Determinants of Latin squares of a given pattern

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Abstract
Cycle structure of autotopisms of Latin squares determine all possible patterns of this kind of design. Moreover, given any isomorphism, the number of Latin squares containing it in its autotopism group only depends on the cycle structure of this isomorphism. This number has been studied in [2] for Latin squares of order up to 7, by following the classification given in [3]. Specifically, regarding each symbol of a Latin square as a variable, any Latin square can be seen as the vector space associated with the solution of an algebraic system of polynomial equations, which can be solved using Gröbner bases, by following the ideas implemented by Bayer [2] to solve the problem of n-colouring a graph. However, computations for orders higher than 7 have been shown to be very difficult without using some other combinatorial tools. In this sense, we will see in this paper the possibility of studying the determinants of those Latin squares related to a given cycle structure. Specifically, since the determinant of a Latin square can be seen as a polynomial of degree $n$ in $n$ variables, it will determine a new polynomial equation that can be included into the previous system. Moreover, since determinants of Latin squares of order up to 7 determine their isoclinic classes [5], we will study the set of isoclinic classes of Latin squares of these orders related to each cycle structure.

Introduction and notation

A Latin square $L$ of order $n$ is an $n \times n$ array with elements chosen from a set of $n$ distinct symbols (in this paper, it will be the set $[n] = \{1, 2, \ldots, n\}$) such that each symbol occurs precisely once in each row and each column. The set of Latin squares of order $n$ is denoted by $LS(n)$. Given $L = (l_{ij}) \in LS(n)$, the orthogonal representation of $L$ is the set of $n^2$ triples $(i, j, l_{ij})$, i, j, $l_{ij} \in [n]$.

The permutation group on $[n]$ by $S_n$. Every permutation $\pi \in S_n$ can be uniquely written as a composition of $m$ pairwise-disjoint cycles, $\pi = C_1 \circ C_2 \circ \cdots \circ C_m$, where for all $i \in [m]$, one has $C_i = (i_1, i_2, \ldots, i_k)$ with $i_1 \leq i_2 \leq \cdots \leq i_k$.

The cycle structure of $\pi$ is the sequence $l_{\pi} = (l_1^1 l_2^1 \cdots l_1^\pi l_2^\pi \cdots)$, where $l_i^\pi$ is the number of cycles of length $i$ for all $i \in [1, 2, \ldots, n]$, thus $l_\pi = (l_1, l_2, \ldots, l_n)$ is the cardinal of the set of fixed points of $\pi$. Fix$(\pi) = \{i \in [n] \mid i \notin \pi(i)\}$. Given $\pi \in S_n$, one defines the complete Latin square $L^\pi = (l_{ij}^\pi)$, where $l_{ij}^\pi = 1$ if $(i, j) \in \pi$. If $F(x_1, x_2, \ldots, x_n)$ is a $S_n$-invariant form, then $F(l_{ij}^\pi)$ is the $S_n$-invariant form $F^\pi$.

Given $L = (l_{ij}) \in LS(n)$, it is defined its associated matrix $X_L$ which is obtained by replacing each element $l_{ij}$ by the variable $x_{ij}$. The determinant $\det(L)$ of $L$ is the homogeneous polynomial of degree $n$ in $n$ variables $\det(X_L)$. Two polynomials $p_1$ and $p_2$ in $\{x_{ij} \mid i, j \in [n]\}$ are said to be similar and it is denoted $p_1 \sim p_2$, if there exists a permutation $\sigma \in S_n$ such that $p_2(x_{\sigma(i), \sigma(j)}) = p_1(x_{ij})$ for all $i, j \in [n]$. Thus, it is verified that $\det(L^\sigma) = \det(L)$ for a Latin square related to similar determinants.

An isomorphism of a Latin square $L = (l_{ij}) \in LS(n)$ is a triple $\alpha = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$. In this way, $\alpha, \beta$ and $\gamma$ are permutations of rows, columns and symbols of $L$, respectively.

The resulting square $L^{\alpha, \beta, \gamma} = (\alpha(i, j), \beta(i, j), \gamma(i, j)) \mid i, j \in [n]$ is also a Latin square and it is said to be isoclinic to $L$. To be isoclinic is an equivalence relation and thus, the set of Latin squares being isoclinic to a given one is the isoclinic class. Given two isoclinic Latin squares $L$ and $L'$, $L' = L^{\alpha, \beta, \gamma}$, it is verified that $\det(L') = \det(L)$ and $\det(L')$ are also isoclinic. Two Latin squares $L_1$ and $L_2$ are said to be of the same isoclinic type if $L_2 = L_1^{\alpha, \beta, \gamma}$. It is also an equivalence relation. The number of isoclinic types and types of the set $LS(n)$ is known for all $n \leq 10$ [5].

Patterns of autotopisms of small Latin squares

Finally, we give the classification of the sets of patterns and determinants corresponding to the cycle structure of all autotopisms of Latin squares of order $2 \leq n \leq 6$. To do so, we have used the classification of all possible cycle structures given in [5] and the following result:

Proposition 3. Given $\pi \in S_n$, it is verified that $\det(L^\pi)$ is smaller than the number of different types of Latin squares of order $n$.

By computing the distinct factorizations of the determinants of the Latin squares related to each cycle structure, we have obtained the following table:

<table>
<thead>
<tr>
<th>$\pi(1,2)$</th>
<th>$\pi(1,3)$</th>
<th>$\det(L)$</th>
<th>$P_1$</th>
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References