflexive if $p > 1$.

Obviously, $H^{1,p}_X(\Omega,w)$ is a closed subspace of $W^{1,p}_X(\Omega,w)$. The main result of the next section is the proof that these two spaces actually coincide.

1.2 The approximation theorem.

Theorem 1.2. *Let $\Omega \subseteq R^n$ and $p \in [1,+\infty)$. Assume that $X = (X_1,\dots,X_m)$ is a family of Lipschitz continuous vector fields defined in Ω and that $w \in A_p$. Then*

$$H^{1,p}_X(\Omega,w) = W^{1,p}_X(\Omega,w).$$

All this section is devoted to the proof of Theorem 1.2. The first step is given by the following technical lemma.

Lemma 1.3. *For $0 < \varepsilon < \varepsilon_1$ let $K_\varepsilon : R^n \longrightarrow R$ be a family of measurable functions supported in $B(0,\varepsilon)$ such that*

$$|K_\varepsilon(z)| \le C\varepsilon^{-n} \qquad \textit{and} \qquad \int_{B(0,\varepsilon)} K_\varepsilon(z)dz = 0.$$

If $1 \le p < \infty$, $w \in A_p$ and $f \in L^p(\Omega,w)$, then

$$\lim_{\varepsilon \longrightarrow 0} \|K_\varepsilon * f\|_{L^p(\Omega,w)} = 0.$$

Here, without loss of generality, we think of f as defined on all of R^n being equal to zero outside of Ω.

Proof. First we assume $p > 1$. Observe that $(K_\varepsilon * f)(x) \longrightarrow 0$ as $\varepsilon \longrightarrow 0+$ for a.e. $x \in \Omega$. Indeed

$$(K_\varepsilon * f)\ (x) = \int_{R^n} K_\varepsilon(x-y)\, f(y)\, dy = \int_{B(x,\varepsilon)} K_\varepsilon(x-y)\ \{f(y) - f(x)\} dy,$$

so that

$$|(K_\varepsilon * f)(x)| \le c\, \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} |f(y) - f(x)|\ dy \longrightarrow 0$$

as $\varepsilon \to 0$ at any Lebesgue point x of f. Then the assertion follows from dominated convergence theorem, since

$$|K_\varepsilon * f|(x) \le c\, Mf(x),$$

where Mf, the Hardy-Littlewood maximal function of f, belongs to $L^p(R^n, w)$ as proved in [M].
If $p = 1$, from the very definition of A_1 $Mw \le cw$ a.e. in Ω, so that we can argue as follows: given $\eta > 0$, choose f_η continuous and compactly supported in Ω such that $\|f - f_\eta\|_{L^1(\Omega,w)} < \eta$.
Since

$$\int_{R^n} |K_\varepsilon * f\ |w\, dx \le \int_{R^n} |K_\varepsilon * (f - f_\eta)\,|w\, dx + \int_{R^n} |K_\varepsilon * f_\eta|\, w\, dx$$

the theorem follows because the first term is small for all $\varepsilon > 0$, whereas the second one is small for ε small because of the uniform continuity of f_η. More precisely

$$\begin{aligned} \int_{R^n} |K_\varepsilon * (f - f_\eta)\,|\, w\, dx &\le \int_{R^n} |f - f_\eta\,|\,(|K_\varepsilon| * w)\, dx \\ &\le c \int_{R^n} |f - f_\eta|\, Mw\, dx \le c \int_{R^n} |f - f_\eta|\, w\, dx < c\,\eta, \end{aligned}$$

and

$$\begin{aligned} \int_{R^n} |K_\varepsilon * f_\eta| w\, dx &= \int_{R^n} |\ \int_{B(x,\varepsilon)} K_\varepsilon(x-y)\ (f_\eta(y) - f_\eta(x))\, dy\ |\ w(x)\, dx \\ &\le c\, h(\varepsilon) \int_\Omega w(x)\, dx \end{aligned}$$

where ε is less than the distance of the support of f_η from $\partial\Omega$ and h is the modulus of continuity of f. Thus the assertion is completely proved. $\square$

Proposition 1.4. *Let $1 \le p < \infty$ and let $w \in A_p$ be given. If $f \in W^{1,p}_X(\Omega, w)$ and if $\Omega' \subset\subset \Omega$, then*

$$\lim_{\varepsilon \to 0} \|f * J_\varepsilon - f\|_{W^{1,p}_X(\Omega', w)} = 0,$$

where $J_\varepsilon(x) = \varepsilon^{-n} J\left(\varepsilon^{-1}|x|\right)$ is a spherically symmetric mollifier supported in $B(0,\varepsilon)$ such that $\int\limits_{R^n} J(|\xi|)d\xi = 1$.

Proof. If $p > 1$, arguing as in [CPSC], it is easy to see that

$$\lim_{\varepsilon \to 0} \|f * J_\varepsilon - f\|_{L^p(\Omega', w)} = 0$$

because $|f * J_\varepsilon|(x) \le Mf(x)$ and using once more that the Hardy-Littlewood maximal function Mf is in $L^p(\Omega, w)$ if $w \in A_p$.

If $p = 1$, for any $\eta > 0$ we can choose a continuous function f_η, such that $\|f - f_\eta\|_{L^1(\Omega,w)} < \eta$. By the definition of A_1-weight, $Mw \le cw$, so that, if $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$,

$$\begin{aligned}
\|J_\varepsilon * (f - f_\eta)\|_{L^1_w(\Omega')} &\le \int\limits_{\Omega'} \int\limits_{R^n} \varepsilon^{-n}\, J(\varepsilon^{-1}|x-y|)\, |f(y) - f_\eta(y)|\, dy\, w(x)dx \\
&\le \int\limits_{\Omega} |f(y) - f_\eta(y)| \int\limits_{R^n} \varepsilon^{-n}\, J(\varepsilon^{-1}|x-y|)\, w(x)dx\, dy \\
&\le \int\limits_{\Omega} |f(y) - f_\eta(y)|\, Mw(y)\, dy \\
&\le c \int\limits_{\Omega} |f(y) - f_\eta(y)|\, w(y)dy \\
&= c\, \|f - f_\eta\|_{L^1(\Omega,w)}.
\end{aligned}$$

Let us fix now η such that both $\|f - f_\eta\|_{L^1(\Omega,w)}$ and $\|J_\varepsilon * (f - f_\eta)\|_{L^1(\Omega,w)}$ are small. Our assertion follows because

$$\|J_\varepsilon * f_\eta - f_\eta\|_{L^1(\Omega,w)} \le c\, h(\varepsilon) \int\limits_{\Omega} w(x)\, dx = C\, h(\varepsilon),$$

where $h(\varepsilon) = \sup\limits_{|x-y|<\varepsilon} |\, f_\eta(x) - f_\eta(y)\,|$.

Then, it will be enough to prove that

$$\lim_{\varepsilon \to 0+} \|X_j\,(f * J_\varepsilon) - X_j f\|_{L^p(\Omega', w)} = 0$$

for $j = 1, \dots, m$. Let $Y = (b_1, \dots, b_n)$ be one of these vector fields. Because

$$\begin{aligned}
\|\, Y(f * J_\varepsilon) - Yf\,\|_{L^p(\Omega',w)} \le\ & \|\,Yf - (Yf) * J_\varepsilon\|_{L^p(\Omega',w)} \\
& + \|(Yf) * J_\varepsilon - Y(f * J_\varepsilon)\|_{L^p(\Omega',w)},
\end{aligned}$$

and $Yf \in L^p(\Omega', w)$ we need only to prove that the last term has limit zero as $\varepsilon \to 0+$. Thus, if $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, arguing as in Proposition 1.2.2 of [FSSC], we have:

$$\int_{\Omega'} |Y(f * f_\varepsilon) - (Yf) * J_\varepsilon|^p w(x)dx$$

$$\leq \sum_{i=1}^n \int_{\Omega'} \left| \int_{B(x,\varepsilon)} (b_i(x) - b_i(y))\, f(y) \partial_i J_\varepsilon(x-y) + \partial_i b_i(y) f(y) J_\varepsilon(x-y)\, dy \right|^p w(x)dx.$$

Now, by Rademacher's theorem, for a.e. $y \in \Omega$

$$b_i(x) - b_i(y) = \sum_{h=1}^n \partial_h b_i(y)\ (x-y)_h + R_i(x,y),$$

where for $i = 1, \dots, n$

$$\lim_{x \to y} \frac{R_i(x,y)}{|x-y|} = 0 \qquad \text{for a.e. } y \in \Omega \tag{1.1}$$

$$\frac{|R_i(x,y)|}{|x-y|} \leq C \qquad \text{for all } x, y \in \Omega. \tag{1.2}$$

Eventually we obtain

$$\|Y(f * J_\varepsilon) - (Yf) * J_\varepsilon\|_{L^p(\Omega', w)}$$

$$\leq \sum_{h \neq i} \left(\int_{\Omega'} \left| \int_{B(x,\varepsilon)} f(y) \partial_h b_i(y) \left[(x-y)_h \, \frac{\partial\, J_\varepsilon(x-y)}{\partial x_i} \right] dy \right|^p w(x)dx \right)^{1/p}$$

$$+ \sum_{i=1}^n \left(\int_{\Omega'} \left| \int_{B(x,\varepsilon)} f(y) \partial_i b_i(y) \ \left[(x-y)_i \frac{\partial\, J_\varepsilon(x-y)}{\partial x_i} + J_\varepsilon(x-y) \right] dy \right|^p w(x)dx \right)^{1/p}$$

$$+ \sum_{i=1}^n \left(\int_{\Omega'} \left| \int_{B(x,\varepsilon)} f(y) R_i(x,y) \ \frac{\partial\, J_\varepsilon(x-y)}{\partial x_i}\, dy \right|^p w(x)dx \right)^{1/p}.$$

By Lemma 2.4, each term of the first two lines has limit zero as $\varepsilon \to 0+$. Indeed both $f \partial_h b_i$ and $f \partial_i b_i$ belong to $L^p(\Omega, w)$ and the kernels in the square parentheses satisfy all the assumptions of the lemma because they can be written as

$$-\frac{\partial}{\partial y_i} \ ((y-x)_h\, J_\varepsilon\,(x-y)) \quad \text{for } i \neq h$$

and

$$\frac{\partial}{\partial y_i} \ ((y-x)_i\, J_\varepsilon\,(x-y))$$

respectively. Thus, we can restrict ourselves to consider the terms containing $R_i(x,y)$. Now, because

$$\left| \frac{\partial}{\partial x_i} J_\varepsilon(x-y) \right| = \left| \frac{1}{\varepsilon^{n+1}} \frac{x_i - y_i}{|x-y|} J'(\varepsilon^{-1}|x-y|) \right|$$
$$\leq \frac{1}{\varepsilon^n} \frac{1}{|x-y|} \sup_{t\geq 0} |J'(t)| = C \frac{1}{\varepsilon^n} \frac{1}{|x-y|},$$

each one of these terms is bounded by

$$I_i = C \left(\int_{\Omega'} \left| \int_{B(x,\varepsilon)} |f(y)| \frac{|R_i(x,y)|}{|x-y|} \frac{1}{\varepsilon^n} dy \right|^p w(x) dx \right)^{1/p}.$$

If $p=1$ this integral is equal to

$$I_i = \int_\Omega |f(y)| \frac{1}{\varepsilon^n} \left(\int_{B(y,\varepsilon)} \frac{|R_i(x,y)|}{|x-y|} w(x)\, dx \right) dy.$$

and, by dominated convergence theorem, it has limit zero as $\varepsilon \to 0$. Indeed

$$\frac{1}{\varepsilon^n} \int_{B(y,\varepsilon)} \frac{|R_i(x,y)|}{|x-y|} w(x) dx$$
$$\leq \frac{1}{\varepsilon^n} \int_{B(y,\varepsilon)} \frac{|R_i(x,y)|}{|x-y|} |w(x) - w(y)|\, dx + w(y) \frac{1}{\varepsilon^n} \int_{B(y,\varepsilon)} \frac{|R_i(x,y)|}{|x-y|} dx$$
$$\leq C \left\{ \frac{1}{\varepsilon^n} \int_{B(y,\varepsilon)} |w(x) - w(y)|\, dx + w(y) \int_{B(0,1)} \frac{|R_i(y+\varepsilon\xi, y)|}{\varepsilon|\xi|} d\xi \right\};$$

here the first term of the last line has limit zero at each Lebesgue point y of w, while the second one tends to zero for a.e. $y \in \Omega$ because of (1.1) and (1.2) and the dominated convergence theorem. Hence

$$\lim_{\varepsilon\to 0} |f(y)| \frac{1}{\varepsilon^n} \int_{B(y,\varepsilon)} \frac{|R_i(x,y)|}{|x-y|} w(x) dx = 0 \quad \text{as } \varepsilon \to 0+, \text{for a.e. } y \in \Omega.$$

Moreover, because $w \in A_1$,

$$|f(y)| \frac{1}{\varepsilon^n} \int_{B(y,\varepsilon)} \frac{|R_i(x,y)|}{|x-y|} w(x) dx \leq C\, |f(y)|\, Mw(y) \leq C\, |f(y)|\, w(y).$$

This concludes the proof in the $p=1$ case.

If $p > 1$, define $f_m = \min\{|f|, m\}$. Then

$$\left(\int_{\Omega'} \left| \int_{B(x,\varepsilon)} |f(y)| \frac{|R_i(x,y)|}{|x-y|} \frac{1}{\varepsilon^n}\, dy \right|^p w(x)dx\right)^{1/p}$$

$$\leq \left(\int_{\Omega'} \left| \int_{B(x,\varepsilon)} (|f(y)| - f_m(y)) \frac{|R_i(x,y)|}{|x-y|} \frac{1}{\varepsilon^n} dy \right|^p w(x)dx\right)^{1/p}$$

$$+ \left(\int_{\Omega'} \left| \int_{B(x,\varepsilon)} f_m(y) \frac{|R_i(x,y)|}{|x-y|} \frac{1}{\varepsilon^n} dy \right|^p w(x)dx\right)^{1/p}$$

$$= I_1 + I_2.$$

Without loss of generality assume that $\operatorname{supp} f,\ \operatorname{supp} f_m \subseteq \Omega$. Then

$$I_1 \leq C \left(\int_{\Omega'} \left(\varepsilon^{-n} \int_{B(x,\varepsilon)} (|f(y)| - f_m(y))\, dy\right)^p w(x)dx\right)^{1/p}$$

$$\leq C \left(\int_{\Omega'} (M(|f| - f_m))^p\, w(x)dx\right)^{1/p}$$

$$\leq C \left(\int_{\Omega} (|f| - f_m)^p\, w(x)\, dx\right)^{1/p}$$

because $w \in A_p$. Hence for any $\theta > 0$ it is possible to choose $m = m_\theta > 0$ such that $I_1 < \theta$ for all $\varepsilon > 0$. Let now this m be fixed. With $y = x + \varepsilon\eta$ we have

$$I_2 = \left(\int_{\Omega'} \left(\int_{B(0,1)} f_m\,(x+\varepsilon\eta)\ \frac{|R_i(x, x+\varepsilon\eta)|}{\varepsilon|\eta|}\, d\eta\right)^p w(x)dx\right)^{1/p}$$

$$\leq \int_{B(0,1)} \left(\int_{\Omega'} f_m^p(x+\varepsilon\eta) \left(\frac{|R_i(x, x+\varepsilon\eta)|}{\varepsilon|\eta|}\right)^p w(x)dx\right)^{1/p} d\eta$$

by Minkowski integral inequality. Hence

$$I_2 \leq \int\limits_{B(0,1)} \left(\int\limits_{\Omega'} f_m^p(x+\varepsilon\eta) \left(\frac{|R_i(x, x+\varepsilon\eta)|}{\varepsilon|\eta|}\right)^p |w(x) - w(x+\varepsilon\eta)|dx\right)^{1/p} d\eta$$

$$+ \int\limits_{B(0,1)} \left(\int\limits_{\Omega'} f_m^p(x+\varepsilon\eta) \left(\frac{|R_i(x, x+\varepsilon\eta)|}{\varepsilon|\eta|}\right)^p w(x+\varepsilon\eta)dx\right)^{1/p} d\eta$$

$$= J_1 + J_2.$$

Now

$$J_1 \leq Cm \int_{B(0,1)} \left(\int_{\Omega} |w(x) - w(x+\varepsilon\eta)|\, dx\right)^{1/p} d\eta < \theta$$

if ε is sufficiently small.

Let now $x + \varepsilon\eta = \chi$ for any $\eta \in B(0,1)$. If $\varepsilon < \mathrm{dist}\,(\Omega', \partial\Omega)$, we get

$$J_2 = \int\limits_{B(0,1)} \left(\int\limits_{\Omega'} f_m^p(\chi) \; \left(\frac{|R_i(\chi - \varepsilon\eta, \chi)|}{\varepsilon|\eta|} \right)^p \; w(\chi)dx \right)^{1/p} d\eta.$$

On the other hand, as $\varepsilon \to 0$ and for a.e. $\chi \in \Omega$

$$f_m^p(\chi) \; \left(\frac{R_i(\chi - \varepsilon\eta, \chi)}{\varepsilon|\eta|} \right)^p w(\chi) \longrightarrow 0$$

and

$$f_m^p(\chi) \left| \frac{R(\chi - \varepsilon\eta, \chi)}{\varepsilon|\eta|} \right|^p w(\chi) \le C\, f_m^p(\chi)\, w(\chi),$$

which belongs to L^1, so that, using twice the dominated convergence theorem, we get that $J_2 \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Thus the assertion is completely proved. $\square$

Proof of Theorem 1.2. Given Proposition 1.4, the full proof of Theorem 1.2 can be achieved following the same arguments of Meyers and Serrin in their original paper [MS]. $\square$

2 Poincaré inequality and approximation theorems

Throughout this section we assume that Ω is bounded and that the family of vector fields $X = (X_1, \dots, X_m)$ is defined and Lipschitz continuous in a neighborhood Ω_0 of $\overline{\Omega}$. Let us recall the following standard definition (see, e.g., [FP], [FL], [NSW]).

Definition 2.1. We say that an absolutely continuous curve $\gamma : [0,T] \longrightarrow \Omega_0$ is a sub-unit curve with respect to X if for any $\xi \in R^n$

$$\langle \dot\gamma(t), \xi \rangle^2 \le \sum_{j=1}^m \langle X_j(\gamma(t)), \xi \rangle^2$$

for a.e. $t \in [0,T]$. If $x_1, x_2 \in \Omega_0$, we define

$$\begin{aligned}\rho(x_1, x_2) = \inf\, \{ T > 0 :\ & \text{there exists a sub-unit curve } \gamma \\ & \gamma : [0,T] \longrightarrow \Omega_0, \quad \gamma(0) = x_1,\ \gamma(T) = x_2 \}.\end{aligned}$$

If the above set of curves is empty, we put $\rho(x_1, x_2) = \infty$.

In all the theorems of this Section we will assume the following hypotheses (H1) and (H2) hold

(H1) $\rho(x, y) < \infty$ for any $x, y \in \Omega_0$, so that ρ is a distance in Ω_0. Moreover, the distance ρ is continuous with respect to the usual topology of R^n.

If $x \in \Omega_0$ and $r > 0$ we will denote by $B_\rho(x,r) = \{y \in \Omega_0 : \ \rho(x,y) < r\}$ the *metric balls* with respect to ρ. The following is known as *doubling property* of ρ.

(H2) For any compact $K \subset \Omega_0$ and for any $r < r_K$ there exists a positive constant C_K such that
$$|B_\rho(x,2r)| \leq C_K\, |B_\rho(x,r)|$$
for any $x \in K$ and $r < r_K$.

¿From now on we will call *geometric constant* any constant depending only on the dimension n, on $C_{\overline{\Omega}}$, on the distance from Ω to Ω_0, and on the $\mathcal{C}^{1,1}$ norm of the vector fields.

Moreover, for the sake of simplicity, we will omit the index ρ in B_ρ when there is no way of misunderstanding, and we will denote by the same letter C different geometric constants.

Remark 2.2. Assumptions (H1) and (H2) are satisfied by several important families of vector fields. For instance:

(i) If the vector fields are smooth and the rank of the Lie algebra generated by $X_1 \dots, X_m$ equals n at any point of Ω_0 (Hörmander condition), then (H1) and (H2) hold ([NSW]).

(ii) If the vector fields are as in [FL], [F1] and [F2], then (H1) and (H2) hold. These assumptions still hold if the vector fields are as in [FGuW], with the strong A_∞ weight identically one.

The following properties of the metric balls follow straightforwardly from (H2).

Proposition 2.3. *Let (H1) and (H2) hold. Then there exist geometric constants $\alpha \geq n,\ r_0 > 0,\ c_1 > 0,\ c_2 > 0,\ c_3 > 0,\ c_4 > 0$ such that*

(i) $|B(x,s)| \geq c_1 \left(\frac{s}{r}\right)^\alpha |B(x,r)| \quad \forall x \in \overline{\Omega},\ \ \forall r,s \quad 0 < s < r \leq r_0;$

(ii) $|B(x,s)| \leq c_2\, s^n \quad \forall x \in \overline{\Omega},\ \ \forall s \quad 0 < s \leq r_0;$

(iii) $c_3|B(x,\rho(x,y))| \leq |B(y,\rho(x,y))| \leq c_4|B(x,\rho(x,y))|$ *for any* $x,y \in \overline{\Omega},\ \ \rho(x,y) \leq r_0$.

Because of (H1), (H2) and following [*C*] we can define A_p classes with respect to the metric ρ and to the Lebesgue measure dx.

Definition 2.4. If w is a weight function, we will say that $w \in A_p = A_p\left(\Omega_0, \rho, dx\right)$ if for all metric balls $B = B_\rho(x,r) \subseteq \Omega_0$

$$\text{(i)} \qquad \fint_B w\, dx \left(\fint_B w^{-1/(p-1)} dx \right)^{p-1} \leq c \qquad \text{when } 1 < p < \infty$$

$$\text{(ii)} \qquad \fint_B w\, dx \leq c\ \operatorname*{ess\,inf}_B\ w \qquad \text{when} \quad p = 1.$$

The smallest constant for which (i) or (ii) hold is the A_p bound of w.

In the following proposition we collect some properties of A_p weights that will be used later. A proof can be found in [C].

Proposition 2.5. *Let $1 \le p < +\infty$ and $w \in A_p(\Omega_0, \rho, dx)$. Then there are constants $c_5, c_6 > 1$ and $\varepsilon_0 > 0$, depending only on the geometric constants of Proposition 2.3 and on the A_p bound of w, such that $\forall B = B_\rho(x, r)$ with $x \in \overline{\Omega}$ and $r < r_0$*

(i) $$w(B_\rho(x, 2r)) \le c_5\, w(B_\rho(x, r))$$

(ii) $$\fint_B w^{1+\varepsilon}(x)\, dx \le c_6 \left(\fint_B w(x)\, dx \right)^{1+\varepsilon}.$$

As usual (i) is called the doubling property of w and (ii) is the reverse Holder inequality. Again, if $w \in A_p(\Omega_0, \rho, dx)$, then $L^p(\Omega_0, w) \subseteq L^1_{\mathrm{loc}}(\Omega_0)$.

Remark 2.6. We stress here that A_p classes with respect to the metric ρ are not trivial (i.e. they contain other functions than constants) and that they differ from A_p classes with respect to the Euclidean metric.

To prove the first assertion, let us prove that, if $\beta > 0$ is such that

$$|B(\bar{x}, \theta r)| \le c\theta^\beta |B(\bar{x}, r)|$$

for some fixed $\bar{x} \in \bar{\Omega}$ and for $\theta \in (0, 1)$, $r \in (0, r_0)$, then $\rho(\bar{x}, \cdot)^\gamma$ belongs to $A_p(\bar{\Omega}, \rho, dx)$ for $-\beta < \gamma < \beta(p-1)$. Note that such β exists and $\beta \ge n$, by Proposition 2.3. Let us prove the assertion when $p > 1$; the case $p = 1$ can be handled in the same way. If $\sigma > -\beta$ and $r \in (0, r_0)$, we have

$$\begin{aligned}
\int_{B(\bar{x}, r)} \rho^\sigma(\bar{x}, x)\, dx &= \sum_{k \le 0} \int_{2^{k-1} r \le \rho(\bar{x}, x) < 2^k r} \rho^\sigma(\bar{x}, x)\, dx \\
&\sim \sum_{k \le 0} 2^{k\sigma} r^\sigma |B(\bar{x}, 2^k r) \setminus B(\bar{x}, 2^{k-1} r)| \\
&\le |B(\bar{x}, r)| r^\sigma \sum_{k \le 0} 2^{k(\sigma + \beta)},
\end{aligned}$$

so that, by choosing successively $\sigma = \gamma$ and $\sigma = -\gamma/(p-1)$, we get

$$A(\bar{x}, r) = \fint_{B(\bar{x}, r)} \rho^\sigma(\bar{x}, x) dx \cdot \left(\fint_{B(\bar{x}, r)} \rho^{-\gamma/(p-1)}(\bar{x}, x) dx \right) \le \text{const.}$$

If now $\rho(\bar{x}, r) < 2r$, then $B(x, r) \subset B(\bar{x}, 3r)$, and $|B(x, r)| \sim |B(\bar{x}, 3r)|$ (by Proposition 2.2), so that $A(x, r) \le$ const. Finally, if $\rho(\bar{x}, x) > 2r$, then, if $y \in B(x, r)$, we have $\frac{1}{2}\rho(\bar{x}, x) < \rho(\bar{x}, y) < \frac{3}{2}\rho(\bar{x}, x)$, so that $A(x, r)$ is bounded and then $\rho^\gamma(\bar{x}, x)$ is an A_p weight with respect to the metric ρ.

If we consider now the vector fields $\partial_\xi + 2\eta\partial_\tau$ and $\partial_\eta - 2\xi\partial_\tau$ in $R^3_{(\xi, \eta, \tau)}$ (Heisenberg group), then it is well known that $|B(x, r)| \sim r^4$ for any $x = (\xi, \eta, \tau) \in R^3$ and for $r > 0$. In addition, $\rho(x, 0) \sim (\xi^2 + \eta^2 + |\tau|)^{1/2}$, so that $\omega(\xi, \eta, \tau) = (\xi^2 + \eta^2 + |\tau|)^{\gamma/2}$ is a A_2 weight for $-4 < \gamma < 4$. Choose now $\gamma \in (2, 4)$; we will show that ω is not an A_2 weight with respect to Euclidean balls. First of all, denoting by

$S = \{(\xi,\eta,\tau);\ \xi^2+\eta^2+\tau^2 \le 2r^2\}$ the Euclidean ball centered at the origin, we have:

$$\int_S (\xi^2+\eta^2+|\tau|)^{\gamma/2}\, d\xi\, d\eta\, d\tau \sim \int_{\substack{\rho^2+\tau^2\le 2r^2\\ \rho>0,\tau>0}} \rho(\rho+\tau)^{\gamma/2} d\rho\, d\tau$$
$$\ge c_\gamma \int_0^r \rho\, d\rho \int_0^r \tau^{\gamma/2} d\tau \sim r^{3+\gamma/2}.$$

On the other hand

$$\int_S (\xi^2+\eta^2+|\tau|)^{-\gamma/2}\, d\xi\, d\eta\, d\tau \ge c\int_0^r d\tau \int_0^r d\rho\, \rho(\rho^2+\tau)^{-\gamma/2}$$
$$= c_\gamma \int_0^r \{\tau^{1-\gamma/2} - (r^2+\tau)^{1-\gamma/2}\}\, d\tau = c_\gamma\{r^{2-\gamma/2} - (r^2+r)^{2-\gamma/2} + r^{4-\gamma}\}$$
$$= c_\gamma r^{2-\gamma/2}\{r(\frac{\gamma}{2} - 2 + o(1)) + r^{2-\gamma/2}\} = c_\gamma r^{4-\gamma}\{1+o(1)\},$$

so that, if r is small enough,

$$\fint_S \omega d\xi\, d\eta\, d\tau\ \fint_S \omega^{-1} d\xi\, d\eta\, d\tau \ge c r^{1-\gamma/2} \to \infty$$

as $r \to 0+$, since $\gamma > 2$.

The main result of this section is the following

Theorem 2.7. *Let $\Omega \subset\subset \Omega_0$ be a bounded open set, $1 < p < +\infty$ and $X = (X_1,\dots,X_m)$ be a family of Lipschitz continuous vector fields defined in Ω_0. Suppose that X satisfy (H1), (H2) and that the following representation formula holds*

$$\begin{aligned}&\textit{there exist geometric constants } c, C > 0 \textit{ such that } \forall B = B_\rho(\bar{x}, r)\\ &\textit{with } cB := B_\rho(\bar{x}, cr) \subseteq \Omega_0, \forall f \in Lip(\overline{cB}) \textit{ and } \forall x \in \bar{B}\\ &|f(x) - \fint_B f(y)dy| \le C \int_{cB} |Xf(y)|\ \frac{\rho(x,y)}{|B(x,\rho(x,y))|}\, dy := T_{cB}(|Xf|)(x).\end{aligned} \tag{2.1}$$

Suppose that $w \in A_p(\Omega_0, \rho, dx)$. Then

$$W^{1,p}_X(\Omega, w) = H^{1,p}_X(\Omega, w).$$

Remark 2.8. If $X = (X_1,\dots,X_m)$ satisfy Hörmander condition (see Remark 2.2(i)), then (2.1) is proved in [FLW], Proposition 2.12. Moreover, if $X_1,\dots,X_m$ satisfy the assumptions of Remark 2.2(ii), the representation formula (2.1) can be deduced respectively from the results of [F1,2] and from Corollary 3.2 of [FGuW] (keeping in mind the arguments of Corollary 3.3 therein).

Proof of Theorem 2.7. The proof will consist of many steps, some of them having independent interest. The first one states that the distance $\rho(\bar{x},\cdot)$ from a fixed point $\bar{x} \in \overline{\Omega}$ belongs to $H^{1,p}_X(\Omega)$ (no weights!) for any $p \ge 1$ and that $|X\rho(\bar{x},\cdot)| \in L^\infty(\Omega)$. This fact will enable us to construct suitable cut-off functions associated with the metric balls B; it can also be used to construct test functions of the same type when dealing with degenerate elliptic equations as

$$\sum_{i,j=1}^{m} X_i^*(a_{ij}X_jf) = 0,$$

$(a_{ij})_{i,j=1,\dots,m}$ being an elliptic matrix. However, we note explicitly that, when $X_1,\dots,X_m$ are as in Remark 2.2 (i) or (ii), cut-off functions associated with the metric balls have already been constructed (see e.g. [CGL], [L1], [L2], [FL], [F1], [FGuW]).

Step 1.

Proposition 2.9. *Let $\Omega \subset\subset \Omega_0$ be a bounded open set, and let $\bar{x} \in \overline{\Omega}$ be fixed. Then the function $\rho(x) = \rho(\bar{x}, x)$ belongs to $H_X^{1,p}(\Omega)$ for any $p \geq 1$. In addition, $|X\rho| \in L^\infty(\Omega)$. Note that this result does not depend on assumption (2.1).*

Proof. Let $j \in \{1,\dots,m\}$ be fixed. For the sake of simplicity, we write $Y = (c_1,\dots,c_n)$ instead of X_j. We will show later that the function $x \longrightarrow \exp_x(tY)$ is Lipschitz continuous for any fixed $t \in R$, $|t| \leq T$, T depending on the distance from Ω to Ω_0 and on the Lipshitz constant L of Y. Moreover its Jacobian determinant $J(x,t)$ satisfies $J(x,t) = 1 + J_1(x,t)$, with

$$|J_1(x,t)| \leq c|t| \tag{2.2}$$

for a.e. $x \in \bar{\Omega}_1$ ($\Omega \subset\subset \Omega_1 \subset\subset \Omega_0$), for $|t| < T$ and with a constant c not depending on x and on t. Let us assume these facts and let us complete the proof. By definition, if $\varphi \in \mathcal{D}(\Omega)$, we have

$$\langle Y\rho, \varphi\rangle = -\int \rho(x)\,(Y\varphi)\ (x)dx - \int \rho(x)\,\varphi(x)\operatorname{div} Y(x)\,dx.$$

The absolute value of the second integral is bounded by $c\operatorname{diam}_\rho(\Omega)\ \|\varphi\|_{L^1(\Omega)}$ where c depends only on the Lipschitz constant L of Y.
The first term is equal to the limit as $t \longrightarrow 0+$ of

$$\int_\Omega \rho(x)\,\frac{1}{t}\ \{\varphi(x) - \varphi\left(\exp_x(tY)\right)\}dx$$
$$= \frac{1}{t}\Big\{\int_\Omega \rho(x)\varphi(x)dx - \int_\Omega \rho(x)\varphi(\exp_x(tY))dx\Big\}.$$

Note now that the map $x \longrightarrow \exp_x(tY)$ is 1-1 for any fixed t, $|t| < T$ by the uniqueness of the Cauchy problem, and that its inverse map is given by $x' \longrightarrow \exp_{x'}(-tY)$. Let $x' = \exp_x(tY)$. Then the difference quotient above can be written as

$$\frac{1}{t}\Big\{\int_\Omega \rho(x)\varphi(x)dx - \int_\Omega \rho(\exp_{x'}(-tY))\,\varphi(x')dx'\Big\}$$
$$-\frac{1}{t}\int_\Omega \rho(\exp_{x'}(-tY))\varphi(x')J_1(x'-t)dx'.$$

Observe that

$$\left| \frac{1}{t} \int_{\Omega} \rho(\exp_{x'}(-tY))\varphi(x')\, J_1(x'-t)dx' \right| \le c\, \mathrm{diam}_\rho(\Omega)\ \|\varphi\|_{L^1(\Omega)},$$

and that, by the triangle inequality, $|\rho(x) - \rho(\exp_x(-tY))| \le \rho(x, \exp_x(-tY)) \le |t|$, because $t \longrightarrow \exp_x(-tY)$ is a sub-unit curve. Hence

$$\begin{aligned}\left| \int_{\Omega} \rho(x)\, Y\varphi(x)dx \right| &\le \limsup_{t\to 0+} \int_{\Omega} \left| \frac{\rho(x) - \rho(\exp_x(-tY))}{t} \right| |\varphi(x)|dx \\ &\quad + c\, \mathrm{diam}_\rho(\Omega)\ \|\varphi\|_{L^1(\Omega)} \\ &\le c_\Omega\ \|\varphi\|_{L^1(\Omega)}.\end{aligned}$$

Thus the functional $\varphi \longrightarrow \langle Y\rho, \varphi\rangle$ is a continuous linear functional on $L^1(\Omega)$, hence $Y\rho$ can be identified with a $L^\infty(\Omega)$-function. This way, we have proved that ρ belongs to $W^{1,p}_X(\Omega)$ for any p, $1 \le p \le +\infty$; hence, given Theorem 1.2, $\rho \in H^{1,p}_X(\Omega)$ for $1 \le p < +\infty$.

We prove now our assertion about the map $x \longrightarrow \exp_x(tY)$. Without loss of generality, we may assume $t > 0$. For the sake of simplicity, let us write $\exp_x(tY) = u_x(t)$. By definition, if $x, y \in \Omega$, we have

$$u_x(t) - u_y(t) = x - y + \int_0^t \left[Y(u_x(\sigma)) - Y\,(u_y(\sigma)) \right] d\sigma,$$

so that

$$|u_x(t) - u_y(t)\,| \le |x-y| + L \int_0^t |u_x(\sigma) - u_y(\sigma)| d\sigma.$$

Then, by Gronwall's lemma, if $t \le T$ we have

$$|u_x(t) - u_y(t)\,| \le C_T\, |x-y|.$$

Let now $t \in]0, T]$ be fixed. By Rademacher's theorem, the map $x \longrightarrow u_x(t)$ is differentiable for a.e. $x \in \Omega$; let x be one of these points and let us consider $(u_x)_j$ the j-th component of u_x. If $k \in \{1, \dots, n\}$ we have for $s > 0$

$$\begin{aligned}\frac{1}{s} \left\{ (u_{x+se_k}(t))_j - (u_x(t))_j \right\} &= \delta_{jk} + \frac{1}{s} \int_0^t \left[c_j\,(u_{x+se_k}(\sigma)) - c_j(u_x(\sigma)) \right] d\sigma \\ &= \delta_{jk} + f_{j,k}(x,t,s).\end{aligned}$$

Since the left-hand side of the above identity converges as $s \longrightarrow 0+$ (by the differentiability), we obtain that $f_{jk}(x,t,s)$ has a limit $f_{j,k}(x,t)$ as $s \longrightarrow 0$. On the other hand

$$\begin{aligned}|f_{j,k}(x,t,s)| &\le \frac{L}{s} \int_0^t |u_{x+se_k}(\sigma) - u_x(\sigma)\,| d\sigma \\ &\le \frac{L}{s}\, C_T\, s\, t = c\, t\end{aligned}$$

so that also $|f_{j,k}(x,t)| \le c\,t$ and (2.2) follows trivially. $\square$

Step 2.

We keep the notations of Theorem 2.7. Let $U \subseteq \Omega_0$ be an open neighborhood of $\overline{cB}$, and let g be a given function belonging to $W^{1,p}_X(U,w)$. As we pointed out above, $L^p(U,w) \subseteq L^1(U)$, so that $g \in W^{1,1}_X(U) = H^{1,1}_X(U)$, by [FSSC], Theorem 1.6 (or Theorem 1.2 above, with $w \equiv 1$). Following [FLW], if $k \in Z$ we put

$$g_k(x) = \begin{cases} 2^{k-1} & \text{if} \quad |g-g_B| \le 2^{k-1} \\ |g-g_B| & \text{if} \quad 2^{k-1} < |g-g_B| \le 2^k \\ 2^k & \text{if} \quad |g-g_B| > 2^k, \end{cases}$$

where

$$g_B = \fint_B g\,dx.$$

Arguing as in [GT], section 7.4, it is easy to see that g_k still belongs to $H^{1,1}_X(U)$ and, if we put

$$\begin{aligned} S_k &= \{x \in B : \quad 2^k < |g(x)-g_B| \le 2^{k+1}\} \\ S_k^* &= \{x \in cB : \quad 2^k < |g(x)-g_B| \le 2^{k+1}\}, \end{aligned}$$

then $|Xg_k| = |Xg|\chi_{S^*_{k-1}}$ a.e. in cB.

Let now $(\varphi_h)_{h\in N}$ be a sequence of smooth functions converging to g_k in $H^{1,1}_X(U)$. Without loss of generality, we assume that

$$\varphi_h \longrightarrow g_k \quad \text{and} \quad |X\varphi_h| \longrightarrow |Xg_k|$$

as $h \longrightarrow \infty$, a.e. in U. Let us prove now that

$$T_{cB}(|X\varphi_h|) \longrightarrow T_{cB}(|Xg_k|)$$

in $L^1(B)$ as $h \longrightarrow \infty$. In fact, we have

$$\begin{aligned} &\int_B \Big| T_{cB}(|X\varphi_h|) - T_{cB}(|Xf_k|) \Big|\,dx \\ &\le \int_B \int_{cB} \Big| |X\varphi_h|(y) - |Xf_k|(y) \Big| \frac{\rho(x,y)}{|B(x,\rho(x,y))|}\,dy\,dx \\ &\le \int_{cB} |X(\varphi_h - f_k)|(y) \int_B \frac{\rho(x,y)}{|B(x,\rho(x,y))|}\,dx\,dy. \end{aligned}$$

By triangle inequality, $B \subseteq B(y,(1+c)r)$ and, by Proposition 2.3 (iii), it follows

$$\begin{aligned} \int_B \frac{\rho(x,y)}{|B(x,\rho(x,y))|}\,dx \le\ & c \int_{B(y,(1+c)r)} \frac{\rho(x,y)}{|B(y,\rho(x,y))|}\,dx \\ &= c\sum_{\ell=0}^{\infty} \int_{\frac{(1+c)r}{2^{\ell+1}} \le \rho(x,y) \le \frac{(1+c)r}{2^\ell}} \frac{\rho(x,y)}{|B(y,\rho(x,y))|}\,dy \\ &\le c\ (1+c)r \sum_{\ell=0}^{\infty} 2^{-\ell}\ \frac{|B(y,2^{-\ell}(1+c)r)|}{|B(y,2^{-\ell-1}(1+r)r)|} \\ &\le c\ r \quad \text{by doubling condition (H2).} \end{aligned}$$

Thus

$$\| T_{cB}(|X\varphi_h|) - T_{cB}(|Xg_k|) \|_{L^1(B)} \le c\, r\, \|\varphi_h - g_k\|_{H^{1,1}_X(U)}$$

and the assertion follows. Again, without loss of generality, we may assume that

$$T_{cB}(|X\varphi_h|)\,(x) \longrightarrow T_{cB}(|Xg_k|)\,(x)$$

as $h \longrightarrow \infty$ for a.e. $x \in B$. On the other hand, by (2.1),

$$|\varphi_h(x) - \fint_B \varphi_h| \le c\, T_{cB}(|X\varphi_h|)\,(x),$$

so that we can take the limit as $h \longrightarrow \infty$ and we obtain that g_k still satisfies the representation formula (2.1) in B.

We can now argue as in [FLW], Proposition 2.12, to prove the following technical lemma:

Lemma 2.10. *With the notations of Theorem 2.7, we have*

$$|g(x) - g_B| \le c\Big\{ T_{cB}(|Xg|\chi_{S^*_{k-1}})\,(x) + \frac{r}{|B|} \int_B |Xg|dy \Big\} \tag{2.3}$$

a.e. in S_k, for any $g \in W^{1,p}_X(\Omega, w)$.

Proof. We have: $2^{k-1} \le g_k(y) \le 2^{k-1} + |g(y) - g_B|$, so that for a.e. $x \in S_k$ we have:

$$\begin{aligned} 2^k = g_k(x) &\le |g_k(x) - (g_k)_B| + (g_k)_B \\ &\le C\, T_{cB}\,(|Xg_k|)\,(x) + 2^{k-1} + \fint_B |g(y) - g_B|dy, \end{aligned}$$

so that, keeping in mind that for a.e. $x \in S_k$ we have $|g(x) - g_B| \le 2^k$, we obtain

$$|g(x) - g_B| \le 2c\, T_{cB}(|Xg|\chi_{S^*_{k-1}})\,(x) + \fint_B |g(y) - g_B|dy.$$

Then, to achieve the proof of the lemma is enough to apply the (unweighted) Poincaré inequality

$$\int_B |g - g_B|dx \le c\, r \int_B |Xg|dx,$$

which can be deduced from (2.1) as follows. First of all, take an arbitrary metric ball $\widetilde{B} = B(x, \theta)$ such that $c\widetilde{B} \subset \Omega$. The representation formula (2.1) holds for g in $\widetilde{B}$; hence arguing as above

$$\int_{\widetilde{B}} |f - f_{\widetilde{B}}|dy \le c \int_{c\widetilde{B}} |Xf(y)| \int_{\widetilde{B}} \frac{\rho(x,y)}{|B(x, \rho(x,y))|}\, dx\, dy \le c\,\theta \int_{c\widetilde{B}} |Xf(y)|dy.$$

Finally we can work as in [FGuW] and[FLW] applying, for instance, Theorem 5.2 and Theorem 5.4 in [FGuW] to get rid of the "enlarging constant"c in the integration domain of $|Xf|$. This completes the proof. □

Given (2.3) and repeating all the arguments of [FLW] the following Sobolev-Poincaré inequality holds:

Theorem 2.11. *Let w_1, w_2 be weight functions and let $1 \le p < q < \infty$ be such that the following balance condition holds for all metric balls I, J with $I \subset J \subset B$ where B is a metric ball with center in $\overline{\Omega}$ and radius $r(B) < r_0$*

$$\frac{r(I)}{r(J)} \left(\frac{w_2(I)}{w_2(J)}\right)^{1/q} \le c \left(\frac{w_1(I)}{w_1(J)}\right)^{1/p} \tag{2.4}$$

Moreover, let us assume that $w_1 \in A_p(\Omega_0, \rho, dx)$, that w_2 is doubling and that (H1), (H2) and (2.1) hold. Then for all $f \in W^{1,p}_X(\Omega, w_1)$ there exists $c(f,B) \in R$ such that

$$\left(\fint_B |f - c(f,B)|^q \, dw_2\right)^{1/q} \le c\, r \left(\fint_B |Xf|^p \, dw_1\right)^{1/p} \tag{2.5}$$

where the constant c is independent of f. In particular, under our assumptions, $W^{1,p}_X(\Omega, w_1)$ is continuously embedded in $L^q(\Omega, w_2)$.

Remark 2.12. If $p > 1$, the above Sobolev-Poincaré inequality has mainly an instrumental interest. Indeed, once our density theorem is proved, the above result follows straightforwardly from the analogous result in [FLW] for regular functions via a limit argument. Because of this, we will not go into more precise statements about the constant $c(B, f)$ that can be found in [FLW]. In the case of vector fields to whom Remark 2.2 does not apply, Theorem 2.11 gives an abstract result which, in turn, is quite obvious looking to the arguments of [FLW].

Remark 2.13. If $1 < p = q$, we can argue as in [FLW], Remark 1.6. Then, inequality (2.5) still holds in this case if there exists $s > 1$ such that w_2^s is a doubling weight and the balance condition (2.4) is replaced by the condition

$$\left(\frac{r(I)}{r(J)}\right)^p \frac{\mathcal{A}_s(I, w_2)}{w_2(J)} \le c \frac{w_1(I)}{w_1(J)} \tag{2.4 bis}$$

for all balls I, J with $I \subset J \subset B$, where

$$\mathcal{A}_s(I, w_2) = |I| \left(\frac{1}{|I|} \int_I w_2^s dx\right)^{\frac{1}{s}}.$$

Moreover, if $p = q = 1$, (2.5) still holds if w_2 is doubling and w_1, w_2 satisfy the condition

$$\frac{1}{w_2(I)} \int_I \frac{\rho(x,y)}{|B(y, \rho(x,y))|} w_2(x) dx \le c \frac{r(I)}{w_1(I)} w_1(y) \text{ a.e. in } I \tag{2.4 ter}$$

for all balls $I \subset B$.
If $w_2 \in A_p$, then (2.4 bis) and (2.4 ter) are equivalent to (2.4). In particular, if $w_1 \equiv w_2 \equiv w \in A_p$, then (2.4.bis) and (2.4.ter) hold, so that

$$\left(\int_B |f(x) - c(f,B)|^p w \, dx\right)^{1/p} \le c\, r \left(\int_B |Xf|^p w \, dx\right)^{1/p} \tag{2.6}$$

for $p \geq 1$.

Step 3.

To prove Theorem 2.7 we still need a few more technical results we will include in this step.

Lemma 2.14. *If $p \geq 1$, we will say that $u \in H^{1,p}_{X,\mathrm{loc}}(\Omega, w)$ if $\psi u \in H^{1,p}_X(\Omega, w)$ for any $\psi \in C_0^\infty(\Omega)$. Then we have*

$$H^{1,p}_{X,\mathrm{loc}}(\Omega, w) \cap W^{1,p}_X(\Omega, w) = H^{1,p}_X(\Omega, w).$$

Proof. The proof is basically the last step of Meyer-Serrin's proof. Let Ω_j, $j = 1, 2 \dots$ be open sets in Ω such that

$$\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega \quad \text{and} \quad \bigcup_j \Omega_j = \Omega.$$

Then let $\{\psi_j : \ j = 0, 1, \dots\}$ be a partition of the unity subordinated to the covering $\{\Omega_{j+1} \backslash \Omega_{j-1} : j = 0, 1, \dots\}$, where $\Omega_0 = \Omega_{-1} = \emptyset$. Then, if $u \in H^{1,p}_{X,\mathrm{loc}}(\Omega, w) \cap W^{1,p}_X(\Omega, w)$, then for any j and for any $\varepsilon_j > 0$ there exists $u_j \in C^\infty(\Omega) \cap W^{1,p}_X(\Omega, w)$ such that

$$\| \ u_j - u\psi_j \ \|_{W^{1,p}_X(\Omega,w)} < \varepsilon_j.$$

Let now $\widetilde{\psi}_j$ be smooth functions supported in $\Omega_{j+1} \backslash \Omega_{j-1}$ such that $0 \leq \widetilde{\psi}_j \leq 1$ and $\widetilde{\psi}_j \equiv 1$ on $K_j := \operatorname{supp} \psi_j$. Moreover, let $b_j > 0$ be such that $|X\widetilde{\psi}_j| \leq b_j$ for $j = 1, \dots$.

Consider now the function $v = \sum_j \widetilde{\psi}_j u_j$, which is smooth since it is a locally finite sum of smooth functions. We have

$$\|v - u\|_{W^{1,p}_X(\Omega,w)} \leq \sum_j \| \ \widetilde{\psi}_j \, u_j - u\psi_j \|_{W^{1,p}_X(\Omega,w)}.$$

Note that

$$\int_{\Omega \backslash K_j} |\widetilde{\psi}_j u_j|^p \ w dx \leq \int_{\Omega \backslash K_j} |u_j|^p w dx \leq \int_\Omega |u_j - u\psi_j|^p w \, dx < \varepsilon_j^p,$$

so that

$$\int |\widetilde{\psi}_j \, u_j - u\psi_j|^p w \, dx = \int_{K_j} |u_j - u\psi_j|^p w \, dx + \int_{\Omega \backslash K_j} |\widetilde{\psi}_j \, u_j|^p w \, dx < 2\varepsilon_j^p.$$

In addition, arguing as above,

$$\int_\Omega |X \, (\widetilde{\psi}_j u_j - u\psi_j)|^p w \, dx \leq \int_{K_j} |X(u_j - u\psi_j)|^p w \, dx$$

$$+ \int_{\Omega \backslash K_j} |\widetilde{\psi}_j|^p \, |X u_j|^p w \, dx + \int_{\Omega \backslash K_j} |u_j|^p \, |X \widetilde{\psi}_j|^p \ w dx \leq (2 + b_j^p)\varepsilon_j^p,$$

so that

$$\| \ \widetilde{\psi}_j u_j - u\psi_j \ \|_{W^{1,p}_X(\Omega,w)} \leq c \, (1 + b_j)\varepsilon_j.$$

Choosing now $\varepsilon_j = 2^{-j}\varepsilon/(1 + b_j)$, the assertion is proved. $\square$

Arguing as in [SC], Lemma 3.3, by slight changes in the proof of Lemma 5.5 in [FGuW], we obtain the following Whitney's lemma:

Lemma 2.15. *Let $\delta \in (0,1/10)$ be given; then there exists a countable family $\{\widetilde{B}_j^\delta = B(\tilde{x}_j^\delta, \tilde{r}_{j\delta}) : j = 1,2,\dots\}$ of metric balls contained in Ω and a geometric constant $\tilde{c}_1$ such that*

$$\bigcup_j 3\widetilde{B}_j^\delta = \Omega, \qquad \widetilde{B}_j^\delta \cap \widetilde{B}_k^\delta = \emptyset \quad \textit{for } j \neq k \tag{2.7}$$

$$\tilde{r}_{j\delta} = \delta \operatorname{dist}_\rho(\widetilde{B}_j^\delta,\ \partial\Omega) \tag{2.8}$$

$$\sum_j \chi_{4\widetilde{B}_j^\delta} \leq \tilde{c}_1 \chi_\Omega \tag{2.9}$$

$$\textit{if} \quad 4\widetilde{B}_i^\delta \cap 4\widetilde{B}_j^\delta = \emptyset, \quad \textit{then} \quad \frac{1}{2} r(\widetilde{B}_i^\delta) \leq r(\widetilde{B}_j^\delta) \leq 2r(\widetilde{B}_i^\delta). \tag{2.10}$$

Step 4.

Let φ be a smooth function from $[0,\infty)$ to R such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on [0,3], $\varphi \equiv 0$ on $[4,\infty)$, $|\varphi'| \leq 2$, and put $\widetilde{\varphi}_j^\delta(x) = \varphi\left(\rho(\tilde{x}_j^\delta, x)/\tilde{r}_{j\delta}\right)$, so that

(i) $\widetilde{\varphi}_j^\delta \in H_X^{1,p}(\Omega, w)$, for any $p \geq 1$ by Proposition 2.9;

(ii) $\widetilde{\varphi}_j^\delta \equiv 1$ on $3\widetilde{B}_j^\delta$, $\operatorname{supp}\widetilde{\varphi}_j^\delta \subseteq 4\widetilde{B}_j^\delta$;

(iii) $|X\widetilde{\varphi}_j^\delta| \leq c/\tilde{r}_{j\delta}$, again by Proposition 2.9.

Now, set $B_j^\delta = 4\widetilde{B}_j^\delta$ and define

$$\varphi_i^\delta = \frac{\widetilde{\varphi}_i^\delta}{\sum_j \widetilde{\varphi}_j^\delta} \qquad \text{for} \quad i \in N.$$

Note that by (2.7), the sum at the denominator is ≥ 1. Moreover only a finite number of balls B_j^δ has non void intersection with B_i^δ, since, if $B_j^\delta \cap B_i^\delta \neq \emptyset$, then both their radii and their Lebesgue measures are comparable (by (2.10) and doubling property), so that

$$\tilde{c}_1|\Omega| \geq \sum_{B_j^\delta \cap B_i^\delta \neq \emptyset} |B_i^\delta| \geq c\, \sharp\{j : B_j^\delta \cap B_i^\delta \neq \emptyset\}\ |\widetilde{B}_i|,$$

which implies that $\sharp\{\cdots\} := \sharp S_i < \infty$. Thus $\operatorname{supp}\varphi_i^\delta \subset B_i^\delta$, $|\varphi_i^\delta| \leq 1$ and hence $\varphi_i^\delta \in L^q(\Omega)$ for any $q \geq 1$. Moreover

$$|X\varphi_i^\delta| \leq \frac{|X\widetilde{\varphi}_i^\delta|}{\sum_j \widetilde{\varphi}_j^\delta} + \frac{\widetilde{\varphi}_i^\delta \sum_{j\in S_i} |X\widetilde{\varphi}_j^\delta|}{\left(\sum_j \widetilde{\varphi}_j^\delta\right)^2} \leq c/\tilde{r}_i^\delta,$$

again by (2.10). Then $\varphi_i^\delta \in W_X^{1,q}(\Omega) = H_X^{1,q}(\Omega)$ for any $q \geq 1$. Let now p be fixed and let ε be as in Step 3. We can choose $(g_h)_{h\in N}$ such that $g_h \in C_0^\infty(5\widetilde{B}_{\varphi j}^\delta)$ for any $h \in N$ and $g_h \overset{h\to\infty}{\longrightarrow} \varphi_i^\delta$ in $W_X^{1,q}(\Omega)$, with $q = p(1+\varepsilon)/\varepsilon$. Then, by Hölder inequality,

$$\|\ \varphi_i^\delta - g_h\ \|_{W_X^{1,p}(\Omega,w)} \leq \|\ \varphi_i^\delta - g_h\|_{W_X^{1,q}(\Omega)} \cdot \Big(\int_{5\widetilde{B}_j^\delta} w^{1+\varepsilon} dx\Big)^{\frac{1}{p(1+\varepsilon)}},$$

so that $g_h \longrightarrow \varphi_i^\delta$ in $W_X^{1,p}(\Omega, w)$. Thus we have proved that $\varphi_i^\delta \in H_X^{1,p}(\Omega, w)$.

With the notations of Theorem 2.11, if $f \in W_X^{1,p}(\Omega, w)$ put now

$$f_\delta = \sum_i c(f, B_i^\delta)\, \varphi_i^\delta,$$

Note now that, if K is a compact subset of Ω only a finite number of B_j^δ intersect K, since, by (2.8), if $B_j^\delta \cap K \neq \emptyset$, then $\tilde r_{j\delta} \geq r_K > 0$ and, in turn, this is possible only for a finite number of balls, by Proposition 2.3. Thus it is easy to see that $f_\delta \in L^1_{\text{loc}}(\Omega)$ and $|X f_\delta| \in L^1_{\text{loc}}(\Omega)$. Moreover,

$$X_j f_\delta = \sum_i c(f, B_i^\delta)\ (X_j \varphi_i^\delta) \quad \text{for} \quad j = 1, \dots, m.$$

and same argument shows that $f_\delta \in H^{1,p}_{X,\text{loc}}(\Omega, w)$, since on any compact set f_δ is a finite linear combination of functions in $H_X^{1,p}(\Omega, w)$.
First of all, let us prove that

$$\int_\Omega |f_\delta - f|^p w\, dx \longrightarrow 0 \quad \text{as} \quad \delta \longrightarrow 0+\,. \tag{2.11}$$

In particular, this will imply that $f_\delta \in L^p(\Omega, w)$ for any $\delta > 0$. Indeed by (2.6), (2.7) and (2.8)

$$\begin{aligned}\int_\Omega |f_\delta - f|^p w\, dx &\leq c\ \sum_i \int_{B_i^\delta} |c(f, B_i^\delta) - f|^p w\, dx \\ &\leq c\ \sum_i \tilde r_{i\delta}^p \int_{B_i^\delta} |Xf|^p w\, dx \leq c\ \delta^p \int_\Omega |Xf|^p w\, dx.\end{aligned}$$

Let us prove now that

$$\int_\Omega |X f_\delta|^p w\, dx \leq c \int_\Omega |Xf|^p w\, dx \quad \text{for all} \quad \delta > 0. \tag{2.12}$$

Indeed for $j = 1, \dots, m$ we have

$$X_j f_\delta = \sum_i c(f, B_i^\delta) X_j \varphi_i^\delta = \sum_i \big(c(f, B_i^\delta) - f\big)\ X_j \varphi_i^\delta,$$

since

$$\sum_i f(X_j \varphi_i^\delta) = f \cdot X_j \left(\sum_i \varphi_i^\delta\right) \equiv 0,$$

because $\{\varphi_i^\delta : i \in N\}$ is a partition of unit. Thus

$$\begin{aligned}\int_\Omega |Xf_\delta|^p w\,dx &\le c \sum_i \int_\Omega |f - c(f, B_i^\delta)|^p\,|X\varphi_i^\delta|^p w\,dx \\ &\le c \sum_i \frac{1}{\tilde{r}_{i\delta}^p} \int_{B_i^\delta} |f - c(f, B_i^\delta)|^p w\,dx \\ &\le c \sum_i \int_{B_i^\delta} |Xf|^p w\,dx \qquad \text{by (2.6)} \\ &\le c \int_\Omega |Xf|^p w\,dx \quad (\text{by (2.9)}).\end{aligned}$$

Now we can complete the proof of Theorem 2.7. First of all, combining (2.11) and (2.12), we obtain that $f_\delta \in W_X^{1,p}(\Omega, w)$, and then that $f_\delta \in H_X^{1,p}(\Omega, w)$, by Lemma 2.14. Again by (2.11) and (2.12) it follows that the set $\{f_\delta, \delta \le 1/10\}$ is bounded in $H_X^{1,p}(\Omega, w)$ which is reflexive for $p > 1$. Thus there exists a sequence $(\delta_n)_{n\in N}$ converging to zero such that $f_n = f_{\delta_n}$ converges weakly in $H_X^{1,p}(\Omega, w)$ to a function g. Because of (2.11), $g \equiv f$, so that f is the weak limit in $W_X^{1,p}(\Omega, w)$ of a sequence of functions in $H_X^{1,p}(\Omega, w)$. Finally (for instance applying Mazur's theorem) there is another sequence converging strongly in $H_X^{1,p}(\Omega, w)$ to f. $\square$

It is clear from the proof that we used assumption (2.1) only to obtain (2.5), so that we might restrict ourselves to assume (2.5) only. Even if (2.5) in many cases is obtained from (2.1) (as in [FLW], [FS], [F1] and [F2]), this remark has some interest because of recent results by [SCos], [MSCos], [BM1] and [BM2], where "abstract Sobolev-Poincaré inequality" are proved without using any representation formula. Moreover, if we already assume that (2.5) holds, then we can weaken the assumptions on w as in [SC].* Indeed we have

Theorem 2.16. *Let $\Omega \subset\subset \Omega_0$ be a bounded open set, $1 \le p < +\infty$ and $X = (X_1, \ldots, X_m)$ be a family of Lipschitz continuous vector fields defined in Ω_0. Suppose that X satisfy (H1), (H2) and (2.5). Suppose that w is a weight function such that $w, w^{1/(1-p)} \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$W_X^{1,p}(\Omega, w) = H_X^{1,p}(\Omega, w).$$

3 Applications: Degenerate elliptic equations and Embedding theorems

Let us consider now the second order degenerate elliptic operator in divergence form

$$\mathcal{L}u := \sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u),$$

*Meanwhile, we stress again that formula (2.1) is supposed to hold only for *regular* functions, which seems much easier to verify, since it does not require any knowledgence of the structure of the functions belonging to W-spaces.

where $a_{ij} = a_{ji} \in L^\infty(\Omega)$ and

$$\nu\, w(x) \sum_{l=1}^{m} \langle X_l(x), \xi\rangle^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \frac{1}{\nu}\, w(x) \sum_{l=1}^{m} \langle X_l(x), \xi\rangle^2$$

for $\nu \in (0,1]$, for a.e. $x \in \Omega$ and for any $\xi \in R^n$. Let us assume that $X_1, \dots, X_m$ satisfy (H1) and (H2) and that $w \in A_2(\Omega, \rho, dx)$.

If $u \in H^{1,2}_{X,\mathrm{loc}}(\Omega, w)$ and $\mathcal{L}u = 0$ in the distribution sense, that is if

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)\, \partial_j u\, \partial_j \varphi\, dx = 0 \qquad \forall \varphi \in \mathcal{C}_0^\infty(\Omega) \tag{3.1}$$

we say that u is local H-solution of $\mathcal{L}u = 0$ in Ω. Analogously, if $u \in W^{1,2}_{X,\mathrm{loc}}(\Omega, w)$ and if (3.1) holds not only for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ but for all $\varphi \in W^{1,2}_X(\Omega, w)$ and compactly supported in Ω we say that u is a local W-solution. Note that even if $\{X_1, \dots, X_m\} = \{\partial_1, \dots, \partial_n\}$ but $w \notin A_2$ it may happen that $H^{1,2}_X(\Omega, w) \neq W^{1,2}_X(\Omega, w)$ and the two notions of solution are actually different. It is clear that $H^{1,2}_X(\Omega, w) \subset W^{1,2}_X(\Omega, w)$ does not imply that H-solutions are W-solutions.

The problem of the regularity of the H and W-solutions when $H \neq W$ seems to be a very difficult one (see e.g. [SC]). However, the following abstract result is now easy to prove using Theorem 2.7, Proposition 2.9 and the standard approach to Harnack's inequality for these degenerate elliptic operators (see, e.g., [FS]).

Theorem 3.1. *Let the hypotheses of Theorem 2.7 be satisfied with $p = 2$. Then*

(i) *u is a local H-solution of $\mathcal{L}u = 0$ if and only if u is a local W-solution;*

(ii) *if u is a local solution of $\mathcal{L}u = 0$ and $u \geq 0$, then there is a geometric constant $c > 1$, depending also on ν and on the A_2 bound of w such that*

$$\sup_{B(x,r)} u \leq c \inf_{B(x,r)} u$$

for any metric ball $B(x,r)$, with $r < \frac{1}{2} dist_\rho(x, \partial\Omega)$.

Proof. Assertion (i) follows from Theorem 2.7. To prove (ii) we can use Moser's iteration techniques as in [FS], since the existence of test functions follows from Proposition 2.9 as shown in Step 5 above. We stress that a Sobolev inequality for functions supported in $B(x,r)$ follows from (2.5) applied to the ball $B(x, 2r)$. $\square$

Again in the spirit of Theorem 2.16, we can prove the following version of Theorem 3.1.

Theorem 3.2. *Suppose (H1), (H2) hold, and let w be a weight function in $A_2(\Omega, \rho, dx)$. Suppose that (2.6) holds with $p = 2$ for any $f \in W^{1,2}_X(\Omega, w)$. Then the conclusions of Theorem 3.1 hold.*

The proof is the same as that of the previous theorem, with the exception of the Sobolev's inequality which now does not follow straightforwardly from our Poincaré inequality, since we do not have any gain of sumability in (2.6). However, we can still manage to obtain this gain by applying Theorem 1 in [BM2] (keeping in mind Remark 1 therein).

Remark 3.3. We note explicitly that the results of Theorems 3.1 and 3.2 are more or less known in the cases of Remark 2.2, except for the statement (i). (See also [SCos]).

We conclude by pointing out that it follows from Theorem 2.11 that $\overset{\circ}{H}{}^{1,p}_X(\Omega, w_1)$ (the closure of $C_0^\infty(\Omega)$ in $H^{1,p}_X(\Omega, w_1)$) is compactly embedded in $L^q(\Omega, w_2)$ for some $q \geq p$. This result extends or partially overlaps with compact embedding theorems proved in other papers. See [BKL], [CDG1], [CDG2], [D], [FS], [GN], [L] and the references therein. More precisely, we have

Theorem 3.4. *Let $\Omega \subset\subset \Omega_0$ be a bounded open set, $1 \leq p \leq q < \infty$ and $X = (X_1, \dots, X_m)$ be a family of Lipshitz continuous vector fields defined in Ω_0. Let w_1 and w_2 be weight functions.*
Assume that X satisfy (H1), (H2) and (2.1).
Assume that w_1, $w_2 \in A_p(\Omega_0, \rho, dx)$ and that balance condition (2.4) holds uniformly on Ω, that is we assume that there is $c_1 > 0$ such that $\forall$ metric balls $I \subseteq J$, centered in $\overline{\Omega}$ with radii $r(I)$, $r(J)$, we have

$$\frac{r(I)}{r(J)} \left(\frac{w_2(I)}{w_2(J)}\right)^{1/q} \leq c_1 \left(\frac{w_1(I)}{w_1(J)}\right)^{1/p}. \tag{3.2}$$

Finally, suppose that $\forall \varepsilon$, $\exists r(\varepsilon) > 0$ such that $\forall I = B_\rho(\bar{x}, r)$ with $\bar{x} \in \overline{\Omega}$ and $r < r(\varepsilon)$

$$r(I)\, w_2(I)^{1/q}\, w_1(I)^{-1/p} < \varepsilon. \tag{3.3}$$

Then $\overset{\circ}{H}{}^{1,p}_X(\Omega, w_1)$ is compactly embedded in $L^q(\Omega, w_2)$.

In particular, if $q_0 > q$ and (3.2) holds with q replaced by q_0, then for all $q \in (p, q_0)$, (3.3) holds, so that $\overset{\circ}{H}{}^{1,p}_X(\Omega, w_1)$ is compactly embedded in $L^q(\Omega, w_2)$.

Proof. First, fix $\bar{r} > 0$ and cover $\overline{\Omega}$ by a finite number of metric balls of radius $\bar{r}$. Then any ball of radius $\bar{r}$ centered at a point of $\overline{\Omega}$ meets one of these balls, and hence, by doubling, both its w_1-measure and its w_2-measure, are equivalent to the corresponding measures of the other ball. Thus, for any ball J with center in $\overline{\Omega}$ and radius $\bar{r}$

$$r(J) w_2(J)^{1/q_0} w_1(J)^{-1/p} \leq c_2.$$

Hence, if I is any ball of radius $r(I) < \bar{r}$ and J is the ball with the same center and radius $\bar{r}$, we get from (3.2)

$$r(I)\, \frac{w_2(I)^{1/q}}{w_1(I)^{1/p}} = r(I)\, \frac{w_2(I)^{1/q_0}}{w_1(I)^{1/p}}\, w_2(I)^{1/q - 1/q_0} \leq c_1\, c_2\, w_2(I)^{1/q-1/q_0},$$

which is small if $r(I)$ is small, since $1/q - 1/q_0 > 0$. Then the assertion is proved. Because of our assumptions, Theorem 2.11 and Remark 2.13 and arguing as in [FLW], Theorem 2, we can choose in (2.5)

$$c(f, B) = \frac{1}{w_2(B)} \int\limits_B f(x) w_2(x)\, dx. \tag{3.4}$$

Let now $(f_h)_{h\in N}$ be a sequence in the unit ball of $\overset{\circ}{H}{}^{1,p}_X(\Omega, w_1)$; without loss of generality, we can assume that $f_h \in C_0^\infty(\Omega)$ for any $h \in N$. By Theorem 2.11, $(f_h)_{h\in N}$ is bounded in $L^q(\Omega, w_2)$, which is reflexive, since $q > p \geq 1$; then, we can suppose that $f_h \rightharpoonup f$ weakly in $L^q(\Omega, w_2)$. We can argue now as in [FS], Theorem 4.6. By a Vitali type argument, we can cover $\overline{\Omega}$ by a finite (because of the compactness) family of metric balls $B(x_1, r), \dots, B(x_{m(r)}, r)$ for $r > 0$ such that

(i) $B(x_k, r/5) \cap B(x_h, r/5) = \emptyset \quad \text{for} \quad k \neq h$;

(ii) $m(r) \leq c\, r^{-\alpha}$, where α is the constant of Proposition 2.3 (i);

(iii) for any i, $\sharp\{k : \ B(x_k, r) \cap B(x_i, r) \neq \emptyset\} \leq M$, where M is a geometric constant.

Thus we have

$$
\begin{aligned}
\int_\Omega |f_n - f_m|^q\, w_2(x)\, dx &\leq \sum_j \int_{B(x_j,r)} |f_n - f_m|^q\, w_2(x)\, dx \\
&\leq c \left\{ \sum_j \int_{B(x_j,r)} \left| (f_n - f_m) - \frac{1}{w_2(B(x_j,r))} \int_{B(x_j,r)} (f_n - f_m) w_2 dy \right|^q w_2(x)\, dx \right. \\
&\qquad \left. + \sum_j w_2(B(x_j,r))^{1-q} \left| \int_{B(x_j,r)} (f_n - f_m) w_2(x)\, dx \right|^q \right\} \\
&= c\, \left\{ \sum_j I_j + \sum_j J_j \right\}.
\end{aligned}
$$

Now, by (2.5),

$$
\sum_j I_j \leq \ c \sum_j \left(r \frac{w_2(B(x_j,r))^{1/q}}{w_1(B(x_j,r))^{1/p}} \right)^q \left(\int_{B(x_j,r)} |X(f_n - f_m)\,|^p w_1(x)\, dx \right)^{q/p},
$$

so that, if $r < r(\varepsilon)$, from (3.3) and (iii) above, we have

$$
\begin{aligned}
\sum_j \ I_j &\leq c\, \varepsilon^q \left(\sum_j \int_{B(x_j,r)} |X(f_n - f_m)|^p w_1(x)\, dx \right)^{1/p} \\
&\leq c\ \varepsilon^q \big\{ \|f_n\|^q_{H^{1,p}_X(\Omega,w_1)} + \|f_m\|^q_{H^{1,p}_X(\Omega,w_1)} \big\} = c\ \varepsilon^q
\end{aligned}
$$

Let now $r < r(\varepsilon)$ be fixed. Since w_2 is doubling there is $\gamma > 0$ such that $w_2(B_\rho(x_j, r)) \geq c r^\gamma$, hence

$$
\sum_j J_j \leq c\ r^{\gamma(1-q)} \ \sum_j \left| \int_{B(x_j,r)} (f_n - f_m) w_2(x)\, dx \right| \leq c\ r^{\gamma(1-q)}\ m(r)\ \varepsilon
$$

because $f_n \rightharpoonup f$. Hence $(f_h)_h$ is a Cauchy sequence in $L^q(\Omega, w_2)$ and the theorem is proved. $\square$

Remark 3.5. We point out that condition $w_2 \in A_p(\Omega_0, \rho, dx)$ can be weakened as in Theorem 2.11 and Remark 2.13.

Remark 3.6. It is easy to see that condition (3.2) is necessary in order for compact immersion to hold (for condition (3.1), see for instance [CW]), at least if the open set Ω satisfies the following geometric condition: if $B = B(x, r)$ is a metric ball with center $x \in \overline{\Omega}$ and radius $r < r(\Omega)$, then there exists $\widetilde{B} = B(y, \tilde{r})$ such that $c(\Omega)r \le \tilde{r} \le C(\Omega)r$, $\widetilde{B} \cap B \neq \emptyset$ and $\widetilde{B} \subset \Omega$, where the constants $r(\Omega)$, $c(\Omega)$, $C(\Omega)$ are geometric constants. This condition is trivially satisfied if, for instance, Ω is a metric ball or, more generally, a John domain (see, e.g., [BKL]). Suppose now by contradiction that the immersion is compact and that (3.3) fails to hold. Then there exists a sequence of metric balls $B_h = B(x_h, r_h)$ such that $r_h \longrightarrow 0$ and $r_h w_2(B_h)^{1/q} w_1(B_h)^{-1/p} \ge \varepsilon_0 > 0$ for all $h \in N$. By our geometrical assumption and by doubling, we can assume that $B_h \subset \Omega$ for any $h \in N$. Moreover, without loss of generality, we can assume that $x_h \longrightarrow \bar{x} \in \overline{\Omega}$. Let now φ be a smooth real function, $0 \le \varphi \le 1$ $\varphi \equiv 1$ on [0,1], supp $\varphi \subseteq [0, 2]$, and let us put

$$f_h(x) = c_h \, \varphi \, \left(\frac{\rho(x_h, x)}{r_h} \right), \quad \text{where} \quad c_h = r_h w_1(B_h)^{-1/p}.$$

Now it is easy to see that, by our choice of c_h, $(f_h)_h$ is bounded in $H^{1,p}_X(\Omega, w_1)$ (note that $f_h \in H^{1,p}_X(\Omega, w_1)$, by Proposition 2.9). Thus, without loss of generality, we can assume by compactedness that $(f_h)_h$ converges strongly in $L^q(\Omega, w_2)$. On the other hand, supp$f_h \subseteq B\left(x_n, \rho(x_n, \bar{x}) + r_n\right)$, so that $f_h \longrightarrow 0$ a.e. in Ω, and hence $f_h \longrightarrow 0$ strongly in $L^q(\Omega, w_2)$. But

$$\|f_h\|^q_{L^q_{w_1}(\Omega)} = c_h^q \int_\Omega \varphi^q \, \left(\frac{\rho(x_h, x)}{r_h} \right) w_2(x) \, dx \ge c_h^q \, w_2(B_h) = r_h^q \, \frac{w_2(B_h)}{w_1(B_h)^{q/p}} \ge \varepsilon_0^q$$

a contradiction.

References

[BH] B. Bojarski and P. Hajłasz, *Pointwise inequalities for Sobolev functions and some applications*, Studia Mathematica **106** (1993), 77–92.

[BKL] S. Buckley, P. Koskela and G. Lu, *Boman equals John*, Proceedings of XVI Rolf Nevanlinna Colloquium (Joensuu, August 1-5, 1995), to appear.

[BM1] M. Biroli and U. Mosco, *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur., to appear.

[BM2] M. Biroli and U. Mosco, *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces*, Proceedings of the Conference "Potential theory and partial differential operators with nonnegative characteristic form," Parma, February 1994, Kluwer, Amsterdam, to appear.

[C] A.P. Calderon, *Inequalities for the maximal function relative to a metric*, Studia Math. **57** (1976), 297–306.

[CDG1] L. Capogna, D. Danielli and N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. Partial Differential Equations **18** (1992), 1765–1794.

[CDG2] L. Capogna, D. Danielli and N. Garofalo, *Subelliptic mollifiers and a characterization of Rellich and Poincaré domains*, Rend. Sem. Mat. Univ. Politec. Torino **51** (1993), 361–386.

[CGL] G. Citti, N. Garofalo and E. Lanconelli, *Harnack's inequality for a sum of squares of vector fields plus a potential*, Amer. J. Math. **115** (1993), 639–73.

[CPSC] V. Chiadò-Piat and F. Serra Cassano, *Relaxation of degenerate variational integrals jour Nonlinear Analysis* **22** (1994), 409–424.

[CW] S. Chanillo and R.L. Wheeden, *Weighted Poincaré and Sobolev inequalities and estimates for the Peano maximal function*, Amer. J. Math. **107** (1985), 1191–1226.

[D] D. Danielli, *A compact embedding theorem for a class of degenerate Sobolev spaces*, Rend. Sem. Mat. Univ. Politec. Torino **49** (1991), 399–420.

[FP] C. Fefferman and D.H. Phong, *Subelliptic eigenvalue problems*, Conference on Harmonic Analysis, Chicago, 1980, W. Beckner et al. ed., Wadsworth, 1981, pp. 590–606.

[F1] B. Franchi, *Weighted Sobolev-Poincare Inequalities and Pointwise Estimates for a Class of Degenerate Elliptic Equations*, Trans. Amer. Math. Soc. **327** (1991), 125–158.

[F2] B. Franchi, *Inégalités de Sobolev pour des champs de vecteurs lipschitziens*, C. R. Acad. Sci. Paris – Ser. A **311** (1990), 329–332.

[FGuW] B. Franchi, C. Gutierrez and R.L. Wheeden, *Weighted Sobolev-Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations **19** (1994), 523–604.

[FL] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa **10** **(4)** (1983), 523–541.

[FLW] B. Franchi, G. Lu and R.L. Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier, Grenoble **45** (1995), to appear.

[FR] K.O.Friedrichs, *The identity of weak and strong extensions of differential operators*, Trans. Am. Math. Soc. **55** (1944), 132–151.

[FS] B. Franchi and R. Serapioni, *Pointwise estimates for a class of strongly degenerate elliptic operators*, Ann. Scuola Norm. Sup. Pisa **(IV)** **14** (1987), 527–568.

[FSSC] B. Franchi, R. Serapioni and F. Serra Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, C.R.Acad.Sci. Paris **t.320, SI** (1995), 695–698.

[GN] N. Garofalo and D.-M. Nhieu, *A general extension theorem for Sobolev spaces arising from systems of non-commuting vector fields* (1995), preprint.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1977.

[H1] P. Hajłasz, *Geometric approach to Sobolev spaces and badly degenerate elliptic equations*, Banach Center Publications (1995).

[H2] P. Hajłasz, *Sobolev Spaces on an Arbitrary Metric Space*, Potential Analysis, to appear.

[H3] P. Hajłasz, *Note on Meyers-Serrin theorem*, Exposition Math., to appear.

[HK] P. Hajłasz and P. Koskela, *Sobolev meets Poincaré*, C.R. Acad. Sci. Paris **320** (1995).

[L1] G. Lu, *Weighted Poincaré and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Revista Mat. Iberoamericana **8** (1992), 367–439.

[L2] G. Lu, *Existence and size estimates for the Green's function of differential operators constructed from degenerate vector fields*, Comm. Partial Differential Equations **17** (1992), 1213–1251.

[MSCos] P. Maheux and L. Saloff-Coste, *Analyse sur les boules d'un opérateur sous-elliptique*, preprint (1994).

[MS] N.G. Meyers, J. Serrin, $H = W$, Proc. Nat. Acad. Sci. USA **51** (1964), 1055–1056.

[M] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[SCos] L. Saloff-Coste, *A note on Poincaré, Sobolev and Harnack inequalities*, Internat. Math. Research Notices (Duke Math. J.)vol 65(2) (1992), 27–38.

[SC] F. Serra Cassano, *On the local boundedness of certain solutions for a class of degenerate elliptic equations*, preprint 1994.

[NSW] A. Nagel, E.M. Stein and S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1985), 103–147.